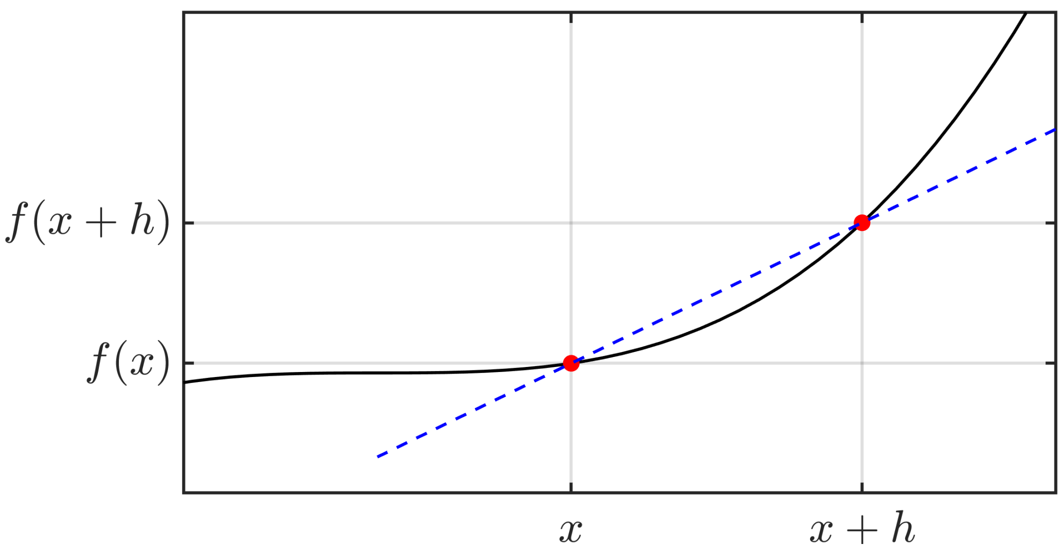
Calculus for Engineers

Lecture Notes for Coursera

Jeffrey R. Chasnov



THE HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY

The Hong Kong University of Science and Technology
Department of Mathematics
Clear Water Bay, Kowloon
Hong Kong

Preface

View the promotional video on YouTube

These lecture notes accompany my online Coursera course, Calculus for Engineers. This course is designed for students who are proficient in mathematics up to the high school level. My course reviews precalculus in the first module, followed by the essential material of derivatives and integrals in the second and third modules. The fourth module introduces the important topic of Taylor series, while the fifth and sixth modules delve into selected applications of calculus.

Each chapter in these notes is a numbered lecture, and corresponds to a specific lecture video on Coursera. Additionally, all Coursera videos are available on YouTube, with links provided at the beginning of each lecture chapter.

At the end of each lecture chapter, you will find problems that highlight the key concepts discussed. While students in a formal university calculus course may be assigned many more problems, these notes adopt a "less is more" philosophy. The provided problems are intended to reinforce understanding without overwhelming students. I encourage all students to attempt these problems; however, should you encounter difficulties, complete solutions are available in the Appendix. Practice quizzes are also included as an extra resource, with solutions provided.

JEFFREY R. CHASNOV
Hong Kong
March 2025

Contents

Module I

Foundations

In this module's lectures, we review the foundational material necessary for understanding calculus. We revisit the set of real numbers and demonstrate how to manipulate complex numbers. We define functions and their inverses, as well as discuss the concepts of limits and continuity. Finally, we introduce the essential functions studied in calculus, including polynomial and rational functions, exponential functions, logarithmic functions, trigonometric functions, and inverse trigonometric functions.

Lecture 1 | Real numbers

View this lecture on YouTube

Calculus is built on the foundation of real numbers, represented by the number line. Let's review these numbers, starting with more basic ones.

Children first learn the counting numbers, one through ten, using their ten fingers as aids. As they grow, they learn to count beyond ten, and imagine the infinite set of counting numbers known as the natural numbers, symbolized by $\mathbb{N}$. Adding 0 to this set creates the whole numbers. Including negative numbers forms the set of integers, denoted by $\mathbb{Z}$. For clarity, natural numbers are sometimes referred to as positive integers, and whole numbers as non-negative integers.

Expanding the set of integers to include ratios of integers (excluding division by zero) gives us the rational numbers, denoted by $\mathbb{Q}$. This set is dense, meaning a rational number exists between any two rational numbers. Engineers work with a finite subset of these numbers, for example, the double precision numbers used in computing.

Mathematicians, however, need numbers that are not rational, and these numbers are called irrational. The Pythagoreans discovered the first known irrational number, the square root of two, denoted as $\sqrt{2}$. This number, together with all the rational numbers, belongs to the larger set of algebraic numbers, which are roots of polynomial equations with integer coefficients. The square root of two, for example, is a root of the equation $x^2 - 2 = 0$.

There exists non-algebraic numbers, and these are called transcendental numbers. The most famous such number is π, first introduced in trigonometry. In calculus, we meet another transcendental number called Euler's number, denoted by the letter e. Transcendental numbers form an infinite set of numbers, and in some well-defined sense, there are more transcendental numbers than there are algebraic numbers.

The union of the sets of rational and irrational numbers forms the set of real numbers, denoted by $\mathbb{R}$. These numbers complete the number line essential to calculus.

Problems for Lecture 1

1. Consider two rational numbers a and b. Identify a rational number that lies between a and b and prove that it is rational.

2. Prove that $\sqrt{2}$ is an irrational number. Start by assuming $\sqrt{2}$ is rational and use this assumption to reach a contradiction.

Solutions to the Problems

Lecture 2 | Complex numbers

View this lecture on YouTube

The real number system can be extended to include complex numbers, denoted by $\mathbb{C}$. We define the imaginary number i as a number satisfying $i^2 = -1$. Formally, we write $i = \sqrt{-1}$. A complex number z is represented as

$$z = x + iy,$$

where x and y are real numbers. Here, x is called the real part of z and y is called the imaginary part of z, denoted by

$$x = \operatorname{Re} z, \qquad y = \operatorname{Im} z.$$

Complex numbers are equal if their real and imaginary parts are equal. The complex conjugate of $z = x + iy$, denoted as $\bar{z}$, is

$$\bar{z} = x - iy.$$

Using z and $\bar{z}$, the real and imaginary parts of z can be expressed as

$$\operatorname{Re} z = \frac{1}{2}\left(z + \bar{z}\right), \qquad \operatorname{Im} z = \frac{1}{2i}\left(z - \bar{z}\right).$$

Furthermore,

$$z\bar{z} = (x + iy)(x - iy) = x^2 - i^2y^2 = x^2 + y^2;$$

and we define the absolute value or modulus of z by

$$|z| = (z\bar{z})^{1/2} = \sqrt{x^2 + y^2}.$$

Complex numbers can be added, subtracted, multiplied, and divided to yield another complex number. Division is the only operation that requires an example,

$$\frac{x + iy}{s + it} = \frac{(x + iy)(s - it)}{(s + it)(s - it)} = \frac{xs + yt}{s^2 + t^2} + i\frac{ys - xt}{s^2 + t^2}.$$

Problems for Lecture 2

1. Write as a complex number $z = x + iy$, where x and y are real:

a) $-(7-i)(-4-2i)(2-i)$;

b) $\dfrac{1+3i}{3-2i}$;

c) $\dfrac{1}{1+i} + \dfrac{1}{1-i}$.

2. Solve the following complex equation for x and y, where x and y are real:

$$2y + ix = 4 + x - i.$$

3. Find two complex numbers for $\sqrt{i}$ by writing $\sqrt{i} = x + iy$, squaring both sides, and solving for x and y.

Solutions to the Problems

Practice Quiz | Numbers

1. Consider $\sqrt{n}$, where n is a natural number between 1 and 20. How many of these square roots are rational?

a) 0

b) 2

c) 4

d) 8

2. The complex number $\dfrac{1}{(1+i)^2}$ is equal to

a) $1+i$

b) $\dfrac{i}{2}$

c) $1-i$

d) $-\dfrac{i}{2}$

3. The solution of $(4+i)(x+iy) = 8-3i$ for x and y is

a) $x = \dfrac{29}{17}, \; y = -\dfrac{20}{17}$

b) $x = \dfrac{20}{17}, \; y = -\dfrac{29}{17}$

c) $x = -\dfrac{29}{17}, \; y = \dfrac{20}{17}$

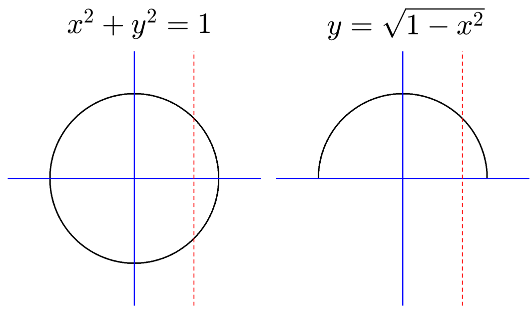
d) $x = -\dfrac{20}{17}, \; y = \dfrac{29}{17}$

Solutions to the Practice quiz

Lecture 3 | Functions and their inverses

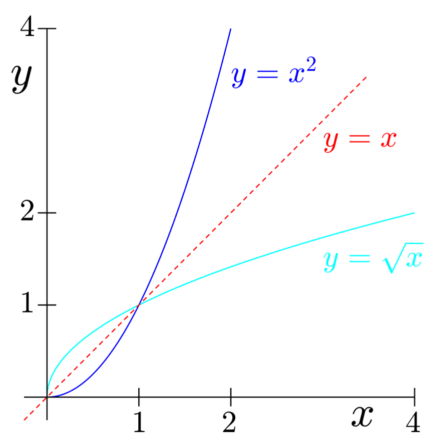
View this lecture on YouTube

In calculus, we study functions that map a set of real numbers, known as the domain, to another set of real numbers, known as the range. A key characteristic of a function is that each element in the domain corresponds to exactly one element in the range.



When graphing a function, it must pass the vertical line test. This means that a vertical line drawn within the domain of the function should intersect the graph at no more than one point. For instance, in the provided figure, a full circle fails this test, whereas a semi-circle passes.

An inverse function is essentially a function reflected over the line $y = x$, swapping the roles of the domain and the range. For an inverse function to exist, the original function must pass both the vertical line test and the horizontal line test. If the function passes both tests, we say that the function is one-to-one.



For example, the quadratic function $y = x^2$ over the domain $\mathbb{R}$ lacks an inverse because values in its range generally correspond to two values in its domain, specifically $y = (\pm x)^2$. However, if we restrict the domain to non-negative real numbers, the function $y = x^2$ does have an inverse, which is the positive square-root function, $y = \sqrt{x}$. The figure illustrates these functions and how they are reflections of each other across the line $y = x$.

If $y = f(x)$ is a function with an inverse, and $y = f^{-1}(x)$ is its inverse, then both of the following conditions hold:

$$x = f(f^{-1}(x)) \quad \text{and} \quad x = f^{-1}(f(x)).$$

Using the quadratic function $y = x^2$ with $x \geq 0$ as an example, we have

$$x = (\sqrt{x})^2 \qquad \text{and} \qquad x = \sqrt{x^2}.$$

Problems for Lecture 3

1. Find the inverse of the linear function given by

$$f(x) = mx + b.$$

2. Find the inverse of the unit quarter-circle function given by

$$f(x) = \sqrt{1 - x^2}, \qquad 0 \leq x \leq 1.$$

Solutions to the Problems

Lecture 4 | Limits and continuity

View this lecture on YouTube

Calculus depends on the concept of limits. We often encounter expressions such as

$$\lim_{x \to a} f(x),$$

which represents the value approached by $f(x)$ as x approaches a.

When calculating limits, direct substitution of a into $f(x)$ often results in an indeterminate form like $0/0$. Through algebraic manipulation or calculus techniques, these limits can often be resolved. For example, consider

$$\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}.$$

Directly substitution of $h = 0$ results in $0/0$, but with some algebraic simplification,

$$\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0} \frac{h(2x+h)}{h} = \lim_{h \to 0}(2x + h) = 2x,$$

where we have canceled the h in the numerator and denominator before setting $h = 0$.

If it happens that $f(a)$ is well-defined and finite, and

$$\lim_{x \to a} f(x) = f(a),$$

then we say that $f(x)$ is continuous at a. Describing a function as continuous over an interval implies that its graph forms an unbroken curve, allowing one to draw it without lifting pen from paper. A continuous function has no holes, no jumps, and no infinite divergences at any finite point.

Continuous functions obey the intermediate value theorem. This states that if a continuous function has values $f(a)$ and $f(b)$ at the endpoints of a closed interval $[a, b]$, it must take on every intermediate value between $f(a)$ and $f(b)$ within that interval. While this is an important theorem in pure mathematics, our use of it will often be implicit.

Problems for Lecture 4

1. Evaluate the following limits:

a) $\lim\limits_{x_2 \to x_1} \dfrac{x_2^2 - x_1^2}{x_2 - x_1}$;

b) $\lim\limits_{h \to 0} \dfrac{(x+h)^3 - x^3}{h}$.

Solutions to the Problems

Lecture 5 | Polynomials and rational functions

View this lecture on YouTube

Polynomial functions, which frequently occur in calculus, are defined as

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n,$$

where n is called the degree of the polynomial.

The domain of polynomial functions is $\mathbb{R}$. According to the fundamental theorem of algebra, an nth degree polynomial has exactly n complex roots, counted with multiplicity. For instance, a quadratic polynomial

$$f(x) = ax^2 + bx + c$$

has roots determined by the quadratic formula

$$x_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The nature of the roots depends on the discriminant, $b^2 - 4ac$, where

- if $b^2 - 4ac > 0$, there are two distinct real roots;
- if $b^2 - 4ac < 0$, there are two complex conjugate roots;
- if $b^2 - 4ac = 0$, there is one real root of multiplicity two.

A rational function is the ratio of two polynomial functions. With a numerator of degree m and a denominator of degree n, a rational function has the form

$$f(x) = \frac{a_0 + a_1 x + \cdots + a_m x^m}{b_0 + b_1 x + \cdots + b_n x^n}.$$

The domain of a rational function is $\mathbb{R}$, excluding the real roots of the denominator. For example, the rational function

$$f(x) = \frac{1}{1 - x},$$

is defined for all x except where $x = 1$.

Problems for Lecture 5

1. Find the ranges of the polynomial functions

a) $f(x) = x^2$;

b) $f(x) = x^3$;

c) $f(x) = x^2 - 4x + 3$.

2. What is the domain and range of the rational function

$$f(x) = \frac{1}{x^2 - 1}\,?$$

Solutions to the Problems

Practice Quiz | Elementary functions

1. The inverse function of $f(x) = (1+x)^2$, where $x \geq -1$, is

a) $\sqrt{1+x}$

b) $\sqrt{1-x}$

c) $\sqrt{x}+1$

d) $\sqrt{x}-1$

2. The limit $\lim\limits_{x\to 4} \dfrac{\sqrt{x}-2}{x-4}$ is

a) 0

b) ∞

c) $\dfrac{1}{4}$

d) $\dfrac{1}{2}$

3. One of the roots of the quadratic polynomial $f(x) = x^2 - x - 1$ is

a) $\dfrac{\sqrt{5}+1}{2}$

b) $\dfrac{\sqrt{5}-1}{2}$

c) $\dfrac{\sqrt{3}+1}{2}$

d) $\dfrac{\sqrt{3}-1}{2}$

Solutions to the Practice quiz

Lecture 6 | Exponential function

View this lecture on YouTube

The general form of an exponential function with a positive base, $a > 0$, is given by

$$f(x) = a^x.$$

The exponential function adheres to the usual rules of exponents,

$$a^x a^y = a^{x+y}, \qquad \frac{a^x}{a^y} = a^{x-y}, \qquad (a^x)^y = a^{xy}.$$

In calculus, when referring to the exponential function, we typically use a specific base denoted by e, also known as Euler's number. The exponential function is then expressed in one of two ways,

$$f(x) = e^x = \exp(x).$$

The number e is transcendental, similar to π, and can be defined by the limit,

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n.$$

The convergence of this limit to the first five decimal places is illustrated in the table below.

n	$\left(1+\frac{1}{n}\right)^n$
1	2
10	2.59374
10^2	2.70481
10^3	2.71692
10^4	2.71815
10^5	2.71827
10^6	2.71828

Problems for Lecture 6

1. Using our definition for e, show that

$$e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$$

Solutions to the Problems

Lecture 7 | Natural logarithm

View this lecture on YouTube

The inverse function to the exponential function $\exp(x)$ is called the natural logarithm, commonly denoted as $\ln x$. Parentheses are optional, but may be used for clarity. The domain of the exponential function is $\mathbb{R}$, while its range consists solely of positive real numbers. Consequently, the domain of the natural logarithm is the positive real numbers. Occasionally we use $\ln|x|$ to include negative real numbers as well. The logarithm of zero is always undefined.

The natural logarithm is simply the logarithm function with base e, and is sometimes written as

$$\ln x = \log_e x.$$

The standard rules of logarithms apply:

$$\ln(xy) = \ln x + \ln y, \quad \ln(x/y) = \ln x - \ln y, \quad \ln(x^r) = r \ln x.$$

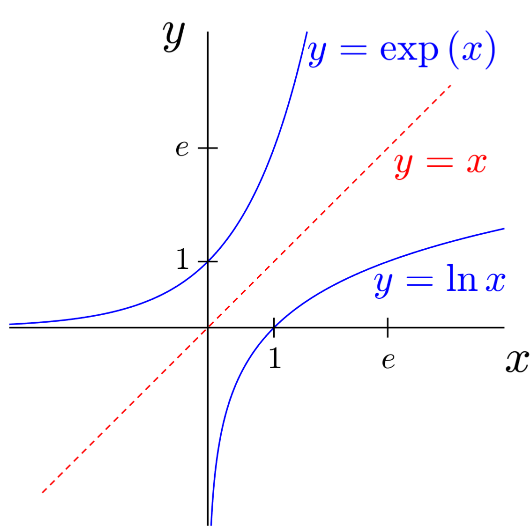
Additionally, we have

$$\ln 1 = 0, \qquad \ln e = 1.$$

The exponential function and the natural logarithm can be used to define the power law function, $y = x^p$, for all real values of p. Since $\exp(x)$ and $\ln x$ are inverse functions, we can write

$$x^p = \exp(\ln x^p) = \exp(p \ln x).$$

A graph of both $y = \exp(x)$ and $y = \ln x$ are shown below. Notice that these functions are reflections of each other across the dashed line, $y = x$.



Problems for Lecture 7

1. Combine the following expression into a single logarithm:

$$\ln\left(\frac{B-C}{B}\right) - \ln\left(\frac{A-C}{A}\right).$$

2. Using the properties of the exponential function, prove that for all real p and q,

a) $x^p x^q = x^{p+q}$

b) $(x^p)^q = x^{pq}$

3. Derive the change of base formula between $\log_a x$ and $\ln x$ by writing $y = \log_a x$ as $a^y = x$ and taking the natural logarithm of both sides.

Solutions to the Problems

Lecture 8 | Trigonometric functions

View this lecture on YouTube

The cosine function, $\cos x$, and the sine function, $\sin x$, are fundamental in calculus. While degrees may have been used in high school, calculus uses radians. An angle in radians is defined as the arc length subtended by that angle divided by the radius of the circle.

Given that the circumference of a circle with radius R is $2\pi R$, a full revolution around a circle corresponds to 2π radians, which is equivalent to $360°$. Since an angle expressed in radians represents the ratio of two lengths, angles in radians are dimensionless numbers. An argument of a function such as $\cos x$ and $\sin x$ should also be dimensionless, which is why we use radians in calculus.

The $\cos x$ and $\sin x$ functions are the prototypical periodic functions. Their period is 2π, meaning that for any integer n,

$$\cos\,(x + 2\pi n) = \cos x, \qquad \sin\,(x + 2\pi n) = \sin x.$$

These functions are also related by a phase shift of $\pi/2$:

$$\cos\,(x - \frac{\pi}{2}) = \sin x, \qquad \sin\,(x + \frac{\pi}{2}) = \cos x.$$

A commonly used Pythagorean identity is

$$\cos^2 x + \sin^2 x = 1.$$

The addition formulas for sine and cosine are

$$\sin\,(x + y) = \sin x \cos y + \cos x \sin y, \qquad \cos\,(x + y) = \cos x \cos y - \sin x \sin y.$$

Other trigonometric functions encountered in calculus include tangent ($\tan x$), cotangent ($\cot x$), secant ($\sec x$) and cosecant ($\csc x$), defined as

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}.$$

It is sometimes useful to parametrize a circle of radius r using polar coordinates. The Cartesian coordinates are given by

$$x = r\cos\theta, \quad y = r\sin\theta, \qquad 0 \le \theta < 2\pi.$$

Problems for Lecture 8

1. Convert the angles 45°, 60°, 90°, and 180° to radians.

2. Use the addition formulas to derive the following phase-shift relations:

$$\cos\left(x - \frac{\pi}{2}\right) = \sin x, \qquad \sin\left(x + \frac{\pi}{2}\right) = \cos x.$$

3. Use the addition formulas to derive the following double-angle formulas:

$$\sin(2x) = 2\sin x \cos x, \qquad \cos(2x) = \cos^2 x - \sin^2 x.$$

4. Use the double-angle formulas and the Pythagorean theorem to derive the following power-reduction formulas:

$$\cos^2 x = \frac{1 + \cos(2x)}{2}, \qquad \sin^2 x = \frac{1 - \cos(2x)}{2}.$$

5. Derive the Pythagorean identities

a) $1 + \tan^2\theta = \sec^2\theta$;

b) $1 + \cot^2\theta = \csc^2\theta$.

6. Use the polar-coordinate relationships,

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

to show that

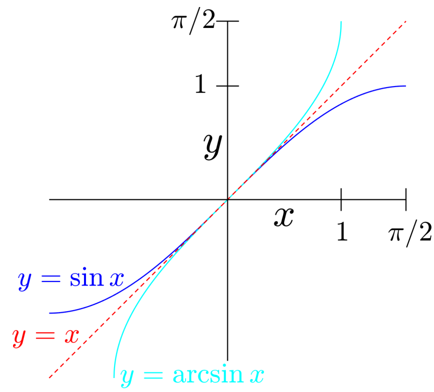
$$r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}.$$

Solutions to the Problems

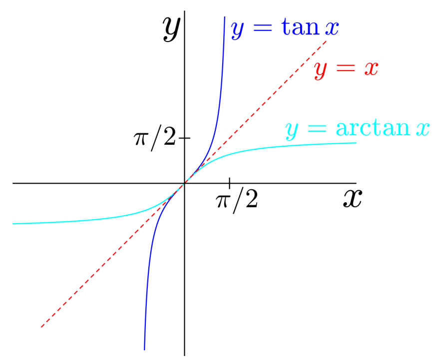
Lecture 9 | Inverse trigonometric functions

View this lecture on YouTube

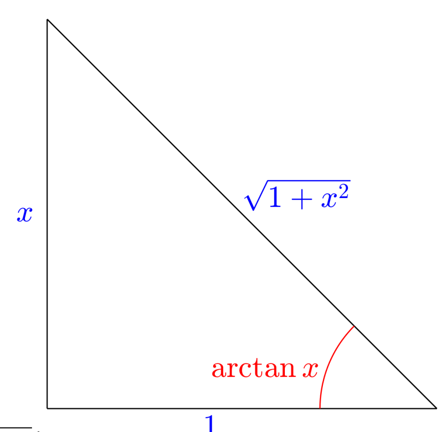
The inverses of trigonometric functions can be defined by restricting their domains to ensure that they are one-to-one. The inverse of the cosine function is defined by restricting the domain of the cosine function to the interval $[0, \pi]$, while the inverse of the sine function is defined by restricting the domain of the sine function to the interval $[-\pi/2, \pi/2]$. These inverse functions are often denoted as $\cos^{-1} x$ and $\sin^{-1} x$, or alternatively as $\arccos x$ and $\arcsin x$. Here, we show a plot of the sine and arcsine functions.



Among the inverse trigonometric functions, the arctangent is perhaps the most useful. By restricting the domain of the tangent function to $(-\pi/2, \pi/2)$, we find that it diverges at the endpoints of this interval. Consequently, the range of this restricted tangent function is all of $\mathbb{R}$, making the domain of the arctangent function also $\mathbb{R}$. The arctangent function has horizontal asymptotes at $\pi/2$ as $x \to \infty$ and at $-\pi/2$ as $x \to -\infty$. Here we show a plot of the tangent and arctangent functions.



Sometimes, it is necessary to find the sine or cosine of an inverse trigonometric function. For example, consider the composite functions, $\sin(\arctan x)$ and $\cos(\arctan x)$. We can visualize this (see plot) by drawing a right triangle where the interior angle is $\arctan x$. In this triangle, the tangent of the angle equals x, allowing us to assign the opposite side a length of x and the adjacent side a length of one. The hypotenuse then has length $\sqrt{1+x^2}$, leading to



$$\sin(\arctan x) = \frac{x}{\sqrt{1+x^2}}, \qquad \cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}.$$

Problems for Lecture 9

1. Sketch a plot of the cosine function restricted to the domain $[0, \pi]$, and its corresponding inverse arccosine.

2. By drawing triangles, simplify the following expressions:

a) $\cos(\arcsin x)$;

b) $\tan(\arcsin x)$;

c) $\sin(\arccos x)$;

d) $\tan(\arccos x)$.

3. Derive the identity, $\arcsin x + \arccos x = \pi/2$.

Solutions to the Problems

Practice Quiz | Transcendental functions

1. Which of the following expressions is not equal to $\exp\left(x^2\right)$?

a) $\lim_{n\to\infty}\left(1+\frac{x^2}{n}\right)^n$

b) $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n^2}$

c) $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{nx}$

d) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{nx^2}$

2. $\ln A - \ln B + \ln C - \ln D$ simplifes to

a) $\ln\left(\frac{AB}{CD}\right)$

b) $\ln\left(\frac{AC}{BD}\right)$

c) $\ln\left(\frac{AD}{BC}\right)$

d) $\ln\left(\frac{BD}{AC}\right)$

3. For the values of x for which the following expressions are defined, which of the following expressions is not equal to the others?

a) $\frac{\sin\left(\arccos x\right)}{\cos\left(\arcsin x\right)}$

b) $\frac{\sin\left(\arctan x\right)}{\cos\left(\arctan x\right)}$

c) $\tan\left(\arcsin x\right)\cos\left(\arcsin x\right)$

d) $\cot\left(\arccos x\right)\sin\left(\arccos x\right)$

Solutions to the Practice quiz

Module II

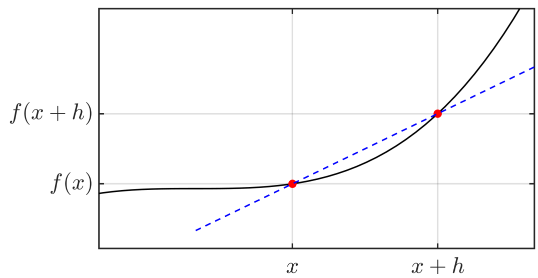
Differentiation

In this module's lectures, we define the derivative and explore methods to differentiate various functions. We begin by learning the power rule to differentiate power functions, followed by learning the sum, product, quotient, and chain rules. We then learn how to differentiate exponential functions, natural logarithms, trigonometric functions, and finally, inverse trigonometric functions.

Lecture 10 | Definition of the derivative

View this lecture on YouTube

Consider a function $y = f(x)$ and two points on its curve: $(x, f(x))$ and $(x + h, f(x + h))$. Connect these points with a secant line (see figure). As h approaches zero, this line becomes the tangent line to the curve at the point $(x, f(x))$. The derivative of $f(x)$ is defined as the slope of this tangent line, with definition



$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$

If we replace h by Δx and $f(x + \Delta x) - f(x)$ by Δy, and introduce what is called the Leibniz notation, the definition of the derivative becomes

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

We refer to dy and dx as differentials. These represent the almost limiting values of Δy and Δx as Δx approaches zero. Intuitively, dx is considered infinitesimally small — greater than zero but smaller than any real number. The differential of $y = f(x)$ is defined by $dy = f'(x)dx$. In this form, the Leibniz notation for the derivative, dy/dx, is treated like a fraction.

The second derivative, or the derivative of the derivative, is denoted in Leibniz notation as

$$\frac{d}{dx}\left(\frac{d}{dx}\right) = \frac{d^2}{dx^2},$$

where dx^2 is a shorthand for $(dx)^2$. Common notations for the first and second derivatives include

$$f'(x), \quad \frac{df}{dx}, \quad y', \quad \frac{dy}{dx}, \qquad f''(x), \quad \frac{d^2 f}{dx^2}, \quad y'', \quad \frac{d^2 y}{dx^2}.$$

For functions of time, such as $x = x(t)$, the dot notation is often used,

$$\dot{x} = \frac{dx}{dt}.$$

The second derivative in this notation is $\ddot{x}$, with additional dots for higher derivatives.

A function is considered differentiable if it is continuous and does not have sharp corners where the tangent slope would vary depending on the direction from which the limit is approached, such as $f(x) = |x|$ at $x = 0$.

Problems for Lecture 10

1. We defined the derivative as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Another common definition is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Show that these two definitions are equivalent.

2. Write the third and fourth derivative of $y = y(x)$ using the Leibniz notation.

3. By examining the graphs of the following functions, determine the values of x for which these functions are differentiable:

a) $y = \ln |x|$;

b) $y = \arctan x$.

4. Evaluate the following limit by examining the graph of the arctangent:

$$\lim_{x \to \infty} \frac{d}{dx} \arctan x.$$

Solutions to the Problems

Lecture 11 | Power rule

View this lecture on YouTube

Consider the power function, x^n, where n is a positive integer. The derivative of this power function is given by

$$\frac{d}{dx}x^n = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}.$$

Using the binomial expansion, we expand $(x+h)^n$, explicitly showing only the first two terms,

$$(x+h)^n = x^n + nhx^{n-1} + \dots,$$

where the omitted terms contain higher powers of h. Thus, the derivative simplifies to

$$\frac{d}{dx}x^n = \lim_{h\to 0}\frac{x^n + nhx^{n-1} + \dots - x^n}{h} = \lim_{h\to 0}\frac{nhx^{n-1} + \dots}{h} = nx^{n-1},$$

where the terms containing nonzero powers of h vanish as h goes to zero. This result is known as the power rule,

$$\frac{d}{dx}x^n = nx^{n-1}.$$

We will later extend this rule to all real exponents. For now, let us prove the power rule for the square-root function. We have

$$\frac{d}{dx}\sqrt{x} = \lim_{h\to 0}\frac{\sqrt{x+h} - \sqrt{x}}{h}.$$

By multiplying the numerator and the denominator by $\sqrt{x+h} + \sqrt{x}$, we obtain

$$\begin{aligned}\frac{d}{dx}\sqrt{x} &= \lim_{h\to 0}\frac{\left(\sqrt{x+h} - \sqrt{x}\right)\left(\sqrt{x+h} + \sqrt{x}\right)}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \lim_{h\to 0}\frac{x+h-x}{h\left(\sqrt{x+h} + \sqrt{x}\right)} \\ &= \lim_{h\to 0}\frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \lim_{h\to 0}\frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.\end{aligned}$$

This can be rewritten following the power rule formula as

$$\frac{d}{dx}x^{1/2} = \frac{1}{2}x^{-1/2}.$$

Problems for Lecture 11

1. Determine the derivative of $f(x) = 1/x$ and show that it satisfies the power rule.

2. Determine the derivative of $f(x) = x^{3/2}$ and show that it satisfies the power rule.

Solutions to the Problems

Lecture 12 | Sum and constant-multiple rules

View this lecture on YouTube

Consider two differentiable functions, $f(x)$ and $g(x)$. The derivative of their sum is given by

$$
\begin{aligned}
(f(x)+g(x))' &= \lim_{h\to 0}\frac{(f(x+h)+g(x+h))-(f(x)+g(x))}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = f'(x)+g'(x).
\end{aligned}
$$

Thus, the derivative of the sum of two functions is the sum of their derivatives. An analogous result also holds true for subtraction.

The derivative of a function multiplied by a constant, c, is given by

$$
(cf(x))' = \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h} = c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = cf'(x).
$$

Thus, the derivative of a constant times a function is the constant times the derivative of the function. Additionally, the derivative of a constant function is zero.

Using the power rule, addition rule, and constant-multiple rule, we can differentiate any polynomial function. For example, applying these differentiation rules, we find

$$
\frac{d}{dx}\left(ax^2+bx+c\right) = 2ax+b.
$$

For the second derivative, we find

$$
\frac{d^2}{dx^2}\left(ax^2+bx+c\right) = \frac{d}{dx}(2ax+b) = 2a.
$$

And for the third (and higher-order derivatives), we find

$$
\frac{d^3}{dx^3}\left(ax^2+bx+c\right) = \frac{d}{dx}(2a) = 0.
$$

Problems for Lecture 12

1. Consider the polynomial function,

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n.$$

Find the constants a_0, a_1, a_2, a_3, $\ldots$, a_n using $f(x)$ and its derivatives at $x = 0$.

Solutions to the Problems

Practice Quiz | Differentiation rules I

1. Which of the following expressions is not a definition of the derivative of $f(x)$?

a) $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

b) $\lim_{h \to 0} \frac{f(x) - f(x-h)}{h}$

c) $\lim_{h \to 0} \frac{f(x+h) - f(x-h)}{h}$

d) $\lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}$

2. The derivative $\frac{d}{dx}\sqrt{x^3}$ is equal to

a) $\frac{3\sqrt{x}}{2}$

b) $\frac{3}{2\sqrt{x}}$

c) $\frac{2\sqrt{x}}{3}$

d) $\frac{2}{3\sqrt{x}}$

3. The second derivative $\frac{d^2}{dx^2}\left(ax^3 + bx^2 + cx + d\right)$ is equal to

a) $3ax^2 + 2bx + c$

b) $6ax + 2b$

c) $6a$

d) 6

Solutions to the Practice quiz

Lecture 13 | Product rule

View this lecture on YouTube

Consider two differentiable functions, $f(x)$ and $g(x)$. Using the definition of the derivative, the derivative of their product is given by

$$[f(x)g(x)]' = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}.$$

By adding and subtracting $f(x)g(x+h)$ in the numerator, we obtain

$$\begin{aligned}
[f(x)g(x)]' &= \lim_{h\to 0} \frac{[f(x+h)g(x+h) - f(x)g(x+h)] + [f(x)g(x+h) - f(x)g(x)]}{h} \\
&= \lim_{h\to 0} \frac{[f(x+h) - f(x)]\,g(x+h) + f(x)\,[g(x+h) - g(x)]}{h} \\
&= \left(\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right)\left(\lim_{h\to 0} g(x+h)\right) + f(x)\left(\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}\right) \\
&= f'(x)g(x) + f(x)g'(x).
\end{aligned}$$

Thus, the derivative of a product follows the rule, "the derivative of the first times the second, plus the first times the derivative of the second."

Example: Find the derivative of the function $(x^2+1)(x^2-1)$.

Both functions (x^2+1) and (x^2-1) have the same derivative $2x$. Using the product rule,

$$\frac{d}{dx}\left[(x^2+1)(x^2-1)\right] = 2x(x^2-1) + (x^2+1)2x = 2x^3 - 2x + 2x^3 + 2x = 4x^3.$$

Alternatively, by expanding and differentiating,

$$\frac{d}{dx}\left[(x^2+1)(x^2-1)\right] = \frac{d}{dx}\left[x^4 - 1\right] = 4x^3.$$

While expanding the product may work for polynomials, the product rule becomes more essential for complex functions.

Problems for Lecture 13

1. Using the product rule for two functions, determine the product rule for three functions.

2. Using the product rule and $\frac{dx}{dx} = 1$, prove that $\frac{d(x^2)}{dx} = 2x$.

Solutions to the Problems

Lecture 14 | Quotient rule

View this lecture on YouTube

Consider two differentiable functions, $f(x)$ and $g(x)$. The derivative of their quotient is

$$\left(\frac{f(x)}{g(x)}\right)' = \lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} = \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x+h)g(x)}.$$

By adding and subtracting $f(x)g(x)$ in the numerator, we obtain

$$\left(\frac{f(x)}{g(x)}\right)' = \lim_{h\to 0}\frac{[f(x+h)g(x) - f(x)g(x)] - [f(x)g(x+h) - f(x)g(x)]}{hg(x+h)g(x)}.$$

$$= \left(\lim_{h\to 0}\frac{1}{g(x+h)g(x)}\right)\left[\left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right)g(x) - f(x)\left(\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right)\right]$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Thus, the derivative of a quotient follows the rule, "the derivative of the top times the bottom, minus the top times the derivative of the bottom, over the bottom squared."

Example: Find the derivative of $(x^2-1)/(x^2+1)$.

Both functions (x^2+1) and (x^2-1) have the derivative $2x$. Using the quotient rule,

$$\frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right) = \frac{2x(x^2+1) - (x^2-1)2x}{(x^2+1)^2} = \frac{2x^3+2x-2x^3+2x}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}.$$

Problems for Lecture 14

1. Derive the quotient rule by writing

$$f(x) = g(x)\left[\frac{f(x)}{g(x)}\right],$$

and differentiating both sides of this equation.

2. Find the derivative of the function $(x^2 + 1)/(x^2 - 1)$.

3. Prove the power rule for all negative integer exponents by using the quotient rule and the power rule for positive integer exponents.

Solutions to the Problems

Lecture 15 | Chain rule

View this lecture on YouTube

Suppose we have a function y that depends on x, and a function x that depends on t. This relationship is expressed as $y = y(x(t))$. The derivative of y with respect to t is obtained using the chain rule,

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt},$$

where, using Leibniz notation, we treat the derivatives as ratios of differentials.

For example, consider

$$y(x) = x^2 \quad \text{and} \quad x(t) = 1 + t^2.$$

Then

$$\frac{dy}{dx} = 2x \quad \text{and} \quad \frac{dx}{dt} = 2t,$$

so that

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = 4tx = 4t(1+t^2).$$

In practice, the chain rule is usually applied directly in the form

$$\left[f\big(g(x)\big)\right]' = f'\big(g(x)\big)g'(x).$$

Thus, the derivative of a composite function follows the rule, "the derivative of the outside times the derivative of the inside." Repeating the above example, the composite function to differentiate is

$$y(t) = (1+t^2)^2.$$

The derivative of the outside is $2(1+t^2)$ and the derivative of the inside is $2t$, so that

$$\frac{dy}{dt} = 4t(1+t^2).$$

Alternatively, we could have expanded the function as

$$y(t) = (1+t^2)^2 = 1 + 2t^2 + t^4,$$

and then differentiated,

$$\frac{dy}{dt} = 4t + 4t^3 = 4t(1+t^2).$$

But often, converting a composite function into a simpler expression is not possible or desirable.

Problems for Lecture 15

1. Use the chain rule and the power rule to find the derivative of

$$\sqrt{1+x^2}.$$

2. Use the chain rule, the power rule, and the product rule to find the derivative of

$$(x^2+1)^2(x^2-1)^2.$$

3. Use the chain rule, the power rule and the quotient rule to find the derivative of

$$\frac{(x^2-1)^2}{(x^2+1)^2}.$$

4. The chain rule can be used for the composition of more than two functions.

a) Use Leibniz notation to determine the chain rule for the derivative of the composition of three functions, $[f(g(h(x)))]'$.

b) Find the time derivative of

$$\left(1+(1+t^2)^2\right)^2.$$

Solutions to the Problems

Practice Quiz | Differentiation rules II

1. What is the derivative of $\sqrt{1-x^2}$?

a) $\dfrac{x}{2\sqrt{1-x^2}}$

b) $\dfrac{-x}{2\sqrt{1-x^2}}$

c) $\dfrac{x}{\sqrt{1-x^2}}$

d) $\dfrac{-x}{\sqrt{1-x^2}}$

2. What is the derivative of $\dfrac{x^2-1}{x^2+1}$?

a) $\dfrac{4x}{(x^2+1)^2}$

b) $\dfrac{-4x}{(x^2+1)^2}$

c) $\dfrac{4x^3}{(x^2+1)^2}$

d) $\dfrac{-4x^3}{(x^2+1)^2}$

3. What is the derivative of $\sqrt{\dfrac{1-x^2}{1+x^2}}$?

a) $\dfrac{2x}{\sqrt{(1-x^2)(1+x^2)^3}}$

b) $\dfrac{-2x}{\sqrt{(1-x^2)(1+x^2)^3}}$

c) $\dfrac{2x^3}{\sqrt{(1-x^2)(1+x^2)^3}}$

d) $\dfrac{-2x^3}{\sqrt{(1-x^2)(1+x^2)^3}}$

Solutions to the Practice quiz

Lecture 16 | Exponential function

View this lecture on YouTube

The derivative of an exponential function with base a is found from

$$\frac{d}{dx}a^x = \lim_{h\to 0}\frac{a^{x+h}-a^x}{h} = \lim_{h\to 0}\frac{a^x a^h - a^x}{h} = a^x \lim_{h\to 0}\frac{a^h-1}{h}.$$

Notice that at $x = 0$, the derivative is given by

$$\left.\frac{d}{dx}a^x\right|_{x=0} = \lim_{h\to 0}\frac{a^{0+h}-e^0}{h} = \lim_{h\to 0}\frac{a^h-1}{h},$$

showing that the derivative of a^x is a^x times its derivative at $x = 0$.

The exponential function with base e has the unique property that its derivative at $x = 0$ equals one. This can be proven using the definition of e, which we rewrite as

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{m\to 0}(1+m)^{1/m}.$$

We then have

$$\left.\frac{d}{dx}e^x\right|_{x=0} = \lim_{h\to 0}\frac{e^h-1}{h} = \lim_{\substack{h\to 0\\ m\to 0}}\frac{\left[(1+m)^{1/m}\right]^h-1}{h} = \lim_{\substack{h\to 0\\ m\to 0}}\frac{\left[(1+m)^{h/m}\right]-1}{h}.$$

If the ratio h/m is a positive integer, we can apply the binomial expansion to obtain

$$\left.\frac{d}{dx}e^x\right|_{x=0} = \lim_{\substack{h\to 0\\ m\to 0}}\frac{\left[(1+m)^{h/m}\right]-1}{h} = \lim_{\substack{h\to 0\\ m\to 0}}\frac{[1+h+\dots]-1}{h} = 1,$$

a result that can be shown to hold true regardless of the value of the ratio h/m. Thus, the derivative of the exponential function with base e is given by

$$\frac{d}{dx}e^x = e^x.$$

Due to the simplicity of its derivative, the exponential function in calculus and advanced mathematics typically refers to the base e, and is expressed as e^x or $\exp(x)$.

Problems for Lecture 16

1. Find the derivatives of the following functions:

a) e^{-x}

b) e^{-x^2}

c) xe^{-x}

d) $\dfrac{e^{-x}}{x}$

2. One way to define the exponential function is

$$\exp(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n.$$

By differentiating inside the limit using the power rule and the chain rule, show that

$$\frac{d}{dx}\exp(x) = \exp(x).$$

Solutions to the Problems

Lecture 17 | Natural logarithm

View this lecture on YouTube

The natural logarithm, $\ln x$, is the inverse of the exponential function, $\exp(x)$, and is defined for $x > 0$. We can express this inverse relationship as

$$x = \exp(\ln x), \qquad x > 0.$$

Because this is not an explicit equation for $\ln x$, differentiating both sides of this equation with respect to x to obtain an equation for the derivative of $\ln x$ is called implicit differentiation. With $dx/dx = 1$, and applying the chain rule to the right-hand side, we obtain

$$1 = \exp(\ln x)\left(\frac{d}{dx}\ln x\right) = x\left(\frac{d}{dx}\ln x\right).$$

Solving for the derivative of $\ln x$, we find

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

Notice that the function $1/x = x^{-1}$ was the only power law unobtainable from the derivative of another power law, as the derivative of a constant (the zero power law) is zero.

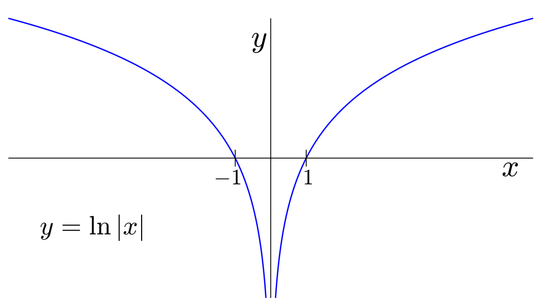
We can also find the derivative of $y = \ln(-x)$ for $x < 0$ using the chain rule,

$$\frac{d}{dx}\ln(-x) = \frac{1}{(-x)}(-1) = \frac{1}{x}.$$

Thus, for $x \neq 0$,

$$\frac{d}{dx}\ln|x| = \frac{1}{x}.$$

The derivative is positive when $x > 0$, indicating an increasing function, and negative when $x < 0$, indicating a decreasing function. We show here a plot of the function $y = \ln|x|$ to illustrate this behavior.



Problems for Lecture 17

1. Find the derivatives of $\ln(ax)$ and $\ln x^r$ using the chain rule, and then show your results are consistent with the identities $\ln(ax) = \ln a + \ln x$ and $\ln x^r = r \ln x$.

2. Find the derivatives of the following functions:

a) $\ln(\ln x)$;

b) $\dfrac{x}{1 + \ln x}$;

c) $\ln(x + \sqrt{1 + x^2})$.

Solutions to the Problems

Lecture 18 | Power rule redux

View this lecture on YouTube

Now that we know how to differentiate the exponential function and the natural logarithm, we can extend the derivative of a power function to all real exponents. For any real number p, and for $x > 0$, we can write

$$x^p = \exp\left(\ln\left(x^p\right)\right) = \exp\left(p \ln x\right).$$

Using the chain rule to differentiate, we find

$$\frac{d}{dx}x^p = \frac{d}{dx}\exp\left(p \ln x\right) = \exp\left(p \ln x\right)\left(\frac{p}{x}\right) = x^p\left(\frac{p}{x}\right) = px^{p-1}.$$

Thus, the power rule is valid for all real values of the exponent.

Problems for Lecture 18

1. Determine the derivative of a^x by differentiating $a^x = \exp(\ln a^x)$.

2. Determine the derivative of $\log_a x$ by differentiating $\log_a x = \ln x/\ln a$.

3. Determine the derivative of x^x.

Solutions to the Problems

Practice Quiz | Exponential and logarithm functions

1. The derivative of xe^{-x^2} is

a) $(1+x)e^{-x^2}$

b) $(1-x)e^{-x^2}$

c) $(1+2x^2)e^{-x^2}$

d) $(1-2x^2)e^{-x^2}$

2. The derivative of $\dfrac{x}{1-\ln x}$ is

a) $\dfrac{1}{1-\ln x}$

b) $\dfrac{1+\ln x}{(1-\ln x)^2}$

c) $\dfrac{2-\ln x}{(1-\ln x)^2}$

d) $\dfrac{2+\ln x}{(1-\ln x)^2}$

3. The derivative of $x^{\ln x}$ is

a) $\dfrac{x^{\ln x}\ln x}{x}$

b) $\dfrac{2x^{\ln x}\ln x}{x}$

c) $\dfrac{x^{\ln x}(1+\ln x)}{x}$

d) $\dfrac{x^{\ln x}(1+2\ln x)}{x}$

Solutions to the Practice quiz

Lecture 19 | Sine and cosine

View this lecture on YouTube

The derivative of the sine function is computed from

$$\begin{aligned}\frac{d}{dx}\sin x &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}\\ &= \cos x\left(\lim_{h\to 0}\frac{\sin h}{h}\right)+\sin x\left(\lim_{h\to 0}\frac{\cos h-1}{h}\right).\end{aligned}$$

The derivative of the cosine function is computed from

$$\begin{aligned}\frac{d}{dx}\cos x &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} = \lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}\\ &= -\sin x\left(\lim_{h\to 0}\frac{\sin h}{h}\right)+\cos x\left(\lim_{h\to 0}\frac{\cos h-1}{h}\right).\end{aligned}$$

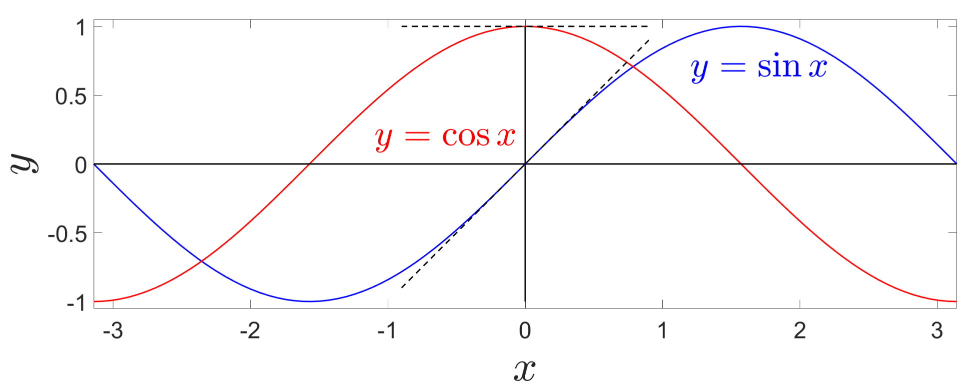
Notice that at $x=0$, the derivatives are given by

$$\left.\frac{d}{dx}\sin x\right|_{x=0} = \lim_{h\to 0}\frac{\sin(0+h)-\sin 0}{h} = \lim_{h\to 0}\frac{\sin h}{h},$$

and

$$\left.\frac{d}{dx}\cos x\right|_{x=0} = \lim_{h\to 0}\frac{\cos(0+h)-\cos 0}{h} = \lim_{h\to 0}\frac{\cos h-1}{h},$$

corresponding to the unknown limits in the above expressions.



From the plot, it is observed that the derivatives of $\sin x$ and $\cos x$ at the origin are one and zero, respectively. These results can be derived from trigonometry, but can also be assumed as part of the definitions of sine and cosine. The key results to remember are

$$\frac{d}{dx}\sin x = \cos x, \qquad \frac{d}{dx}\cos x = -\sin x.$$

Problems for Lecture 19

1. Given $\dfrac{d}{dx}\sin x = \cos x$, determine $\dfrac{d}{dx}\cos x$ by writing $\cos x = \sin\left(x + \pi/2\right)$.

2. Find the derivatives of the following functions:

a) $\ln\left(\sin x\right)$;

b) $\ln\left(\cos x\right)$;

c) $\sin\left(\ln x\right)$;

d) $\cos\left(\ln x\right)$.

3. Let $y = A\cos x + B\sin x$, where A and B are constants. Show that

$$\frac{d^2y}{dx^2} = -y.$$

4. Let $z = \cos x + i\sin x$, where $i = \sqrt{-1}$. Show that

$$\frac{dz}{dx} = iz.$$

Solutions to the Problems

Lecture 20 | Trigonometric functions

View this lecture on YouTube

Using the derivatives,

$$\frac{d}{dx}\sin x = \cos x, \qquad \frac{d}{dx}\cos x = -\sin x,$$

we can derive the derivatives of all other trigonometric functions. We make use of the definitions,

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}.$$

Applying the quotient rule, we find

$$\frac{d}{dx}\tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x;$$

$$\frac{d}{dx}\cot x = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x;$$

$$\frac{d}{dx}\sec x = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{\sin x}{\cos^2 x} = \tan x \sec x;$$

$$\frac{d}{dx}\csc x = \frac{d}{dx}\left(\frac{1}{\sin x}\right) = -\frac{\cos x}{\sin^2 x} = -\cot x \csc x.$$

Problems for Lecture 20

1. Find the derivative of

$$y = \frac{\tan x - 1}{\sec x}$$

in two ways:

a) using the quotient rule and differentiating $\tan x$ and $\sec x$ directly. Try to simplify your resulting solution;

b) converting $y = y(x)$ into a function containing only $\cos x$ and $\sin x$ and then differentiating.

Solutions to the Problems

Lecture 21 | Inverse trigonometric functions

View this lecture on YouTube

To find derivatives of the inverse trigonometric functions, we use implicit differentiation. Starting with the arctangent, we write

$$\tan\left(\arctan x\right) = x.$$

Differentiating both sides with respect to x and applying the chain rule, we obtain

$$\frac{1}{\cos^2(\arctan x)}\frac{d}{dx}(\arctan x) = 1.$$

Solving for the derivative of the arctangent, we find

$$\frac{d}{dx}(\arctan x) = \cos^2(\arctan x).$$

Using the identity

$$\cos(\arctan x) = \frac{1}{\sqrt{1+x^2}},$$

we derive

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2},$$

which is defined for all real x.

For arccosine and arcsine, we write

$$\cos\left(\arccos x\right) = x, \qquad \sin\left(\arcsin x\right) = x.$$

Implicit differentiation gives

$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sin\left(\arccos x\right)}, \qquad \frac{d}{dx}(\arcsin x) = \frac{1}{\cos\left(\arcsin x\right)}.$$

With the identities

$$\sin\left(\arccos x\right) = \cos\left(\arcsin x\right) = \sqrt{1-x^2},$$

we derive

$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}, \qquad \frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}.$$

These formulas are valid for $-1 < x < 1$.

Problems for Lecture 21

1. Given $\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$ and $\arcsin x + \arccos x = \pi/2$, find $\frac{d}{dx}\arccos x$.

2. Find the derivative of

$$\arcsin\left(\frac{1}{\sqrt{1+x^2}}\right).$$

Solutions to the Problems

Practice Quiz | Trigonometric functions

1. The derivative of $\cos(x - \pi/2)$ is

a) $\cos x$

b) $-\cos x$

c) $\sin x$

d) $-\sin x$

2. The second derivative of $\tan x$ is

a) $2\tan^2 x \sec x$

b) $2\tan x \sec^2 x$

c) $\tan^2 x \sec x$

d) $\tan x \sec^2 x$

3. The second derivative of $\arctan x$ is

a) $\dfrac{x}{(1+x^2)^2}$

b) $\dfrac{-x}{(1+x^2)^2}$

c) $\dfrac{2x}{(1+x^2)^2}$

d) $\dfrac{-2x}{(1+x^2)^2}$

Solutions to the Practice quiz

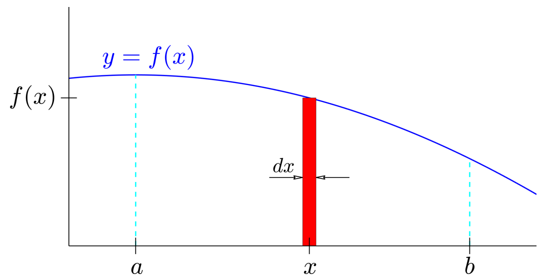
Module III

Integration

In this module's lectures, we define the integral and explore methods to integrate various functions. We begin by learning how the definite integral is used to calculate areas. We then find a connection between integration and differentiation by proving the first and second fundamental theorems of calculus. These theorems motivate us to define an indefinite integral as an anti-derivative. Throughout the module, we will examine various integration techniques, including integration by substitution, integration by parts, integration of trigonometric functions, trigonometric substitution, and integration by partial fractions.

Lecture 22 | Definite integral

View this lecture on YouTube



The definite integral of a positive function $y = f(x)$ from $x = a$ to $x = b$ (where $b > a$) is defined as the area between the vertical lines $x = a$ and $x = b$, the x-axis, and the curve $y = f(x)$ (see figure).

To find the area under the curve, consider an infinitesimally thin rectangle of width dx at position x with height $f(x)$, as shown. The area of this rectangle is given by $dA = f(x)\,dx$. The total area under the curve is found from summing all such infinitesimal areas. The symbol we use for this infinite sum is the integral sign, and we write

$$A = \int dA = \int_a^b f(x)\,dx\,.$$

The integral is read as "the integral from a to b of f of x dee x." The function inside the integral, $f(x)$, is called the integrand, and the dx indicates integration with respect to x. The terms a and b are called the lower and upper limits of the integral, respectively. The variable x in the integral is called a dummy variable, and could be replaced by any symbol such as x' or s.

The integral definition can also be extended to handle cases where $b \le a$ and $f(x) \le 0$:

$$\int_b^a f(x)dx = -\int_a^b f(x)dx \qquad \text{and} \qquad \int_a^b (-f(x))\,dx = -\int_a^b f(x)dx.$$

When $f(x) < 0$, then the area above the curve and the x-axis is counted as negative area. Additionally, the integral from a to c can be split into two integrals, i.e.,

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx,$$

which is interpreted as the total area from a to c is the sum of the areas from a to b and from b to c.

If an integrand exhibits an even symmetry, $f(-x) = f(x)$, or an odd symmetry $g(-x) = -g(x)$, then symmetric integrals around the origin simplify to

$$\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx \qquad \text{and} \qquad \int_{-a}^a g(x)\,dx = 0.$$

Problems for Lecture 22

1. Compute the integral

$$\int_{-\pi}^{\pi} \cos mx \sin nx \, dx \, .$$

2. Simplify the following integrals using the integrand's even or odd symmetry.

a) $\displaystyle\int_{-\pi}^{\pi} x \cos mx \, dx;$

b) $\displaystyle\int_{-\pi}^{\pi} x \sin mx \, dx;$

c) $\displaystyle\int_{-\pi}^{\pi} x^2 \cos mx \, dx;$

d) $\displaystyle\int_{-\pi}^{\pi} x^2 \sin mx \, dx.$

Solutions to the Problems

Lecture 23 | First fundamental theorem of calculus

View this lecture on YouTube

We now consider the definite integral of a function $y = f(x)$ from a constant a to a variable upper limit x. To avoid confusion between the true variable x and the dummy variable of integration, we will use s for the dummy variable, and write

$$y(x) = \int_a^x f(s)ds.$$

To differentiate y with respect to x, we use the definition of the derivative. We have

$$\begin{aligned} y'(x) = \frac{d}{dx}\int_a^x f(s)ds &= \lim_{h\to 0}\frac{\int_a^{x+h} f(s)ds - \int_a^x f(s)ds}{h} \\ &= \lim_{h\to 0}\frac{\int_x^{x+h} f(s)ds}{h}. \end{aligned}$$

As h approaches zero, this integral approaches the area of a single rectangle of width h and height $f(x)$. That is,

$$\frac{d}{dx}\int_a^x f(s)ds = \lim_{h\to 0}\frac{hf(x)}{h} = f(x).$$

This result is called the first fundamental theorem of calculus, and establishes a crucial link between differentiation and integration.

Problems for Lecture 23

1. Suppose a and b are constants. Determine

$$\frac{d}{dx}\int_a^b f(x)\,dx.$$

2. Suppose a is a constant and $f(x)$, $g(x)$, $g_1(x)$, and $g_2(x)$ are functions of x. Use the first fundamental theorem of calculus to simplify the following integrals.

a) $\displaystyle\frac{d}{dx}\int_x^a f(s)\,ds$

b) $\displaystyle\frac{d}{dx}\int_a^{g(x)} f(s)\,ds$

c) $\displaystyle\frac{d}{dx}\int_{g_1(x)}^{g_2(x)} f(s)\,ds$

3. Consider the integral $\displaystyle\int_a^x f(x)\,dy$, where a is a constant. The function f may also depend on the integration variable y, making it a function of two variables, but let's not worry about that here. Using the definition of the derivative, show that

$$\frac{d}{dx}\left[\int_a^x f(x)\,dy\right] = f(x) + \int_a^x f'(x)\,dy.$$

Solutions to the Problems

Lecture 24 | Second fundamental theorem of calculus

View this lecture on YouTube

Antidifferentiation is the reverse process of differentiation. If $F(x)$ is a function such that $F'(x) = f(x)$, then $F(x)$ is an antiderivative of $f(x)$. Antiderivatives are unique except for the addition of a constant.

The first fundamental theorem of calculus states

$$\frac{d}{dx}\int_a^x f(s)\,ds = f(x),$$

which means that the antiderivative of $f(x)$ is given by

$$F(x) = \int_a^x f(s)\,ds + C,$$

where C is a constant. Now, since

$$F(a) = \int_a^a f(s)\,ds + C = C,$$

and $F(b) = \int_a^b f(s)ds + C$, we have derived the second fundamental theorem of calculus,

$$\int_a^b f(s)\,ds = F(b) - F(a),$$

where $F(x)$ is an antiderivative of $f(x)$. Sometimes this theorem is written more simply as

$$\int_a^b f'(x)\,dx = f(x)\Big|_a^b = f(b) - f(a),$$

where the vertical bar indicates evaluation of the antiderivative at the limits of the integral.

Problems for Lecture 24

1. Using the second fundamental theorem of calculus, compute the following integrals.

a) $\displaystyle\int_a^b x^n\,dx$ (for n a positive integer);

b) $\displaystyle\int_a^b \frac{1}{x}\,dx$ (for $0 < a < b$);

c) $\displaystyle\int_0^{\pi/2} \cos x\,dx\,;$

d) $\displaystyle\int_{-\infty}^1 e^x\,dx\,.$

Solutions to the Problems

Lecture 25 | Indefinite integral

View this lecture on YouTube

The second fundamental theorem of calculus connects the evaluation of definite integrals with finding antiderivatives. This relationship leads us to define the indefinite integral.

An antiderivative of $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$. We define the indefinite integral of $f(x)$ as

$$\int f(x)\,dx = F(x) + C,$$

where C is called the constant of integration. We use the integral symbol here, but without lower and upper limits. The same variable x appears on both sides of the equation.

From our knowledge of the derivatives of some elementary functions, we can already compile a short list of antiderivatives. For example, the antiderivative of a power law function, with $n \neq -1$, is

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C.$$

Using the sum rule and constant-multiple rule, the antiderivative of any polynomial function can also be found. The special case $n = -1$ is given by

$$\int \frac{1}{x}\,dx = \ln|x| + C.$$

The exponential function and sine and cosine functions have antiderivatives

$$\int \exp(x)\,dx = \exp(x) + C, \quad \int \sin x\,dx = -\cos x + C, \quad \int \cos x\,dx = \sin x + C.$$

We will learn techniques to find antiderivatives of more complicated functions in upcoming lectures.

Problems for Lecture 25

1. Compute the indefinite integral of the polynomial function,

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n.$$

2. Compute the indefinite integral,

$$\int (\cos x - \sin x)\, dx.$$

3. Compute the indefinite integral,

$$\int (\exp(x) + \exp(-x))\, dx.$$

Solutions to the Problems

Practice Quiz | Fundamental theorems

1. The definite integral $\int_{-\pi}^{\pi} e^{x^2} (\cos mx + \sin mx)\, dx$ simplifies to

a) 0

b) $2\int_{0}^{\pi} e^{x^2} (\cos mx + \sin mx)\, dx$

c) $2\int_{0}^{\pi} e^{x^2} \cos mx\, dx$

d) $2\int_{0}^{\pi} e^{x^2} \sin mx\, dx$

2. The derivative $\frac{d}{dx}\int_{0}^{x^2} f(s)\, ds$ is equal to

a) 0

b) $2x$

c) $2xf(x)$

d) $2xf(x^2)$

3. The definite integral $\int_{0}^{1} x^2\, dx$ is equal to

a) 0

b) $\frac{1}{3}$

c) $\frac{2}{3}$

d) 1

4. The indefinite integral $\int \left(ax^2 + bx + c\right) dx$ is equal to

a) C

b) $2ax + b + C$

c) $ax^2 + bx + C$

d) $\frac{ax^3}{3} + \frac{bx^2}{2} + cx + C$

Solutions to the Practice quiz

Lecture 26 | Integration by substitution

View this lecture on YouTube

Integration by substitution is a powerful technique that uses the chain rule,

$$\frac{d}{dx}f\big(g(x)\big) = f'\big(g(x)\big) \cdot g'(x),$$

to integrate functions of the form

$$\int f'\big(g(x)\big) \cdot g'(x)dx = f\big(g(x)\big) + C.$$

Typically, substitution proceeds by letting $u = g(x)$, and finding the differential $du = g'(x)\,dx$. This transforms the integral to

$$\int f'\big(g(x)\big)g'(x)\,dx = \int f'(u)\,du = f(u) + C = f\big(g(x)\big) + C.$$

Substitution also applies to definite integrals, requiring a change of limits,

$$\int_a^b f'\big(g(x)\big)g'(x)\,dx = \int_{g(a)}^{g(b)} f'(u)\,du = f(g(b)) - f(g(a)).$$

Example: Integrate $\int xe^{-x^2}\,dx$.

Let $u = x^2$, $du = 2x\,dx$. Then the integral transforms as

$$\int xe^{-x^2}\,dx = \frac{1}{2}\int e^{-u}\,du = -\frac{1}{2}e^{-u} + C = -\frac{1}{2}e^{-x^2} + C.$$

Example: Integrate $\int_0^{\pi/2} \sin^3\theta\cos\theta\,d\theta$.

Let $u = \sin\theta$, $du = \cos\theta\,d\theta$. To change limits, we substitute $\theta = 0$ in the lower limit to get $u = \sin 0 = 0$, and $\theta = \pi/2$ in the upper limit to get $u = \sin(\pi/2) = 1$. Then the integral transforms as

$$\int_0^{\pi/2} \sin^3\theta\cos\theta\,d\theta = \int_0^1 u^3\,du = \frac{u^4}{4}\bigg|_0^1 = \frac{1}{4}.$$

Problems for Lecture 26

1. Integrate the following definite and indefinite integrals using a simple substitution.

a) $\displaystyle\int_0^t e^{rs}\,ds;$

b) $\displaystyle\int \cos(\omega t + \phi)\,dt.$

2. Determine the correct substitution and then integrate.

a) $\displaystyle\int_0^x \frac{ds}{(1+\frac{s}{R})^2};$

b) $\displaystyle\int_0^1 x\sqrt{1-x^2}\,dx;$

c) $\displaystyle\int \frac{\ln x}{x}\,dx;$

d) $\displaystyle\int \frac{dx}{x\ln x}.$

Solutions to the Problems

Lecture 27 | Integration by parts

View this lecture on YouTube

Integration by parts is based on the product rule for differentiation. Since

$$(fg)' = f'g + fg',$$

we can rearrange it as

$$f'g = (fg)' - fg'.$$

Integrating and using the fundamental theorem of calculus, we obtain the integration by parts formula

$$\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx.$$

Typically, integration by parts is applied by setting

$$\begin{aligned} u &= g(x), & dv &= f'(x)dx, \\ du &= g'(x)dx, & v &= f(x). \end{aligned}$$

Then, the formula to be remembered is

$$\int u\,dv = uv - \int v\,du.$$

The corresponding formula for definite integrals is

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du.$$

The key idea here is that $\int v\,du$ should be simpler to integrate than $\int u\,dv$.

Example: Integrate $\int xe^x\,dx$.

Let

$$\begin{aligned} u &= x, & dv &= e^x dx, \\ du &= dx, & v &= e^x. \end{aligned}$$

Then

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C = (x-1)e^x + C.$$

Problems for Lecture 27

1. Integrate the following integrals using integration by parts.

a) $\int x \sin x \, dx$;

b) $\int \ln x \, dx$;   (*Hint: let $dv = dx$.*)

c) $\int e^x \sin x \, dx$;   (*Hint: Integrate by parts twice and then solve for the integral.*)

d) $\int \arcsin x \, dx$.   (*Hint: let $dv = dx$. You will also need to integrate by substitution.*)

2. Define the gamma function as the definite integral

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt.$$

a) Using integration by parts, show that $\Gamma(x+1) = x\Gamma(x)$.

b) By direct integration, show that $\Gamma(1) = 1$.

c) Using parts (a) and (b), show that for n a non-negative integer, the Gamma function is related to the factorial function by $\Gamma(n) = (n-1)!$.

Solutions to the Problems

Practice Quiz | Elementary techniques of integration

1. The definite integral $\int_0^{\pi/2} \cos^3\theta \sin\theta \, d\theta$ is equal to

a) 0

b) $\frac{1}{4}$

c) $\frac{1}{2}$

d) 1

2. The definite integral $\int_0^{a/b} \frac{dx}{a+bx}$ is equal to

a) $2\ln a$

b) $2\ln b$

c) $\frac{1}{a}\ln 2$

d) $\frac{1}{b}\ln 2$

3. The definite integral $\int_1^x s \ln s \, ds$ is equal to

a) $\frac{1}{4}x^2\ln x + \frac{1}{2}(x^2-1)$

b) $\frac{1}{2}x^2\ln x + \frac{1}{4}(x^2-1)$

c) $\frac{1}{4}x^2\ln x - \frac{1}{2}(x^2-1)$

d) $\frac{1}{2}x^2\ln x - \frac{1}{4}(x^2-1)$

Solutions to the Practice quiz

Lecture 28 | Trigonometric integrals

View this lecture on YouTube

A common type of integral has the form

$$\int \sin^m \theta \cos^n \theta \, d\theta,$$

where m and n are whole numbers. If both m and n are even, we can integrate using the reduction formulas,

$$\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta), \qquad \cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta).$$

Example: Integrate $\displaystyle\int_0^{2\pi} \cos^2 \theta \, d\theta$.

Using the reduction formula for $\cos^2 \theta$,

$$\int_0^{2\pi} \cos^2 \theta \, dx = \frac{1}{2} \int_0^{2\pi} (1 + \cos 2\theta) \, d\theta = \frac{1}{2} \left[\theta + \frac{1}{2} \sin 2\theta \right]_0^{2\pi} = \pi.$$

If either m or n is odd, we can integrate using the Pythagorean theorem,

$$\sin^2 \theta + \cos^2 \theta = 1.$$

Example: Integrate $\displaystyle\int_0^{\pi/2} \cos^3 \theta \, d\theta$.

Write $\cos^3 \theta = (\cos^2 \theta)(\cos \theta)$ and substitute $\cos^2 \theta = 1 - \sin^2 \theta$ to obtain

$$\int_0^{\pi/2} \cos^3 \theta \, d\theta = \int (1 - \sin^2 \theta) \cos \theta \, d\theta.$$

Integrate by substitution with $u = \sin \theta$, $du = \cos \theta \, d\theta$. We then have

$$\int_0^{\pi/2} \cos^3 \theta \, d\theta = \int_0^1 (1 - u^2) \, du = \left[u - \frac{1}{3} u^3 \right]_0^1 = \frac{2}{3}.$$

Problems for Lecture 28

1. Compute the following definite integrals.

a) $\displaystyle\int_0^{2\pi} \sin^2\theta\, d\theta;$

b) $\displaystyle\int_0^{\pi/2} \sin^3\theta\, d\theta\,;$

c) $\displaystyle\int_0^{2\pi} \cos^4\theta\, d\theta\,.$

Solutions to the Problems

Lecture 29 | Trigonometric substitution

View this lecture on YouTube

Certain integrands with square roots can be solved using a trigonometric substitution. These square roots are of the form

$$\sqrt{a^2 - x^2}, \qquad \sqrt{a^2 + x^2}, \qquad \sqrt{x^2 - a^2}.$$

To integrate expressions containing these square roots, we make use of what is called an inverse substitution, where, respectively, we let

$$x = a\sin\theta, \qquad x = a\tan\theta, \qquad x = a\sec\theta.$$

The differential dx is found through differentiation. We now illustrate this technique by solving a simple definite integral.

Example: Integrate $\int_0^1 \sqrt{1-x^2}\, dx$.

Here, we let $x = \sin\theta$ and find the differential $dx = \cos\theta\, d\theta$. We change the limits by observing that as x goes from zero to one, θ goes from zero to $\pi/2$. The integral then becomes

$$\begin{aligned}\int_0^1 \sqrt{1-x^2}\, dx &= \int_0^{\pi/2} \cos\theta\sqrt{1-\sin^2\theta}\, d\theta \\ &= \int_0^{\pi/2} \cos^2\theta\, d\theta.\end{aligned}$$

Using the reduction formula for cosine, we find

$$\int_0^1 \sqrt{1-x^2}\, dx = \frac{1}{2}\int_0^{\pi/2} (1+\cos 2x)\, dx = \frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right]_0^{\pi/2} = \frac{\pi}{4}.$$

Problems for Lecture 29

1. The secant integral is tricky, but we will need it for the following problem.

a) Verify the secant integral by differentiation:

$$\int \sec\theta\, d\theta = \ln\left(\sec\theta + \tan\theta\right) + C.$$

You may assume that θ lies in the first quadrant so that $\sec\theta + \tan\theta > 0$.

b) By using the substitution $x = \tan\theta$ and the result of Part (a), solve the definite integral

$$\int_0^1 \frac{dx}{\sqrt{1+x^2}}.$$

Solutions to the Problems

Lecture 30 | Integration by partial fractions

View this lecture on YouTube

Partial fraction decomposition is a useful technique for integrating rational functions. Although there are several special cases one could consider, here we illustrate this method with only a single simple example.

Example: Integrate $\displaystyle\int_{x_0}^{x} \frac{ds}{s(1-s)}$, *where* $0 < x_0, x < 1$.

The partial fraction decomposition of the integrand proceeds by writing

$$\frac{1}{s(1-s)} = \frac{A}{s} + \frac{B}{1-s}.$$

To find A and B, we use the cover-up method. We multiply both sides of the equation by s and set $s = 0$ to find $A = 1$. Similarly, we multiply both sides of the equation by $1-s$ and set $s = 1$ to find $B = 1$. Therefore,

$$\frac{1}{s(1-s)} = \frac{1}{s} + \frac{1}{1-s},$$

and the integral becomes

$$\begin{aligned}\int_{x_0}^{x} \frac{ds}{s(1-s)} &= \int_{x_0}^{x} \frac{ds}{s} + \int_{x_0}^{x} \frac{ds}{1-s} = \ln s\Big|_{x_0}^{x} - \ln(1-s)\Big|_{x_0}^{x} \\ &= [\ln x - \ln x_0] - [\ln(1-x) - \ln(1-x_0)] = \ln\left(\frac{x}{x_0}\right) - \ln\left(\frac{1-x}{1-x_0}\right).\end{aligned}$$

This simplifies to

$$\int_{x_0}^{x} \frac{ds}{s(1-s)} = \ln\left(\frac{x(1-x_0)}{x_0(1-x)}\right).$$

Problems for Lecture 30

1. Integrate $\displaystyle\int_0^x \frac{ds}{(a-s)(b-s)}$, with $x < a, b$ and $a \neq b$.

Solutions to the Problems

Practice Quiz | Advanced techniques of integration

1. The definite integral $\int_0^{2\pi} \sin^4\theta\, d\theta$ is equal to

a) $\dfrac{\pi}{4}$

b) $\dfrac{\pi}{2}$

c) $\dfrac{3\pi}{4}$

d) π

2. The definite integral $\int_0^1 \sqrt{1+x^2}\, dx$ is equal to which of the following definite integrals?

a) $\int_0^{\pi/4} \sec\theta\, d\theta$

b) $\int_0^{\pi/4} \sec^2\theta\, d\theta$

c) $\int_0^{\pi/4} \sec^3\theta\, d\theta$

d) $\int_0^{\pi/4} \sec^4\theta\, d\theta$

3. The definite integral $\int_a^b \dfrac{dx}{x(1-x)(1+x)}$ is equal to

a) $\ln\left(\dfrac{a\sqrt{1-a^2}}{b\sqrt{1-b^2}}\right)$

b) $\ln\left(\dfrac{b\sqrt{1-b^2}}{a\sqrt{1-a^2}}\right)$

c) $\ln\left(\dfrac{a\sqrt{1-b^2}}{b\sqrt{1-a^2}}\right)$

d) $\ln\left(\dfrac{b\sqrt{1-a^2}}{a\sqrt{1-b^2}}\right)$

Solutions to the Practice quiz

Module IV

Infinite series

In this module's lectures, we explore sequences and series. We learn how an infinite power series can converge to a function. These convergent series are known as Taylor series, and we will determine the Taylor series for the most important functions of calculus, including the exponential function, sine and cosine functions, the natural logarithm, and the arctangent. We also learn L'Hospital's rule, a very useful tool for finding indeterminate limits.

Lecture 31 | Sequences and series

View this lecture on YouTube

A sequence is an ordered list of numbers that can be finite or infinite. We typically represent a sequence with a letter and a subscript, such as $a_1, a_2, a_3, \ldots$.

Example: Compute the first ten numbers of the Fibonacci sequence, defined by the recursion relation $F_{n+2} = F_{n+1} + F_n$, with starting values $F_1 = F_2 = 1$.

We have $F_1 = F_2 = 1$. The next numbers in the sequence are $F_3 = F_2 + F_1 = 1 + 1 = 2$ and $F_4 = F_3 + F_2 = 2 + 1 = 3$. Continuing this calculation, we can obtain the first ten Fibonacci numbers: $\{F_1, \ldots, F_{10}\} = \{1, 1, 2, 3, 5, 8, 13, 21, 34, 55\}$.

A series is the sum of a sequence of numbers. A series can be converted to a sequence by computing its partial sums. For example, if the series is given by

$$\sum_{k=1}^{n} a_k,$$

then the sequence of partial sums, denoted as $s_1, s_2, \ldots, s_n$, is defined as

$$s_1 = a_1, \quad s_2 = a_1 + a_2, \quad \ldots, \quad s_n = a_1 + a_2 + \cdots + a_n.$$

Example: Compute the partial sums of the Fibonacci sequence.

Consider the following identities:

$$\begin{aligned} F_n &= F_{n+2} - F_{n+1} \\ F_{n-1} &= F_{n+1} - F_n \\ \vdots\ &\quad \vdots \\ F_2 &= F_4 - F_3 \\ F_1 &= F_3 - F_2. \end{aligned}$$

Adding all the left-hand sides yields the sum of the first n Fibonacci numbers. Adding all the right-hand sides results in a telescoping series, where all the terms except the first and last cancel. Using $F_2 = 1$, the partial sum of the first n Fibonacci numbers is

$$\sum_{k=1}^{n} F_k = F_{n+2} - 1.$$

As an example, $\sum_{k=1}^{5} F_k = 1 + 1 + 2 + 3 + 5 = 12 = F_7 - 1$.

An infinite series is the sum of an infinite sequence of numbers. An infinite series may converge to a number, diverge to infinity, or never settle. If the infinite sequence of partial sums converges to a number, then we say that the infinite series converges.

Problems for Lecture 31

1. Prove that the partial sum of the squares of the Fibonacci numbers is given by

$$\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}.$$

(*Hint: Begin with the right-hand side and repeatedly use the recursion relation* $F_{k+1} = F_k + F_{k-1}$.)

2. An often needed partial sum is $\sum\limits_{k=1}^{n} k = 1 + 2 + \cdots + n$. Derive a formula for this sum by adding two such sums in reverse order.

3. Derive a formula for $\sum\limits_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2$ by using the identity

$$(k+1)^3 - k^3 = 3k^2 + 3k + 1.$$

Write this identity for each k from 1 to n and then sum all the resulting equations.

Solutions to the Problems

Lecture 32 | p-Series

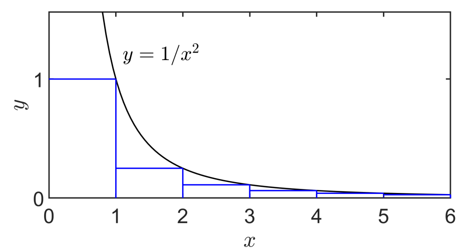
View this lecture on YouTube

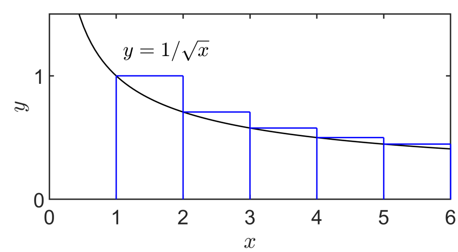
The p-series is defined as

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \frac{1}{5^p} + \dots.$$

The partial sums of this series are increasing, and either converge, or diverge to infinity. We want to find the values of p for which this series converges.

Consider the graphs of $y = 1/x^p$, where rectangles of width one are drawn with heights corresponding to the terms in the p-series. The sum of the areas of these rectangles represents the sum of the p-series.





In the top graph, the sum of the areas of the rectangles is less than one (the area of the first rectangle) plus the area under the curve from one to infinity. In the bottom graph, the sum of the areas of the rectangles is greater than the area under the curve from one to infinity. We can use the definite integral to find the area under the curve, which gives us bounds for the p-series:

$$\int_1^{\infty} \frac{1}{x^p}\,dx < \sum_{n=1}^{\infty} \frac{1}{n^p} < 1 + \int_1^{\infty} \frac{1}{x^p}\,dx\,.$$

For $p \neq 1$, we calculate the integral using the power rule. We find

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \int_1^{\infty} x^{-p}\,dx = \left.\frac{x^{-p+1}}{-p+1}\right|_1^{\infty} = \frac{1}{1-p}\left(\lim_{x\to\infty} x^{1-p} - 1\right).$$

This integral converges when $p > 1$ and diverges when $p < 1$. When $p = 1$, the integral becomes a logarithm, and we have

$$\int_1^{\infty} \frac{1}{x}dx = \ln x\Big|_1^{\infty} = \lim_{x\to\infty} \ln x = \infty,$$

which diverges.

We have shown that when $p > 1$, the p-series is bounded above by a convergent integral, and we conclude that the p-series converges. And when $p \leq 1$, the p-series is bounded below by a divergent integral, so the the p-series diverges.

Problems for Lecture 32

1. Some of the exact and approximate values of the p-series are

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.6449341\,, \qquad \sum_{n=1}^{\infty} \frac{1}{n^3} \approx 1.2020569\,,$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \approx 1.0823232\,, \qquad \sum_{n=1}^{\infty} \frac{1}{n^5} \approx 1.0369278\,,$$

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945} \approx 1.0173431\,, \qquad \sum_{n=1}^{\infty} \frac{1}{n^7} \approx 1.0083493\,.$$

Show that by calculating the integral, that these six p-series satisfy

$$\sum_{n=1}^{\infty} \frac{1}{n^p} < 1 + \int_1^{\infty} \frac{1}{x^p}\,dx\,.$$

Solutions to the Problems

Lecture 33 | Harmonic series

View this lecture on YouTube

The p-series with $p = 1$ is called the harmonic series, and is given by

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots .$$

The harmonic series lies on the boundary between convergent and divergent p-series, and is itself divergent.

For another proof of its divergence, we can group terms in the series as follows:

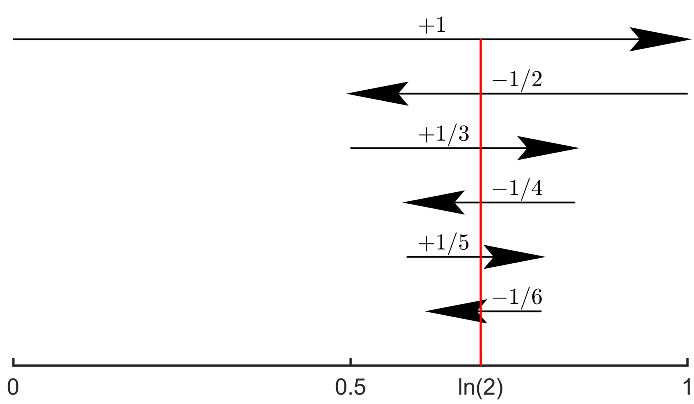
$$\begin{aligned} &1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) + (\frac{1}{9} + \dots + \frac{1}{16}) + \dots \\ &\geq 1 + \frac{1}{2} + (\frac{1}{4} + \frac{1}{4}) + (\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}) + (\frac{1}{16} + \dots + \frac{1}{16}) + \dots \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots . \end{aligned}$$

In the last expression the $1/2$ terms continue indefinitely. This final series clearly diverges, and since the harmonic series is larger than this divergent series, it must also diverge.

A closely related series is the alternating harmonic series, given by

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots .$$

By examining the graph shown here, we can see why this series converges. The partial sums alternate between increasing and decreasing, with the magnitude of change decreasing towards zero. Later, we will learn that the alternating harmonic series converges to $\ln 2$.



Problems for Lecture 33

1. An Egyptian fraction is a finite sum of fractions, where each fraction has a numerator of one and a denominator with a different positive integer. Because of the divergence of the harmonic series, any positive rational number can be expressed as an Egyptian fraction.

As an example, we can express the integer two as an Egyptian fraction:

$$2 - \frac{1}{1} = 1, \qquad 1 - \frac{1}{2} = \frac{1}{2}, \qquad \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

Therefore, $2 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$.

An amusing application of Egyptian fractions is in dividing food into equal shares. For instance, suppose you need to divide five pizzas among eight diners. We can find the Egyptian fraction for each diners share of $5/8$:

$$\frac{5}{8} - \frac{1}{2} = \frac{1}{8}.$$

Therefore $5/8 = 1/2 + 1/8$. To divide the five pizzas, cut four pizzas in half and give each diner one-half of a pizza, then cut the fifth pizza into eighths and give each diner one-eighth of a pizza.

Using Egyptian fractions, divide

a) six pizzas among eight diners;

b) seven pizzas among eight diners.

2. A infinite series is said to converge absolutely if the series formed by taking the absolute value of each term also converges. For absolutely convergent series, rearranging the terms does not change the sum. However, this is not true for series that do not converge absolutely. The alternating harmonic series does not converge absolutely.

The alternating harmonic series is given by

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \dots .$$

Consider summing a rearranged series with the same terms. Notice that terms with odd denominators are positive, and terms with even denominators are negative. In this rearranged series, each odd-denominator term is followed by two even-denominator terms:

$$\frac{1}{1} - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \dots .$$

Now group the terms as follows:

$$(\frac{1}{1} - \frac{1}{2}) - \frac{1}{4} + (\frac{1}{3} - \frac{1}{6}) - \frac{1}{8} + (\frac{1}{5} - \frac{1}{10}) - \frac{1}{12} + \dots .$$

By simplifying the terms in parentheses, show that this new series sums to half the value of the original alternating harmonic series.

Solutions to the Problems

Lecture 34 | Geometric series

View this lecture on YouTube

A geometric series is a series where each term is a constant multiple of the previous term. We consider here the most basic geometric series, given by

$$\sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n.$$

To find the sum, we multiply the series by x and write

$$x \sum_{k=0}^{n} x^k = x + x^2 + x^3 + \cdots + x^{n+1}.$$

Subtracting the second equation from the first results in

$$(1-x)\sum_{k=0}^{n} x^k = 1 - x^{n+1} \qquad \text{or} \qquad \sum_{k=0}^{n} x^k = \frac{(1-x^{n+1})}{1-x}.$$

In the limit as $n \to \infty$, the geometric series diverges when $|x| \geq 1$ because the terms do not approach zero. However, when $|x| < 1$, $x^{n+1} \to 0$, and we have

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}, \qquad (|x| < 1).$$

Example: Determine the value of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$.

The partial sums of this series start at one and with each additional term moves halfway towards two. Intuitively, this series should converge to two. Our intuition is confirmed by using the formula for the infinite geometric series with $x = 1/2$:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \frac{1}{1 - \frac{1}{2}} = 2.$$

Problems for Lecture 34

1. Convert the repeating decimal, $0.333\ldots$, into a fraction using a geometric series.

2. A dose of D mg of a drug is taken every 12 hours. The half-life of the drug, which is the time it takes for the blood concentration to decrease to half its initial value, is also 12 hours. At steady state, determine the maximum and minimum drug concentration in the patient's blood. Use concentration units equivalent to the dose, so that immediately after the initial dose of D mg, the drug concentration in the blood is D.

Solutions to the Problems

Practice Quiz | Sequences and series

1. The sum of the first 50 natural numbers squared plus twice the sum of the first 100 natural numbers is

a) 53025

b) 53030

c) 53035

d) 53040

2. The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges only when

a) $p > 0$

b) $p \geq 0$

c) $p > 1$

d) $p \geq 1$

3. A dose of D mg of a drug is taken every eight hours. The drug has a half-life of eight hours. At the steady state, what is the range of drug dosage d in the body in units equivalent to the dose?

a) $D/2 \leq d \leq D$

b) $D \leq d \leq 2D$

c) $D \leq d \leq 3D/2$

d) $3D/2 \leq d \leq 2D$

Solutions to the Practice quiz

Lecture 35 | Ratio test

View this lecture on YouTube

Consider a general infinite series in the form

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \dots .$$

To test for convergence, we compare this series to the geometric series, given by

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots .$$

The ratio of successive terms in the geometric series is x, and we know it converges when $|x| < 1$ and diverges when $|x| > 1$.

The convergence of an infinite series depends only on the behavior of its terms for large n. We define L as the limit of the absolute value of the ratio of successive terms as $n \to \infty$, that is,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L.$$

By comparison to the geometric series, we conclude that the series converges if $L < 1$ and diverges if $L > 1$. The boundary case $L = 1$ turns out to be indeterminate.

Example: Apply the ratio test for convergence to the following series:

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots .$$

We compute the ratio of the successive terms as $n \to \infty$. We have

$$\lim_{n\to\infty} \frac{1/(n+1)!}{1/n!} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

By the ratio test, we conclude that the series converges. In fact, we will see later that it converges to Euler's number, e.

Problems for Lecture 35

1. Use the ratio test to find a number r such that if $0 < a < r$, the following infinite series converges:

$$\sum_{n=1}^{\infty} \frac{a^n n!}{n^n}$$

(*Hint: You will need to use the definition of e from Lecture 6.*)

Solutions to the Problems

Lecture 36 | Power series

View this lecture on YouTube

A power series is an infinite series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots .$$

The ratio test shows that this power series converges when

$$\lim_{n\to\infty}\left|\frac{c_{n+1}x^{n+1}}{c_n x^n}\right| = |x| \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| < 1 \qquad \text{or} \qquad |x| < \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|.$$

The $n \to \infty$ limit of $|c_n/c_{n+1}|$ is called the radius of convergence. If the radius of convergence is infinite, the power series converges for all x. If the radius is zero, the series only converges for $x = 0$.

When $|x|$ is smaller than the radius of convergence, the power series can be differentiated or integrated term-by-term. The derivative of the power series is

$$\frac{d}{dx}\sum_{n=0}^{\infty} c_n x^n = \sum_{n=1}^{\infty} n c_n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \dots .$$

The indefinite integral of the power series is

$$\int \left(\sum_{n=0}^{\infty} c_n x^n\right) dx = C + \sum_{n=0}^{\infty} \frac{c_n x^{n+1}}{n+1} = C + c_0 x + \frac{c_1 x^2}{2} + \frac{c_2 x^3}{3} + \dots,$$

where C is the integration constant.

Example: Find the radius of convergence of the following power series, and then find its derivative.

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots .$$

The values of x for which the series converges are found using the ratio test:

$$|x| < \lim_{n\to\infty} \frac{1/n!}{1/(n+1)!} = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} (n+1) = \infty.$$

The radius of convergence is therefore infinite, and the power series converges for all values of x. The derivative of the power series is

$$\frac{d}{dx}\sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

which is equal to the original power series.

Problems for Lecture 36

1. Consider the function $f(x)$ defined by the power series

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots .$$

a) Show that $f'(x) = f(x)$.

b) Define

$$g(x) = e^{-x} f(x).$$

Show that $g'(x) = 0$, and therefore $g(x)$ is a constant function.

c) Show that $g(0) = 1$, and thus $g(x) = 1$.

d) Conclude that

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x.$$

2. Consider two functions, $f(x)$ and $g(x)$, defined by the power series

$$f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots, \qquad g(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots .$$

a) Show that $f'(x) = -g(x)$ and $g'(x) = f(x)$.

b) Define

$$h(x) = (\cos x - f(x))^2 + (\sin x - g(x))^2 .$$

Show that $h'(x) = 0$, and therefore $h(x)$ is a constant function.

c) Show that $h(0) = 0$, and thus $h(x) = 0$.

d) Conclude that

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \cos x, \qquad x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sin x.$$

3. Consider the power series equation

$$\sum_{n=0}^{\infty} c_n x^n = 0.$$

Prove using calculus that $c_n = 0$ for all n.

Solutions to the Problems

Practice Quiz | Power series

1. Consider the three power series, defined as

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots,$$
$$g(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots,$$
$$h(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots.$$

Which of the following statements is false?

a) $f'(x) = f(x)$

b) $g'(x) = h(x)$

c) $h'(x) = g(x)$

d) $g(-x) + h(-x) = g(x) - h(x)$

2. Suppose $\sum_{n=1}^{\infty} a_n$ converges. Which of the following statements must be true?

A. $\lim_{n\to\infty} a_n = 0$ B. $|a_{k+1}| < |a_k|$ for all k C. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \leq 1$

a) A and B only

b) A and C only

c) B and C only

d) all of the above

3. The power series equation given by

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + a_n\right] x^n = 0$$

results in the recursion relation given by

a) $a_n = \frac{a_{n+2}}{(n+2)(n+1)}$

b) $a_n = -\frac{a_{n+2}}{(n+2)(n+1)}$

c) $a_{n+2} = \frac{a_n}{(n+2)(n+1)}$

d) $a_{n+2} = -\frac{a_n}{(n+2)(n+1)}$

Solutions to the Practice quiz

Lecture 37 | Taylor series

View this lecture on YouTube

A Taylor series is a power series representation of a function $f(x)$ about a point $x = a$. It is constructed so that the derivatives of $f(x)$ at a match the derivatives of the power series. To determine the Taylor series for $f(x)$, we write

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots .$$

By taking derivatives of $f(x)$ and evaluating them at $x = a$, we obtain

$$f(a) = c_0, \quad f'(a) = c_1, \quad f''(a) = 2c_2, \quad f'''(a) = 3 \cdot 2c_3, \quad \dots, \quad f^{(n)}(a) = n!c_n.$$

Solving for the coefficients, the Taylor series for $f(x)$ is

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots .$$

Commonly, the Taylor series is developed with $a = 0$, resulting in the simpler form,

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots .$$

Sometimes, it is useful to approximate a function such as $f(x+\epsilon)$ when ϵ is small. Here, ϵ is the variable, and x is a constant. The Taylor series around $\epsilon = 0$ is given by

$$f(x+\epsilon) = f(x) + f'(x)\epsilon + \frac{f''(x)}{2!}\epsilon^2 + \frac{f'''(x)}{3!}\epsilon^3 + \dots .$$

This form provides a low-order polynomial approximation for $f(x+\epsilon)$.

Example: Find a linear approximation in ϵ for $\dfrac{1}{1+\epsilon}$.

We use the Taylor series approximation, $f(x+\epsilon) \approx f(x) + f'(x)\epsilon$, with $f(x) = 1/x$ and $x = 1$. The derivative is $f'(x) = -1/x^2$. Substituting these, we get for our linear approximation,

$$\frac{1}{1+\epsilon} \approx 1 - \epsilon.$$

Problems for Lecture 37

1. Use Taylor series to find a linear approximation in ϵ for the following functions:

a) $\dfrac{1}{1-\epsilon}$;

b) $\sqrt{1+\epsilon}$;

c) $\dfrac{1}{\sqrt{1+\epsilon}}$;

d) $\ln(1+\epsilon)$.

Solutions to the Problems

Lecture 38 | L'Hospital's rule

View this lecture on YouTube

L'Hospital's rule is a technique for finding limits when direct substitution results in the indeterminate forms $0/0$ or ∞/∞. Suppose two functions, $f(x)$ and $g(x)$, have convergent Taylor series around $x = a$:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots,$$
$$g(x) = g(a) + (x-a)g'(a) + \frac{(x-a)^2}{2!}g''(a) + \cdots.$$

If $g(a) \neq 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)}.$$

However, if $f(a) = g(a) = 0$, then direct substitution yields the indeterminate form $0/0$. In this case, we use the Taylor series to factor out $(x-a)$ from both the numerator and denominator. We obtain

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{(x-a)\left(f'(a) + \frac{(x-a)}{2!}f''(a) + \cdots\right)}{(x-a)\left(g'(a) + \frac{(x-a)}{2!}g''(a) + \cdots\right)} = \frac{f'(a)}{g'(a)}.$$

Here, we cancel the common factor $(x-a)$ and assume $g'(a) \neq 0$.

L'Hopital's rule states that if direct substitution results in $0/0$, then take the derivatives of the numerator and denominator separately and try direct substitution again. If another $0/0$ results, continue applying the rule until the limit is well-defined.

Example: Find $\lim_{x\to 0} \sin ax/x$.

Direct substitution results in $0/0$, so we apply L'Hospital's rule:

$$\lim_{x\to 0} \frac{\sin ax}{x} = \lim_{x\to 0} \frac{a\cos ax}{1} = a.$$

L'Hospital's rule also applies to the indeterminate form ∞/∞. Indeterminate forms, like $0 \times \infty$ and $\infty - \infty$, can often be algebraically manipulated into $0/0$ or ∞/∞.

Problems for Lecture 38

1. Suppose direct substitution of $x = a$ into the limit expression $\lim_{x\to a} f(x)/g(x)$ results in the indeterminate form ∞/∞. By considering the reciprocals of these functions, apply L'Hospital's rule to the resulting indeterminate form $0/0$ to show that L'Hospital's rule also applies to ∞/∞.

2. Use L'Hospital's rule to calculate the following limits:

a) $\displaystyle\lim_{x\to 0} \frac{x}{\tan x}$;

b) $\displaystyle\lim_{x\to 1} \frac{\ln x}{x-1}$;

c) $\displaystyle\lim_{x\to 0} x \ln x$;

3. Prove that the exponential function grows faster than any power law by showing that the following limit is zero using L'Hospital's rule:

$$\lim_{x\to\infty} \frac{x^n}{e^x}.$$

4. Prove that the logarithm function grows slower than any positive power law by showing that the following limit is zero using L'Hospital's rule:

$$\lim_{x\to\infty} \frac{\ln x}{x^n}, \quad n > 0.$$

Solutions to the Problems

Lecture 39 | Taylor series of elementary functions I

View this lecture on YouTube

We derive the Taylor series for the elementary functions $\exp(x)$, $\sin x$, and $\cos x$. The general form of a Taylor series centered at zero is

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots.$$

Exponential function

The exponential function satisfies $\exp(0) = 1$. Since it is its own derivative, all higher-order derivatives at zero are also one. Thus, the Taylor series for $\exp(x)$ is

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots.$$

The radius of convergence of this series is infinite, so it converges for all x.

Sine function

For $\sin x$, the derivatives are

$$f(x) = \sin x, \quad f'(x) = \cos x, \quad f''(x) = -\sin x, \quad f'''(x) = -\cos x, \quad \text{and so on;}$$

Evaluating at $x = 0$, we have

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = 0, \quad f'''(0) = -1, \quad \text{and so on.}$$

The Taylor series for sine includes only odd powers of x (since sine is an odd function) and has alternating signs,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots.$$

This series also converges for all x.

Cosine function

The Taylor series for $\cos x$ is derived by differentiating the sine series term-by-term,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots.$$

This series includes only even powers of x (since cosine is an even function) and also has alternating signs. Again, this series converges for all x.

Problems for Lecture 39

1. Use the Taylor series expansions to show that

a) $\dfrac{d}{dx}\exp(x) = \exp(x)$;

b) $\dfrac{d}{dx}\sin x = \cos x$;

c) $\dfrac{d}{dx}\cos x = -\sin x$.

2. Some authors define e^x, $\sin x$, and $\cos x$ using their Taylor series. Using these definitions, prove the following limit formulas:

a) $\displaystyle\lim_{h\to 0}\frac{e^h - 1}{h} = 1$;

b) $\displaystyle\lim_{h\to 0}\frac{\sin h}{h} = 1$;

c) $\displaystyle\lim_{h\to 0}\frac{\cos h - 1}{h} = 0$.

3. Verify that the following addition formulas are satisfied by the first two non-zero terms in the Taylor series expansions.

a) $\sin(x+y) = \sin x\cos y + \cos x\sin y$.

b) $\cos(x+y) = \cos x\cos y - \sin x\sin y$

Hint: Start with the right-hand sides of the identities.

4. Define the hyperbolic trigonometric functions by

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}.$$

a) Using the Taylor series for $\exp(x)$, find the Taylor series for $\cosh x$ and $\sinh x$

b) Using the Taylor series for $\cosh x$ and $\sinh x$, show that $\dfrac{d}{dx}\sinh x = \cosh x$ and $\dfrac{d}{dx}\cosh x = \sinh x$.

Solutions to the Problems

Lecture 40 | Taylor series of elementary functions II

View this lecture on YouTube

Our earlier summation of the geometric series gives us the two Taylor series,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots, \qquad \frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots .$$

These series converge for $|x| < 1$. If we integrate the second of these series term-by-term, and use $\ln 1 = 0$ to determine the integration constant, we obtain

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots .$$

Replacing x with x^2 in the series for $1/(1+x)$, we obtain

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots .$$

Integrating this series and using $\arctan 0 = 0$ to determine the integration constant gives

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots .$$

The series for $\ln(1+x)$ and $\arctan x$, both evaluated at $x = 1$, result in the remarkable formulas

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots, \qquad \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots .$$

The Taylor series for $\sqrt{1+x}$ and $\sqrt{1-x}$ are often useful when $|x| << 1$, and sometimes only the first two or three terms are needed. Calculating the first two derivatives of $\sqrt{1+x}$, and evaluating at $x = 0$, we have

$$\begin{aligned} f(x) &= (1+x)^{1/2}, & f'(x) &= \frac{1}{2}(1+x)^{-1/2}, & f''(x) &= -\frac{1}{4}(1+x)^{-3/2}, \\ f(0) &= 1, & f'(0) &= \frac{1}{2}, & f''(0) &= -\frac{1}{4}. \end{aligned}$$

Thus, the first three terms in the Taylor series are

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \dots, \qquad \sqrt{1-x} = 1 - \frac{1}{2}x - \frac{1}{8}x^2 + \dots .$$

Problems for Lecture 40

1. The binomial theorem allows us to expand the function $(1+x)^n$, where n is a positive integer, as an nth-degree polynomial in x. Derive the binomial theorem for $(1+x)^n$ using a finite Taylor series.

2. Find the first three non-zero terms in the Taylor series expansion of $\tan x$.

3. Integrate the Taylor series for e^{-x^2} to find the first five terms in a series expansion of

$$\int_0^1 \exp\left(-x^2\right) dx.$$

Solutions to the Problems

Practice Quiz | Taylor series

1. The linear approximation in ϵ for $\sqrt[3]{1+\epsilon}$ is

a) $1+\epsilon$

b) $1+\dfrac{1}{2}\epsilon$

c) $1+\dfrac{1}{3}\epsilon$

d) $1+\dfrac{1}{4}\epsilon$

2. Evaluate $\displaystyle\lim_{x\to 0}\frac{3\sin x-\sin 3x}{x-\sin x}$. The result is

a) 0

b) 1

c) 12

d) 24

3. Consider the three power series, defined as

$$f(x)=1+x+\frac{x^2}{2!}+\dots,\quad g(x)=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\dots,\quad h(x)=x+\frac{x^3}{3!}+\frac{x^5}{5!}+\dots.$$

Which of the following statements is true?

A. $f'(x)=f(x)$ B. $g'(x)=h(x)$ C. $h'(x)=g(x)$

a) A only

b) A and B only

c) B and C only

d) all of the above

4. The fourth term in the Taylor series for $\sqrt{1+x}$ is

a) $\dfrac{x^3}{4}$

b) $\dfrac{x^3}{8}$

c) $\dfrac{x^3}{16}$

d) $\dfrac{x^3}{32}$

Solutions to the Practice quiz

Module V

Applications I

In this module's lectures, we begin to apply the calculus. Using Taylor series, we define the complex exponential function and use it to prove key trigonometric identities. We employ calculus to derive the circumference and area of a circle, as well as the surface area and volume of a sphere. Finally, we show how calculus can be used in numerical methods to find the roots of equations and to integrate and differentiate functions.

Lecture 41 | Complex exponential function

View this lecture on YouTube

We have derived the Taylor series for the exponential function,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots,$$

and for the cosine and sine functions,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots, \qquad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots.$$

These three series can be connected using the imaginary unit $i = \sqrt{-1}$, with the relations $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, and so on. We can express the complex exponential function as

$$e^{ix} = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} \dots.$$

Writing the Taylor series for e^{ix} in the form of a complex number, we find

$$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right).$$

Recognizing the Taylor series for cosine and sine, we obtain Euler's formula,

$$e^{ix} = \cos x + i \sin x.$$

Using the identities $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$, we also have

$$e^{-ix} = \cos x - i \sin x.$$

Combining these two exponential functions, we find an expression for cosine and sine,

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

Euler's identity, sometimes seen on T-shirts, derives from Euler's formula by setting $\theta = \pi$, and using $\cos \pi = -1$ and $\sin \pi = 0$,

$$e^{i\pi} + 1 = 0.$$

This identity links the five most fundamental numbers of mathematics — 0, 1, i, π, and e — using the three most basic mathematical operations: addition, multiplication and exponentiation.

Problems for Lecture 41

1. Consider the function
$$g(x) = e^{-ix}(\cos x + i\sin x).$$

a) Show that $g'(x) = 0$, and therefore $g(x)$ is a constant function.

b) Show that $g(0) = 1$, and thus $g(x) = 1$.

c) Conclude that $\cos x + i\sin x = e^{ix}$.

2. Some trigonometric integrals are easier to evaluate using the complex exponential function. Integrate
$$\int_0^{2\pi} \cos^4 x\, dx$$
by first writing $\cos x = (e^{ix} + e^{-ix})/2$.

3. Using the identities,
$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i},$$
prove that $\cos^2 x + \sin^2 x = 1$.

4. Define the hyperbolic trigonometric functions by
$$\cosh x = \frac{e^{x} + e^{-x}}{2}, \qquad \sinh x = \frac{e^{x} - e^{-x}}{2}.$$
Prove the following:

a) $\cosh x = \cos ix, \qquad \sinh x = -i\sin ix;$

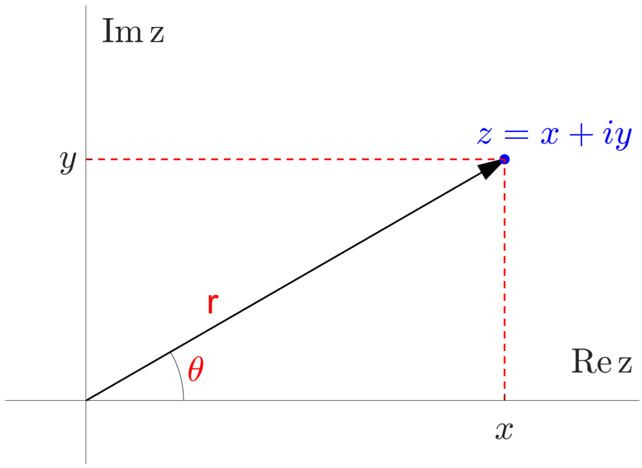
b) $e^x = \cosh x + \sinh x;$

c) $\cosh^2 x - \sinh^2 x = 1.$

Solutions to the Problems

Lecture 42 | Polar form of complex numbers

View this lecture on YouTube



A complex number z can be represented in the complex plane, with $\operatorname{Re} z$ on the x-axis and $\operatorname{Im} z$ on the y-axis (see figure). Using the definitions of cosine and sine, we have

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

with inverse

$$r = \sqrt{x^2 + y^2}, \qquad \tan\theta = y/x.$$

With $z = x + iy = r(\cos\theta + i\sin\theta)$, we obtain the polar form of a complex number,

$$z = x + iy = re^{i\theta}.$$

Note that θ is not unique, since $e^{i\theta} = e^{i(\theta + 2\pi k)}$ for any integer k. The usual convention is to choose θ such that $-\pi < \theta \leq \pi$, and $\theta = 0$ when $r = 0$.

The polar form of a complex number is useful for multiplication. For example, if

$$z_1 = r_1 e^{i\theta_1} \quad \text{and} \quad z_2 = r_2 e^{i\theta_2},$$

then

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}.$$

In particular, if $r_2 = 1$, multiplying z_1 by z_2 rotates the representation of z_1 in the complex plane by an angle θ_2 counterclockwise.

Problems for Lecture 42

1. The polar form is useful for simplifying complex expressions. Let

$$z = \frac{\alpha + i\beta}{\alpha^2 + \beta^2} e^{i\gamma}.$$

By writing $\alpha + i\beta$ in polar form, show that

$$\operatorname{Re} z = \frac{1}{\sqrt{\alpha^2 + \beta^2}} \cos(\gamma + \phi), \quad \text{with} \quad \tan\phi = \beta/\alpha\,.$$

2. The polar form is useful for finding roots of complex numbers.

a) Show that $e^{i\theta} = e^{i(\theta + 2\pi k)}$ when k is an integer.

b) Using three different polar forms for the number one, find the three complex cube roots of unity.

Solutions to the Problems

Lecture 43 | Trigonometric identities

View this lecture on YouTube

We can derive useful trigonometric identities using Euler's formula and properties of the exponential function. Consider the exponential identities

$$e^{i(x+y)} = e^{ix}e^{iy}, \qquad e^{inx} = (e^{ix})^n.$$

Using Euler's formula, the first exponential identity becomes

$$\begin{aligned}\cos(x+y) + i\sin(x+y) &= (\cos x + i\sin x)(\cos y + i\sin y)\\ &= (\cos x\cos y - \sin x\sin y) + i(\sin x\cos y + \cos x\sin y).\end{aligned}$$

By equating the real and imaginary parts, we can derive the well-known addition formulas,

$$\cos(x+y) = \cos x\cos y - \sin x\sin y, \qquad \sin(x+y) = \sin x\cos y + \cos x\sin y.$$

For the second exponential identity, Euler's formula results in

$$\cos(nx) + i\sin(nx) = (\cos x + i\sin x)^n.$$

For example, if $n = 2$, we have

$$\cos 2x + i\sin 2x = (\cos x + i\sin x)^2 = (\cos^2 x - \sin^2 x) + 2i\cos x\sin x.$$

Equating the real and imaginary parts then gives us

$$\cos 2x = \cos^2 x - \sin^2 x, \qquad \sin 2x = 2\cos x\sin x.$$

Making additional use of $\cos^2 x + \sin^2 x = 1$, we can also derive from the first identity the reduction formulas,

$$\cos^2 x = \frac{1+\cos 2x}{2}, \qquad \sin^2 x = \frac{1-\cos 2x}{2},$$

which were useful for integrating trigonometric functions.

Problems for Lecture 43

1. Define the Kronecker delta, δ_{nm} as

$$\delta_{nm} = \begin{cases} 1 & \text{if } n = m; \\ 0 & \text{otherwise.} \end{cases}$$

Using the Kronecker delta, with n and m positive integers, the orthogonality relations for cosine and sine are given by

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \pi\delta_{nm}, \quad \int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \pi\delta_{nm}, \quad \int_{-\pi}^{\pi} \cos mx \sin nx \, dx = 0.$$

Prove these relations using the even and odd symmetry of the cosine and sine functions, and the addition formulas for cosine, given by

$$\begin{aligned} \cos\left(mx + nx\right) &= \cos mx \cos nx - \sin mx \sin nx, \\ \cos\left(mx - nx\right) &= \cos mx \cos nx + \sin mx \sin nx. \end{aligned}$$

2. Prove the addition and double angle formulas for the hyperbolic trigonometric functions. You may assume the analogous identities for sine and cosine.

a) $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$;

b) $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$;

c) $\cosh 2x = \cosh^2 x + \sinh^2 x$;

d) $\sinh 2x = 2 \cosh x \sinh x$.

Solutions to the Problems

Practice Quiz | Complex exponential function

1. The expression $\sin x + i\cos x$ is equal to

a) e^{ix}

b) e^{-ix}

c) ie^{ix}

d) ie^{-ix}

2. The expression $\left(\frac{1+i}{2}\right)e^{i\pi/4}$ simplifies to

a) $-\frac{i\sqrt{2}}{2}$

b) $\frac{i\sqrt{2}}{2}$

c) $-\frac{i}{2}$

d) $\frac{i}{2}$

3. The expression $\cosh(x+y) + \sinh(x+y)$ simplifies to

a) 1

b) e^{x+y}

c) e^{x-y}

d) e^{y-x}

Solutions to the Practice quiz

Lecture 44 | Circumference of a circle

View this lecture on YouTube

To find the arc length of a curve γ in the x-y plane, we divide γ into infinitesimal segments of length ds, and sum over all these lengths. This gives us an integral, which we write symbolically as

$$\int_\gamma ds.$$

There are two methods to determine ds. First, if the curve can be written as $y = y(x)$, then using infinitesimals and the Pythagorean theorem, we have

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (dy/dx)^2}\, dx,$$

which converts the integral over ds to an integral over dx.

Second, if the curve can be parameterized by a parameter t, say, then we can write $x = x(t)$ and $y = y(t)$. Then using

$$dx = (dx/dt)\, dt, \qquad dy = (dy/dt)\, dt,$$

we have

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt,$$

which converts the integral over ds to an integral over dt.

As an example, we can compute the circumference of a circle. A circle of radius R can be parametrized by

$$x(\theta) = R\cos\theta, \qquad y(\theta) = R\sin\theta, \qquad \text{for } 0 \le \theta \le 2\pi.$$

Then

$$ds = \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}\, d\theta = \sqrt{R^2\sin^2\theta + R^2\cos^2\theta}\, d\theta = Rd\theta;$$

and the circumference of a circle of radius R, denoted by C, is given by

$$C = \int_0^{2\pi} Rd\theta = 2\pi R.$$

Problems for Lecture 44

1. Compute the circumference of a circle with radius R using the infinitesimal arc length given by

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{(1 + (dy/dx)^2}\, dx,$$

and the equation for a circle given by $x^2 + y^2 = R^2$.

Solutions to the Problems

Lecture 45 | Area of a circle

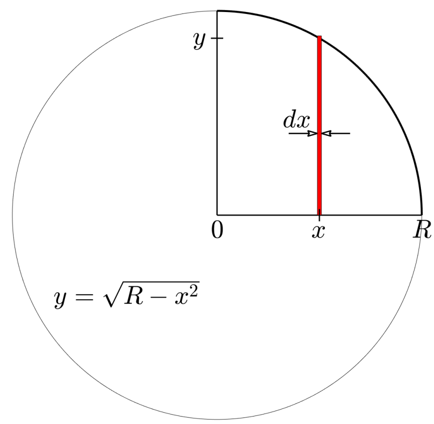
View this lecture on YouTube

The equation for a circle of radius R is given by

$$x^2 + y^2 = R^2.$$

To compute the area of a circle, we first find the area in the first quadrant and multiply by four. In the first quadrant, y is given by

$$y = \sqrt{R^2 - x^2}, \qquad 0 \le x \le R.$$



The area of a circle can then be found by integrating over infinitesimally thin rectangles, as shown in the figure. We have

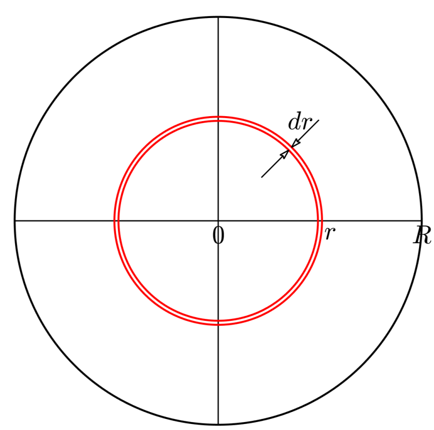
$$A = 4\int_0^R y\,dx = 4\int_0^R \sqrt{R^2 - x^2}\,dx.$$

We use a trigonometric substitution to integrate. Let $x = R\sin\theta$, so $dx = R\cos\theta\,d\theta$. When $x = 0$, $\theta = 0$, and when $x = R$, $\theta = \pi/2$. Therefore,

$$\begin{aligned}
A &= 4\int_0^R \sqrt{R^2 - x^2}\,dx = 4\int_0^{\pi/2} \sqrt{R^2 - R^2\sin^2\theta}\,(R\cos\theta\,d\theta) \\
&= 4R^2\int_0^{\pi/2} \sqrt{1 - \sin^2\theta}\,\cos\theta\,d\theta = 4R^2\int_0^{\pi/2} \cos^2\theta\,d\theta = 2R^2\int_0^{\pi/2} (1 + \cos 2\theta)\,d\theta \\
&= 2R^2\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \pi R^2.
\end{aligned}$$

A simpler derivation uses the formula for the circumference of a circle of radius r, given by $C = 2\pi r$. As shown in the figure, the area of a circle can be found by integrating over infinitesimally thin rings. We have

$$A = \int_0^R 2\pi r\,dr = 2\pi \frac{r^2}{2}\Big|_0^R = \pi R^2.$$



Problems for Lecture 45

1. By integration, determine the area of an ellipse defined by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

2. Given that a circle of radius r has area $A = \pi r^2$, determine the circumference of the circle by considering the area of a thin ring.

Solutions to the Problems

Lecture 46 | Surface area of a sphere

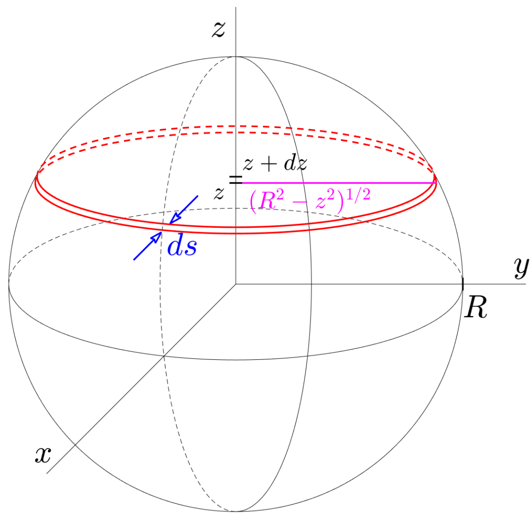
View this lecture on YouTube

We can use the formula for the circumference of a circle to compute the surface area of a sphere. With the center of the sphere at the origin, the equation for the sphere is given by

$$x^2 + y^2 + z^2 = R^2.$$

We draw a circle on the sphere parallel to the x-y-plane, centered at a height $z > 0$, with a radius $\sqrt{x^2 + y^2} = \sqrt{R^2 - z^2}$, and circumference

$$C = 2\pi\sqrt{R^2 - z^2}.$$



Next, we draw a second circle on the sphere an infinitesimal distance dz above the first circle. The two circles are separated by a distance ds along the surface of the sphere. Where $y = 0$, the infinitesimals dx, dz and ds form a right triangle with ds as the hypotenuse. Therefore,

$$ds = \sqrt{dz^2 + dx^2} = \sqrt{1 + (dx/dz)^2}\, dz.$$

Also at $y = 0$, the surface of the sphere satisfies $x^2 + z^2 = R^2$, so that

$$x = \pm\sqrt{R^2 - z^2} \quad \text{and} \quad \frac{dx}{dz} = \mp\frac{z}{\sqrt{R^2 - z^2}}.$$

Thus,

$$ds = \sqrt{1 + (dx/dz)^2}\, dz = \sqrt{1 + \frac{z^2}{R^2 - z^2}}\, dz = \frac{R}{\sqrt{R^2 - z^2}}\, dz.$$

The surface area of this infinitesimal ring on the surface of the sphere is therefore

$$dS = C\, ds = \left(2\pi\sqrt{R^2 - z^2}\right)\left(\frac{R}{\sqrt{R^2 - z^2}}\, dz\right) = 2\pi R\, dz.$$

Integrating from $z = 0$ to R, and multiplying by two, we find the surface area of a sphere with radius R,

$$S = 2\int_0^R 2\pi R\, dz = 4\pi R^2.$$

Problems for Lecture 46

1. Compute the surface area of a cone with base radius r and height h by integrating the surface areas of infinitesimally thick rings. Use the equation for the cone given by

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = \left(\frac{z}{h}\right)^2.$$

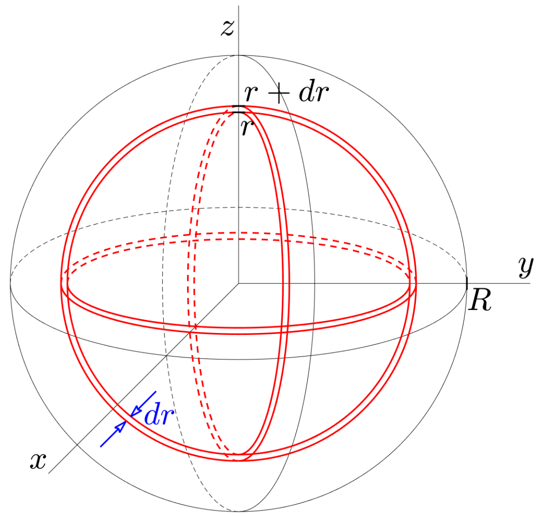
Solutions to the Problems

Lecture 47 | Volume of a sphere

View this lecture on YouTube

To determine the volume of a sphere of radius R, consider a spherical shell with the same center as the sphere, having radius r and an infinitesimal thickness dr (see figure). The surface area of the spherical shell is $4\pi r^2$, and the infinitesimal volume dV of the shell is given by

$$dV = 4\pi r^2 dr.$$



To find the volume of the sphere, we integrate dV from $r = 0$ to the radius R of the sphere. We have

$$V = \int dV = \int_0^R 4\pi r^2 \, dr = \frac{4}{3}\pi R^3.$$

Problems for Lecture 47

1. Given that a sphere of radius r has volume $V = \frac{4}{3}\pi r^3$, determine the surface area of the sphere by considering the volume of a spherical shell of width dr.

2. Compute the volume of a cone with base radius r and height h by integrating over the volumes of infinitesimally thick rings. Use the equation for the cone given by

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = \left(\frac{z}{h}\right)^2.$$

Solutions to the Problems

Practice Quiz | Areas and volumes

1. The arc length of the parabola $y = x^2$ for $0 \leq x \leq 1$ is given by the integral

a) $\int_0^1 \sqrt{1+2x}\, dx$

b) $\int_0^1 \sqrt{1+4x}\, dx$

c) $\int_0^1 \sqrt{1+2x^2}\, dx$

d) $\int_0^1 \sqrt{1+4x^2}\, dx$

2. Let $A = 4\int_0^r \sqrt{r^2 - x^2}\, dx$. Then dA/dr is equal to

a) πr

b) $2\pi r$

c) $4\pi r$

d) $8\pi r$

3. A sphere of radius R with center at the origin has a circle drawn on it at fixed z. The circumference of the circle is

a) $2\pi R$

b) $2\pi z$

c) $\pi(R+z)$

d) $2\pi\sqrt{R^2 - z^2}$

4. A sphere of radius r has surface area $S = S(r)$ and volume $V = V(r)$. Which of the following relationships are true?

A. $S = \dfrac{dV}{dr}$ B. $S = \int_0^r V(r')\, dr'$ C. $V = \dfrac{dS}{dr}$ D. $V = \int_0^r S(r')\, dr'$

a) A and C only

b) B and D only

c) A and D only

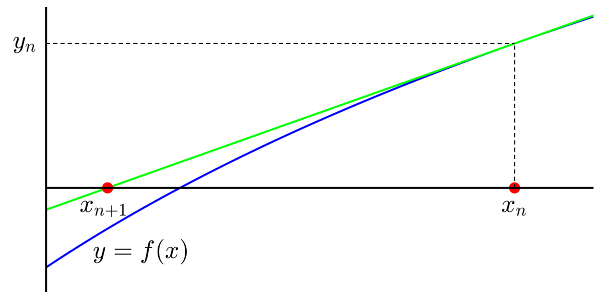
d) B and C only

Solutions to the Practice quiz

Lecture 48 | Newton's method

View this lecture on YouTube

The problem of root finding is to solve $f(x) = 0$ for a root $x = r$. If an analytical computation of the derivative of $f(x)$ is possible, then Newton's method is one of the fastest methods for root finding. Newton's method approximates the function $y = f(x)$ by the tangent line to the curve at the point (x_n, y_n), where $y_n = f(x_n)$. The slope of the tangent line is $f'(x_n)$, and the equation of the tangent line is given by



$$y - f(x_n) = f'(x_n)(x - x_n).$$

The next approximation to the root occurs where the line intercepts the x-axis. At this point, $y = 0$ and $x = x_{n+1}$. We have

$$-f(x_n) = f'(x_n)(x_{n+1} - x_n) \qquad \text{or} \qquad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Newton's Method requires an initial guess x_0, which should be chosen as close as possible to the root $x = r$.

Example: Estimate $\sqrt{2} = 1.41421\ldots$ *using Newton's method with* $x_0 = 1$.

We solve $f(x) = 0$, where $f(x) = x^2 - 2$. To implement Newton's Method, we use $f'(x) = 2x$. Therefore, Newton's Method is the iteration

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n} = \frac{x_n^2 + 2}{2x_n}.$$

Choosing an initial guess $x_0 = 1$, we have

$$x_1 = \frac{(1)^2 + 2}{2 \times 1} = \frac{3}{2} = 1.5, \qquad x_2 = \frac{(\frac{3}{2})^2 + 2}{2 \times \frac{3}{2}} = \frac{17}{12} = 1.416667,$$
$$x_3 = \frac{(\frac{17}{12})^2 + 2}{2 \times (\frac{17}{12})} = \frac{577}{408} = 1.41426,$$

and so on. With only three iterations, Newton's method has already converged to $\sqrt{2}$ to four decimal places.

Problems for Lecture 48

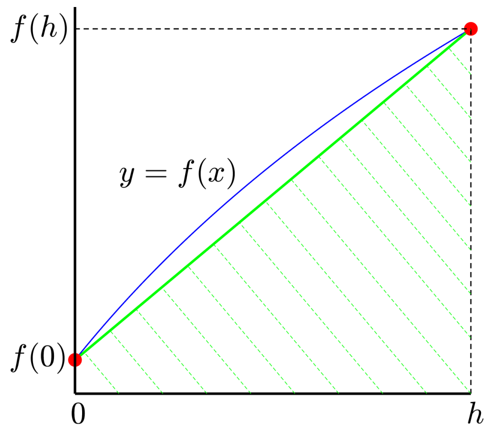
1. Using Newton's method, estimate $\sqrt{3} = 1.73205\ldots$. Use $x_0 = 1$ and iterate to find the value of x_3.

Solutions to the Problems

Lecture 49 | Trapezoidal rule

View this lecture on YouTube

The trapezoidal rule is a widely used numerical integration method. To derive it, we first approximate the function $y = f(x)$ from $x = 0$ to $x = h$ by a straight line connecting the points $(0, f(0))$ and $(h, f(h))$ (see figure). This gives



$$f(x) \approx f(0) + \frac{f(h) - f(0)}{h} x.$$

Integrating from 0 to h, we obtain

$$\begin{aligned}\int_0^h f(x)\,dx &\approx \int_0^h \left(f(0) + \frac{f(h) - f(0)}{h} x \right) dx \\ &= \left[f(0)x + \frac{f(h) - f(0)}{2h} x^2 \right]_0^h .\end{aligned}$$

Evaluating the definite integral, we obtain

$$\int_0^h f(x)\,dx \approx f(0)h + \frac{1}{2}\left(f(h) - f(0)\right) h = \frac{h}{2}\left(f(0) + f(h)\right),$$

which is just the area of the trapezoid.

Most commonly, we want to integrate a function from a to b. We divide the integration interval into n subintervals, each of width $h = (b - a)/n$. We then define the function values at $n + 1$ evenly spaced points as

$$f_0 = f(a), \quad f_1 = f(a + h), \quad f_2 = f(a + 2h), \quad \ldots, \quad f_n = f(b).$$

The trapezoidal rule applied to each pair of neighboring points and summed becomes

$$\begin{aligned}\int_a^b f(x)\,dx &\approx \frac{h}{2}\left(f_0 + f_1\right) + \frac{h}{2}\left(f_1 + f_2\right) + \cdots + \frac{h}{2}\left(f_{n-1} + f_n\right) \\ &= \frac{h}{2}\left(f_0 + 2f_1 + \cdots + 2f_{n-1} + f_n\right).\end{aligned}$$

All function evaluations, except the first and last, are multiplied by two and added together, and the final sum is multiplied by $h/2$.

Problems for Lecture 49

1. Use the trapezoidal rule to estimate $\ln 2 \approx 0.693147$ by numerically integrating

$$\ln 2 \approx \int_1^2 \frac{dx}{x},$$

with $h = 0.1$.

2. The trapezoidal rule is often used when a function's values are known only at specific points, which may not be evenly spaced. Suppose a function $f(x)$ is known at the $n+1$ points specified by

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

Denote $f(x_k) = f_k$. Show that applying the trapezoidal rule to each pair of neighboring points and summing the results approximates the integral as

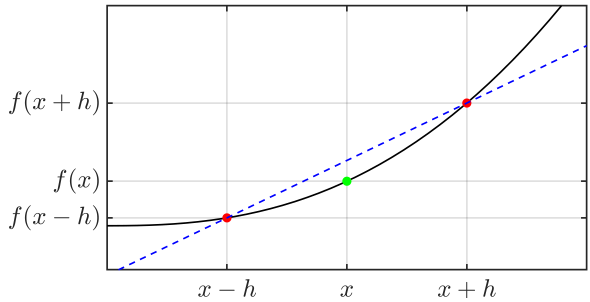
$$\int_a^b f(x)\,dx \approx \frac{1}{2}\Big[(x_1 - x_0)f_0 + (x_2 - x_0)f_1 + \cdots + (x_{n-1} - x_{n-3})f_{n-1} + (x_n - x_{n-1})f_n\Big].$$

Solutions to the Problems

Lecture 50 | Centered differences

View this lecture on YouTube

In numerical calculations, it is often more accurate to define the derivative as a centered difference. The point $(x, f(x))$ is positioned between two points $(x-h, f(x-h))$ and $(x+h, f(x+h))$, as shown in the figure. The derivative is then defined as



$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}.$$

We can also derive a centered-difference formula for the second derivative. Starting with the centered difference for the first derivative, we reduce the step size h by half,

$$f'(x) = \lim_{h \to 0} \frac{f(x+\frac{h}{2}) - f(x-\frac{h}{2})}{h}.$$

The second derivative is then defined as

$$\begin{aligned} f''(x) = \lim_{h \to 0} \frac{f'(x+\frac{h}{2}) - f'(x-\frac{h}{2})}{h} &= \lim_{h \to 0} \frac{\frac{f(x+h)-f(x)}{h} - \frac{f(x)-f(x-h)}{h}}{h} \\ &= \lim_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}. \end{aligned}$$

Problems for Lecture 50

1. Consider the differential equation $y'' = 0$ on the domain $0 \leq x \leq 1$. Divide this domain into a grid with n intervals and $n + 1$ grid points, with a grid spacing of $h = 1/n$. Show that using the centered-difference formula for the second derivative on this grid makes the value of y at any internal grid point equal to the average of y at its two neighboring grid points.

Solutions to the Problems

Practice Quiz | Numerical methods

1. Use Newton's method to estimate $\sqrt{5}$ with an initial guess of $x_0 = 2$. What is the value of x_2?

a) $\dfrac{9}{4}$

b) $\dfrac{20}{9}$

c) $\dfrac{161}{72}$

d) $\dfrac{559}{250}$

2. Use the trapezoidal rule with $h = 0.5$ to estimate $\int_0^1 \exp\left(-x^2\right) dx$. The result accurate to four decimal places is

a) 0.7300

b) 0.7314

c) 0.7468

d) 0.7500

3. The centered difference approximation for the derivative is given by

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}.$$

For which functions is this centered difference approximation exact?

A. a B. $ax + b$ C. $ax^2 + bx + c$ D. $ax^3 + bx^2 + cx + d$

a) A only

b) A and B only

c) A, B, and C only

d) all of the above

Solutions to the Practice quiz

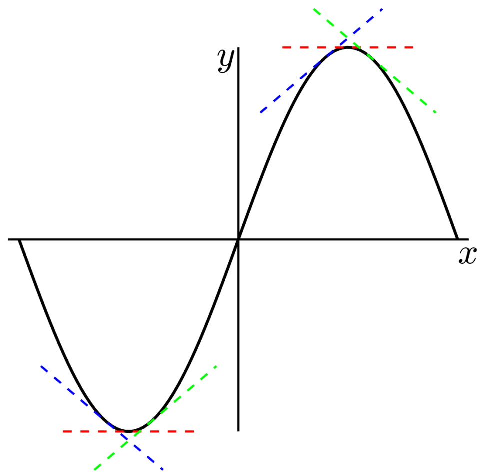
Module VI

Applications II

In this module's lectures, we continue exploring applications of calculus. We learn how to use derivatives to find local extrema of functions. We prove that among rectangles with a given perimeter, the one that maximizes the area is a square. We find the shortest path between two villages after collecting water from a river. We determine the optimal position on the beach for a lifeguard to enter the sea to rescue a swimmer in distress. We discuss how calculus is used in physics to define velocity and acceleration, and how to determine the position and velocity of an object falling under gravity. Lastly, we explore differential equations related to growth, decay, and oscillation, including equations for compound interest and the oscillating pendulum.

Lecture 51 | Maxima and minima

View this lecture on YouTube



The derivative is a powerful tool for finding local maxima and minima of functions. Consider the function shown in the figure. We have drawn tangent lines to the graph at points before, on, and after the function's minimum and maximum values. The slopes of these tangent lines represent the derivatives of the function at those points.

First, examine the local minimum. A function with a local minimum decreases to the left of the minimum and increases to the right. Mathematically, this means the derivative is negative to the left of the minimum, positive to the right, and zero exactly at the minimum. Points where the derivative is zero are called critical points.

Second, examine the local maximum. Here, the derivative is positive to the left of the maximum, negative to the right, and zero at the maximum.

At a local minimum, the first derivative increases with x, indicating a positive second derivative. Conversely, at a local maximum, the first derivative decreases with x, indicating a negative second derivative. This leads to the second derivative test: a positive second derivative at a critical point implies a local minimum, while a negative second derivative implies a local maximum.

The sign of the second derivative also indicates a function's concavity. A function with a positive second derivative is concave up (e.g., $y = x^2$), while a function with a negative second derivative is concave down (e.g., $y = -x^2$). A point where the second derivative is zero and changes sign is called an inflection point.

The first derivative is often used to find local maxima and minima, but some functions with zero derivative may have neither. A simple example is the function $y = x^3$, which has a zero derivative at $x = 0$ but no local maximum or minimum.

Problems for Lecture 51

1. Determine the condition for a general cubic polynomial to have no local maxima or minima. Explain the case where the function has only one critical point.

2. Consider the cubic polynomial given by

$$f(x) = x^3 - px + q, \qquad \text{with} \quad p > 0.$$

a) Find the two critical points of the polynomial.

b) Find the values of $f(x)$ at the two critical points.

c) Find a condition on the coefficients of this polynomial such that it has three distinct real roots. (*Hint: The polynomial will have three distinct real roots if the local maximum and minimum occur on opposite sides of the x-axis.*)

Solutions to the Problems

Lecture 52 | Maximum area of a rectangle

View this lecture on YouTube

We can use calculus to show that for a given perimeter, the rectangle with the maximum area is a square. Let a rectangle have side lengths x and y, with perimeter P and area A. The formula for the perimeter is

$$P = 2x + 2y;$$

and the formula for the area is

$$A = xy.$$

We aim to maximize the area A, while keeping the perimeter P fixed. First, we solve the perimeter equation for y and substitute it into the area equation to express A as a function of x alone:

$$A(x) = x\left(\frac{P-2x}{2}\right) = \frac{1}{2}Px - x^2.$$

To find the maximum area, we take the derivative of A with respect to x, and set it equal to zero:

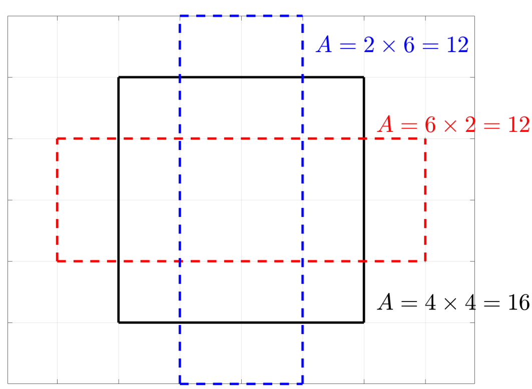
$$A'(x) = \frac{1}{2}P - 2x = 0.$$

Solving for x, and then for y, we find

$$x = y = \frac{1}{4}P.$$

Since x equals y, we have shown that a square maximizes the area for a given perimeter.

The figure illustrates this result. The perimeters of these three rectangles are equal, and the square is observed to have the maximum area.



Problems for Lecture 52

1. A farmer has a length L of fencing and wants to construct a rectangular enclosure for his animals of maximum area, using one side of his barn as part of the enclosure. Determine the dimensions of the enclosure in terms of L, and describe the relationship between the length of the single fenced side parallel to the barn wall and the lengths of the two other fenced sides.

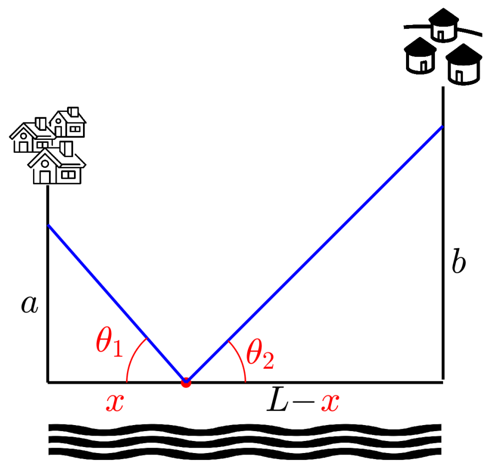
Solutions to the Problems

Lecture 53 | Shortest path

View this lecture on YouTube

The new village and the old village are located on the same side of a straight river. You live in the new village and your elderly relatives live in the old village. They ask you to walk to the river to fetch them water, and to save time and effort, you use calculus to determine the shortest walking path.

The parameters of the problem are as follows: a is the distance from the new village to the river; b is the distance from the old village to the river; and L is the distance along the river from the new village to the old village. The geometry of the problem is shown in the figure.



The unknowns are x, the distance along the river from the new village to the point of water collection, and the angles θ_1 and θ_2 between the river and the two walking paths. Using the Pythagorean theorem, the walking distance from the new village to the water collection point is $\sqrt{a^2 + x^2}$, and the distance from the water collection point to the old village is $\sqrt{(L-x)^2 + b^2}$.

To find x we minimize the total distance walked. The derivative should be zero at the minimum, so we solve

$$\frac{d}{dx}\left(\sqrt{a^2 + x^2} + \sqrt{(L-x)^2 + b^2}\right) = 0.$$

Taking the derivative, we obtain

$$\frac{x}{\sqrt{a^2 + x^2}} - \frac{L-x}{\sqrt{(L-x)^2 + b^2}} = 0.$$

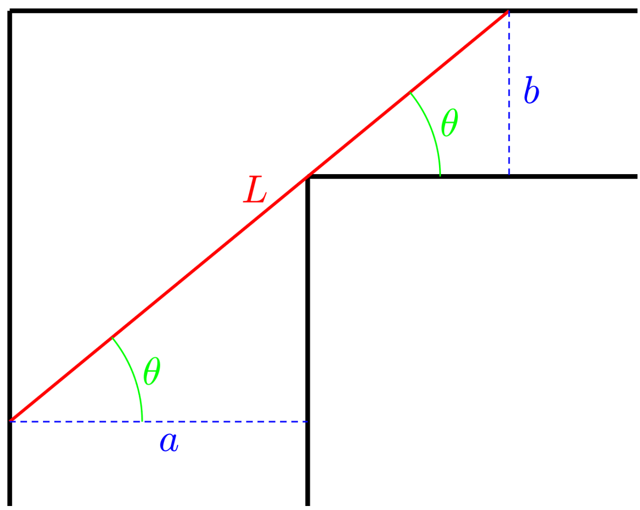
With

$$\frac{x}{\sqrt{a^2 + x^2}} = \cos\theta_1 \qquad \text{and} \qquad \frac{L-x}{\sqrt{(L-x)^2 + b^2}} = \cos\theta_2,$$

minimization of the total distance walked is achieved when $\cos\theta_1 = \cos\theta_2$. The cosine functions are equal when the angles themselves are equal, so making these angles equal will minimize the total walking path. This problem is believed to have been first solved by Heron of Alexandria during the Roman era.

Problems for Lecture 53

1. How long can a pipe be to move it around a corner without getting stuck? Consider two hallways: one with width a and the other with width b. Let L be the length of the pipe. Use the angle θ, as shown in the figure, to solve this problem using calculus.



Solutions to the Problems

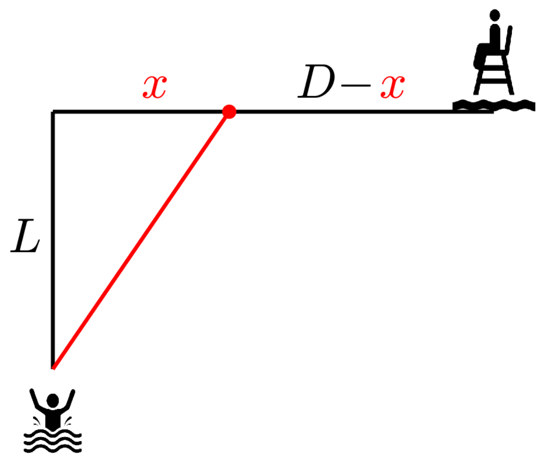
Lecture 54 | Lifeguard problem

View this lecture on YouTube

A lifeguard patrolling the water's edge spots a swimmer in distress. The lifeguard can run along the shore before entering the sea to rescue the swimmer. The goal is to find the optimal point to enter the water to minimize the time taken to reach the swimmer.

The parameters of the problem are as follows: L is the perpendicular distance from the swimmer to the shore; D is the distance from the lifeguard to the point on the shore closest to the swimmer; r is the running speed of the lifeguard; and s is the swimming speed of the lifeguard. We assume that lifeguards run faster than they swim.

Let x be the distance from the water entry point to the point on the shore closest to the swimmer. Thus, $D - x$ is the distance the lifeguard runs before entering the water. The distance the lifeguard swims is $\sqrt{x^2 + L^2}$. The geometry of the problem is shown in the figure.



Using the relationship that distance equals velocity times time, the total time T spent both running and swimming is given by

$$T = \frac{D - x}{r} + \frac{\sqrt{x^2 + L^2}}{s}.$$

To minimize this time, we differentiate T with respect to x and set the derivative equal to zero. We have

$$\frac{dT}{dx} = -\frac{1}{r} + \frac{x}{s\sqrt{x^2 + L^2}} = 0.$$

Solving for x, we find

$$x = \frac{(s/r)L}{\sqrt{1 - (s/r)^2}}.$$

Typically, a person runs about four times faster than they swim, so a good approximation is $(s/r) = 1/4$. The square-root factor is nearly one, so a good approximation for the distance the lifeguard should run is

$$D - x \approx D - L/4.$$

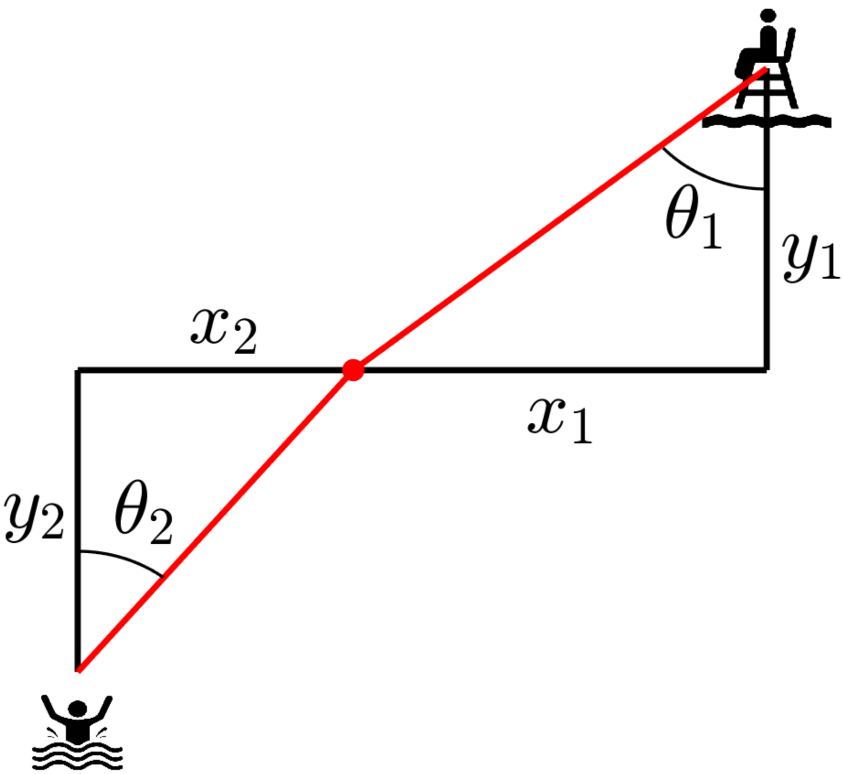
This means the lifeguard should not run the entire distance D along the shore but should shorten the run by about $1/4$ of the distance the swimmer is out to sea. For example, if the swimmer is very close to shore, the lifeguard should run most of the way along the beach; if the swimmer is far out to sea, the lifeguard should start swimming right away.

Problems for Lecture 54

1. Consider a lifeguard at his post and a swimmer in distress as shown in the figure below. The lifeguard is posted a distance y_1 from the shore, while the swimmer is in the water a distance y_2 from the shore. To minimize the time it takes to reach the swimmer, the lifeguard should enter the water at the distance x_1 along the shore, leaving a remaining distance x_2 along the shore to the swimmer. The total distance along the shore, $x = x_1 + x_2$, is given. Also given is the lifeguard's running speed u_1 and the lifeguard's swimming speed u_2. With the angles θ_1 and θ_2 defined in the figure, show that for the lifeguard to minimize his time to reach the swimmer, the angles must satisfy

$$\frac{\sin\theta_1}{u_1} = \frac{\sin\theta_2}{u_2}.$$

This relationship is similar to Snell's law in optics.



Solutions to the Problems

Practice Quiz | Local extrema

1. A local maximum of the cubic polynomial $f(x) = x^3 - 3x + 2$ occurs at

a) $x = -2$

b) $x = -1$

c) $x = 0$

d) $x = 1$

2. A farmer has 100 meters of fencing to build a rectangular enclosure, using one side of his barn as part of the enclosure. The dimension of the fence parallel to the barn wall that maximizes the area is

a) 25 m

b) 33 m

c) 50 m

d) 75 m

3. Two villages are located on the same side of a straight river, both the same distance D from the river, and a distance L along the river from one village to the other. What is the length of the shortest walking path from one village to the river to the second village?

a) $2\sqrt{D^2 + \frac{1}{4}L^2}$

b) $2\sqrt{L^2 + \frac{1}{4}D^2}$

c) $2\sqrt{D^2 + (D - L)^2}$

d) $2\sqrt{L^2 + (L - D)^2}$

4. A lifeguard patrolling the water's edge spots a swimmer in distress about the same distance from shore L as the lifeguard is from the point on the shore closest to the swimmer. Approximately, how far should the lifeguard run along the shore before entering the water?

a) $\frac{1}{4}L$

b) $\frac{1}{2}L$

c) $\frac{3}{4}L$

d) L

Solutions to the Practice quiz

Lecture 55 | Position, velocity, and acceleration

View this lecture on YouTube

Consider a mass moving in one direction. We define the x-axis along the direction of motion, specifying positive and negative directions. The x-position of the mass indicates its location in space, while its velocity indicates both speed and direction. A positive velocity means the mass is moving in the positive direction, whereas a negative velocity means it is moving in the negative direction.

Acceleration describes how the velocity changes over time. A positive acceleration means the velocity is increasing, which can mean that a mass with positive velocity is speeding up, or that a mass with negative velocity is slowing down. Conversely, a negative acceleration indicates the opposite.

The position of a mass at time t is denoted by $x = x(t)$. Using calculus, we can define the instantaneous velocity of the mass as the first derivative of its position with respect to time. We denote the velocity $v = v(t)$ as

$$v(t) = \dot{x}(t),$$

where the dot notation denotes a time derivative (two dots denote the second time derivative). The instantaneous acceleration $a = a(t)$ is the derivative of the velocity, or the second derivative of the position,

$$a(t) = \dot{v}(t) = \ddot{x}(t).$$

According to Newton's second law, $F = ma$, we can derive an equation for the acceleration of a mass by considering all the forces acting on it:

$$\ddot{x} = F/m.$$

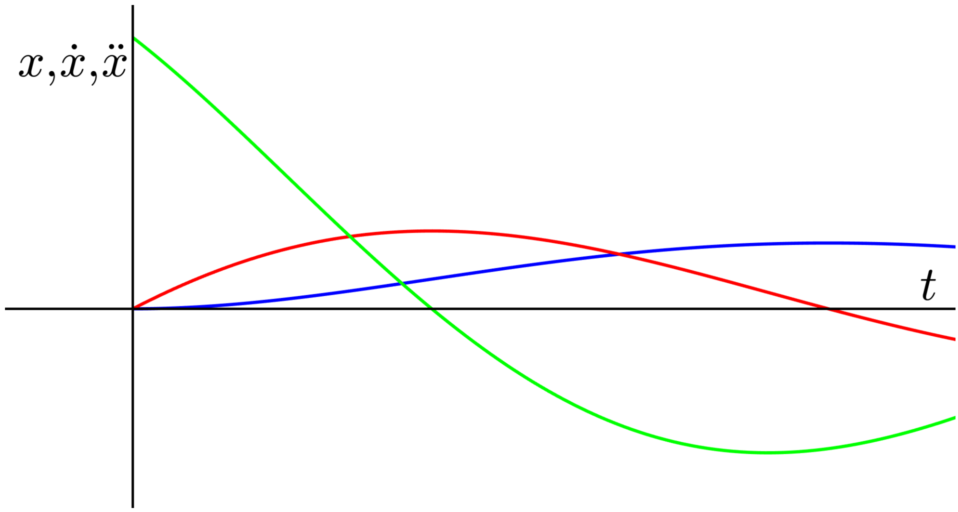
Sometimes, it is useful to consider the velocity of the mass as a function of its position, or $v = v(x)$. Since $x = x(t)$, the chain rule gives us the acceleration as

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}.$$

This equation relates acceleration to the rate of change of velocity with respect to position.

Problems for Lecture 55

1. The figure below shows three graphs: red, green and blue. One represents the position of a mass, one its velocity, and one its acceleration. Identify each curve and explain your reasoning.



Solutions to the Problems

Lecture 56 | Gravity of Earth

View this lecture on YouTube

Consider a mass falling downward, such as an apple falling from a tree. On the surface of the Earth, the mass falls with a constant acceleration, described by the differential equation,

$$\ddot{x} = -g,$$

where x is the height of the mass above the ground, and $g = 9.8\ \text{m/s}^2$ is the constant gravitational acceleration. The minus sign indicates that the x-axis points upward while the acceleration of the mass is directed downward.

The first integration, obtained by anti-differentiation, yields

$$\dot{x} = A - gt,$$

where A is the first constant of integration. The second integration yields

$$x = B + At - \frac{1}{2}gt^2,$$

where B is the second constant of integration. The constants A and B can be determined from initial conditions. If the initial height of the mass is x_0, and the initial velocity is v_0, the initial conditions are

$$x(0) = x_0, \quad \dot{x}(0) = v_0.$$

Substituting these initial conditions into the equations for x and $\dot{x}$ allows us to find $B = x_0$ and $A = v_0$. Thus, the solution becomes

$$x(t) = x_0 + v_0 t - \frac{1}{2}gt^2.$$

For example, suppose we drop a ball from the top of a 50-meter building and want to determine how long it will take for the ball to hit the ground. We need to find the time T such that $x(T) = 0$ given $x_0 = 50\,\text{m}$ and $v_0 = 0$. Solving for T, we find

$$T = \sqrt{\frac{2x_0}{g}} = \sqrt{\frac{2 \times 50}{9.8}}\ \text{s} \approx 3.2\,\text{s}.$$

Problems for Lecture 56

1. A cannon ball is shot from the ground with velocity v, making an angle α with the horizontal (see figure). Assume the horizontal velocity of the cannonball is constant, and the vertical acceleration is $-g$.

a) Show that the path of the cannon ball is a parabola by demonstrating that $y = ax^2 + bx + c$ for some constants a, b and c.

b) Find the angle α that maximizes the horizontal distance traveled before the cannonball hits the ground.

Solutions to the Problems

Practice Quiz | Newton's second law

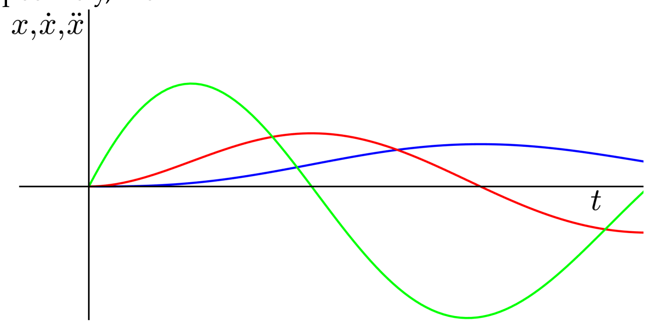
1. The figure shows three graphs: red (r), green (g), and blue (b). One represents the position of a mass, one its velocity, and one its acceleration. The colors of the position, velocity, and acceleration, respectively, are

a) r, g,b

b) g, b, r

c) b, r, g

d) g, r, b



2. The position of a mass at time t is given by $x = x(t)$. The velocity and acceleration are given by $v = v(t)$ and $a = a(t)$. Which of the following equations are true?

A. $a = \dfrac{dv}{dt}$ B. $a = \dfrac{d^2x}{dt^2}$ C. $a = x\dfrac{dv}{dx}$ D. $a = v\dfrac{dv}{dx}$

a) A only

b) A and B only

c) A, B and C only

d) A, B and D only

3. A stone is dropped into a deep well and is heard hitting the water about 4.5 seconds after being dropped. The approximate depth of the well (in meters) is

a) 50 m

b) 75 m

c) 100 m

d) 125 m

Solutions to the Practice quiz

Lecture 57 | Growth, decay, and oscillation

View this lecture on YouTube

We can determine the differential equations for growth, decay, and oscillation. The order of a differential equation is defined as the highest order derivative it contains.

A function $x = x(t)$ grows linearly if it has the form,

$$x(t) = at + b.$$

This linear function satisfies the second-order differential equation,

$$\ddot{x} = 0, \quad \text{with} \quad x(0) = b \quad \text{and} \quad \dot{x}(0) = a.$$

A function $x = x(t)$ grows exponentially ($r > 0$) or decays exponentially ($r < 0$) if it has the form,

$$x(t) = Ae^{rt}.$$

Since the derivative of an exponential function is also an exponential function, it satisfies the first-order differential equation

$$\dot{x} = rx, \quad \text{with} \quad x(0) = A.$$

The most important periodic functions are the cosine and sine functions, and the general form is

$$x(t) = A\cos\omega t + B\sin\omega t.$$

Since the second derivatives of cosine and sine are negative cosine and negative sine, this periodic function satisfies the second-order differential equation,

$$\ddot{x} + \omega^2 x = 0.$$

Only the cosine function is obtained with the initial conditions $x(0) = A$ and $\dot{x}(0) = 0$; and only the sine function is obtained with the initial conditions $x(0) = 0$ and $\dot{x}(0) = aB$.

Problems for Lecture 57

1. Consider the first-order complex differential equation,

$$y' = iy, \qquad \text{with} \qquad i = \sqrt{-1}.$$

a) Solve the differential equation with the initial condition $y(0) = 1$.

b) Express the complex function y as $y = u + iv$, where $u = u(x)$ and $v = v(x)$ are real functions. Determine u and v.

c) Find two coupled first-order real differential equations for u and v. What initial conditions correspond to $y(0) = 1$?

d) Derive second-order differential equations for both u and v. What initial conditions correspond to $y(0) = 1$?

Solutions to the Problems

Lecture 58 | Compound interest

View this lecture on YouTube

Compound interest occurs when interest is earned on both the initial deposit and the interest that has already been earned. Let $S(t)$ be the value of the account at time t, with $S(0) = S_0$ as the initial deposit, and let r be the annual interest rate.

If compounding occurs once per year, then the account balance after t years is

$$S(t) = S_0(1+r)^t.$$

But if compounding occurs n times per year, the interest awarded during each period will be at the rate of r/n, and the account balance after t years becomes

$$S(t) = S_0\left(1+\frac{r}{n}\right)^{nt}.$$

For continuous compounding, we consider the limit as $n \to \infty$. From the Taylor series for the exponential function, we have

$$1+\frac{r}{n} \quad \to \quad \exp\left(\frac{r}{n}\right).$$

Thus, the account value at time t is

$$S(t) = S_0 \exp(rt).$$

Another way to derive this equation is to assume compounding occurs after every time interval Δt. The account value at time $t + \Delta t$ can be written as

$$S(t+\Delta t) = S(t) + r\Delta t S(t),$$

where the interest credited to the account at the end of the time interval Δt is $r\Delta t S(t)$. Rearranging terms to isolate the derivative yields

$$\frac{S(t+\Delta t) - S(t)}{\Delta t} = rS(t).$$

Continuous compounding is obtained in the limit as $\Delta t \to 0$, resulting in the differential equation

$$\frac{dS}{dt} = rS.$$

With the initial condition $S(0) = S_0$, we arrive at the same exponential solution as before.

Problems for Lecture 58

1. Suppose an initial amount S_0 is invested at an annual compound interest rate r.

a) Find an exact formula for the time T required for the original amount S_0 to double in value.

b) Using the approximation $100 \ln 2 \approx 72$, complete the following table of T versus r:

interest rate (%)	years to double
2	
4	
6	
8	9
9	
12	

2. Newton's law of cooling states that the rate of change of an object's temperature is proportional to the difference between the object's temperature and that of its surroundings. Let $T = T(t)$ be the temperature of the object at time t, and let T_s be the temperature of its surroundings. Newton's law of cooling is expressed by the differential equation

$$\frac{dT}{dt} = -k(T - T_s),$$

where k is a positive constant.

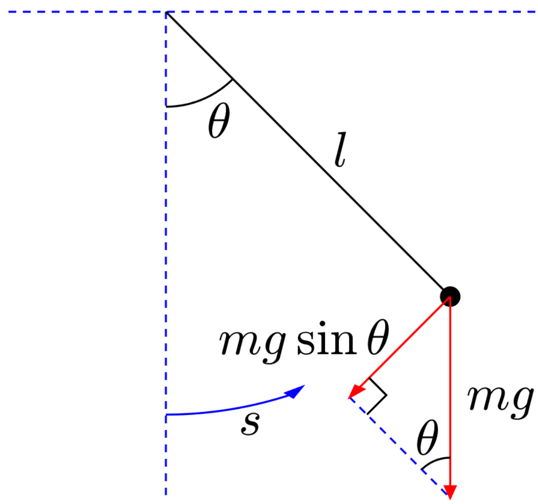
a) Explain why the differential equation indicates that the temperature of the object decreases when it is higher than the surrounding temperature, and increases when it is lower. What happens to the object's temperature when it equals the surrounding temperature?

b) Solve the differential equation by assuming the initial temperature of the object is $T(0) = T_0$, and by making the substitution $x = T - T_s$.

Solutions to the Problems

Lecture 59 | Simple pendulum

View this lecture on YouTube



The simple pendulum consists of a mass attached to a massless, rigid rod, oscillating without friction along an arc of a circle centered at the pivot point (see figure). Let l be the fixed length of the rod, and let θ be the angle the rod makes with the vertical.

We can apply Newton's equation, $F = ma$, to the mass, with the origin at the bottom and a curved axis along the arc, with the positive direction to the right. The position s of the mass along the arc is given by $s = l\theta$. The relevant gravitational force, F_g, on the pendulum is the component along the arc, which is $-mg\sin\theta$. Using Newton's equation, $m\ddot{s} = F_g$, we have

$$ml\ddot{\theta} = -mg\sin\theta.$$

For small oscillation amplitudes, we can approximate $\sin\theta \approx \theta$, and the differential equation becomes

$$\ddot{\theta} + \frac{g}{l}\theta = 0.$$

This is the equation for oscillations with an angular frequency of $\omega = \sqrt{g/l}$. If the pendulum is released with zero velocity and a small amplitude $\theta(0) = \theta_0$, then the solution is a cosine function, given by

$$\theta(t) = \theta_0 \cos\omega t.$$

We say that the simple pendulum, at small amplitudes of oscillation, undergoes simple harmonic motion.

Problems for Lecture 59

1. Consider a simple pendulum when the mass is launched from the bottom with an initial velocity u_0 to the right. Find the solution for $\theta = \theta(t)$.

2. Consider a pendulum with a frictional force opposing the velocity, and an external force driving the pendulum motion. Use Newton's law to determine the differential equation.

Solutions to the Problems

Practice Quiz | Growth, decay, and oscillations

1. The differential equation and initial conditions satisfied by $x(t) = \cosh t$ are

a) $\ddot{x} - x = 0; \quad x(0) = 1, \quad \dot{x}(0) = 0$

b) $\ddot{x} - x = 0; \quad x(0) = 0, \quad \dot{x}(0) = 1$

c) $\ddot{x} + x = 0; \quad x(0) = 1, \quad \dot{x}(0) = 0$

d) $\ddot{x} + x = 0; \quad x(0) = 0, \quad \dot{x}(0) = 1$

2. An investment account earning 6% annually will double in value in approximately how many years?

a) 8

b) 10

c) 12

d) 14

3. A simple pendulum is launched from the bottom with an initial velocity u_0 to the right. Under the small angle approximation, what should the value of u_0 be so that the maximum angle attained by the pendulum is $\theta_m = \pi/4$?

a) $\dfrac{\pi\sqrt{gl}}{4}$

b) $\dfrac{4\sqrt{gl}}{\pi}$

c) $\dfrac{\pi\sqrt{gl}}{2}$

d)

e) $\dfrac{2\sqrt{gl}}{\pi}$

Solutions to the Practice quiz

Appendices

Appendix A | Problem and practice quiz solutions

Solutions to the Problems for Lecture 1

1. To find a rational number between two rational numbers, a and b, consider their average, $(a+b)/2$. Let's assume $a = m/n$ and $b = p/q$, where m, n, p, and q are integers. Their average can be expressed as

$$\frac{a+b}{2} = \frac{\frac{m}{n} + \frac{p}{q}}{2} = \frac{mq+np}{2nq}.$$

Since both $mq + np$ and $2nq$ are integers, the number $(mq+np)/(2nq)$ is rational. Thus, $(a+b)/2$ is a rational number that lies between a and b.

2. To prove that $\sqrt{2}$ is an irrational number, we start by assuming it is a rational number. Let's express it as

$$\sqrt{2} = \frac{m}{n},$$

where m and n are positive integers with no common factors, meaning the fraction is in its simplest form. Squaring both sides gives us

$$2 = \frac{m^2}{n^2} \qquad \text{or} \qquad m^2 = 2n^2.$$

From this, we see m^2 is even. Since the square of an odd number is odd, m must be even. We can set $m = 2p$, where p is a positive integer. Substituting back, we get

$$\sqrt{2} = \frac{2p}{n}.$$

Squaring this equation results in

$$2 = \frac{4p^2}{n^2} \qquad \text{or} \qquad n^2 = 2p^2.$$

This implies n^2 is even, hence n must also be even. Since both m and n are even, the fraction m/n is not in its simplest form, contradicting our initial assumption. Therefore, we conclude that $\sqrt{2}$ is irrational.

Solutions to the Problems for Lecture 2

1.

a) $-(7-i)(-4-2i)(2-i) = (-7+i)(-8-2) = (-7+i)(-10) = 70-10i.$

b) $\dfrac{1+3i}{3-2i} = \dfrac{(1+3i)(3+2i)}{(3-2i)(3+2i)} = \dfrac{-3+11i}{13} = -\dfrac{3}{13} + \dfrac{11}{13}i.$

c) $\dfrac{1}{1+i} + \dfrac{1}{1-i} = \dfrac{1-i}{(1+i)(1-i)} + \dfrac{1+i}{(1+i)(1-i)} = \dfrac{1}{2}(1-i) + \dfrac{1}{2}(1+i) = 1.$

2. $2y + ix = 4 + x - i;$
$\mathrm{Re}\{2y+ix\} = \mathrm{Re}\{4+x-i\}, \quad \mathrm{Im}\{2y+ix\} = \mathrm{Im}\{4+x-i\};$
$2y = 4+x, \qquad x = -1;$
$x = -1, \qquad y = 3/2.$

3. Squaring both sides of $\sqrt{i} = x + iy$, we obtain

$$i = x^2 - y^2 + 2ixy.$$

By equating the real and imaginary parts, we obtain the two equations,

$$x^2 - y^2 = 0, \qquad 2xy = 1.$$

The first equation gives $y^2 = x^2$, or $y = \pm x$.

1. **Case** $y = x$:
   The second equation becomes $2x^2 = 1$, resulting in $x = \pm\sqrt{2}/2$.
2. **Case** $y = -x$
   The second equation becomes $-2x^2 = 1$, which has no solution for real x.

Therefore, the two complex numbers for $\sqrt{i}$ are

$$\sqrt{i} = \begin{cases} \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}, \\ -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}. \end{cases}$$

Solutions to the Practice quiz: Numbers

1. c. Only the square roots of natural numbers that are perfect squares are rational. In this case, the perfect squares among the numbers from 1 to 20 are 1, 4, 9, and 16.

2. d.

$$\frac{1}{(1+i)^2} = \frac{1}{1+2i+i^2} = \frac{1}{2i} = -\frac{i}{2}.$$

3. a.

$$(4+i)(x+iy) = 8-3i$$
$$x + iy = \frac{8-3i}{4+i} = \frac{(8-3i)(4-i)}{17} = \frac{32-3}{17} - \frac{12+8}{17}i = \frac{29}{17} - \frac{20}{17}i.$$

Thus,

$$x = \frac{29}{17}, \qquad y = -\frac{20}{17}.$$

Solutions to the Problems for Lecture 3

1. For a linear function, we have

$$y = mx + b.$$

Solving for x, we have

$$x = \frac{y-b}{m}.$$

The linear function and its inverse are therefore

$$f(x) = mx + b, \qquad f^{-1}(x) = \frac{x-b}{m}.$$

2. The unit quarter circle satisfies

$$x^2 + y^2 = 1, \qquad x \geq 0, \quad y \geq 0.$$

Solving for y yields $y = \sqrt{1-x^2}$, and solving for x yields $x = \sqrt{1-y^2}$. The unit quarter-circle function and its inverse are one and the same, and are given by

$$f(x) = f^{-1}(x) = \sqrt{1-x^2}, \qquad 0 \leq x \leq 1.$$

Indeed, the unit quarter-circle is invariant on reflexion around $y = x$.

Solutions to the Problems for Lecture 4

1.

a)

$$\lim_{x_2 \to x_1} \frac{x_2^2 - x_1^2}{x_2 - x_1} = \lim_{x_2 \to x_1} \frac{(x_2 - x_1)(x_2 + x_1)}{x_2 - x_1} = \lim_{x_2 \to x_1} (x_2 + x_1) = 2x_1.$$

b)

$$\begin{aligned} \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} &= \lim_{h \to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h} = \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\ &= \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2)}{h} = \lim_{h \to 0} (3x^2 + 3xh + h^2) = 3x^2. \end{aligned}$$

Solutions to the Problems for Lecture 5

1.

a) The range of the function $f(x) = x^2$ is the non-negative real numbers.

b) The range of the function $f(x) = x^3$ is $\mathbb{R}$.

c) The function factors as $f(x) = (x-3)(x-1)$. The function crosses the x-axis at $x = 1$ and $x = 3$, and by symmetry, reaches a minimum value at $x = 2$. Since $f(2) = -1$, the range of the function is $x \geq -1$.

2. We have

$$f(x) = \frac{1}{x^2-1} = \frac{1}{(x+1)(x-1)}.$$

The domain of the function $f(x)$ is the real numbers excluding the values where the denominator vanishes, $x = \pm 1$. Since the function diverges to $\pm\infty$ at the excluded points in its domain, the range is $\mathbb{R}$.

Solutions to the Practice quiz: Elementary functions

1. d. To find the inverse function of $f(x) = (1+x)^2$, with $x \geq -1$, we set $y = (1+x)^2$ and solve for x. Since $x \geq -1$, we have

$$y = (1+x)^2 \implies 1 + x = \sqrt{y} \implies x = \sqrt{y} - 1.$$

Thus, the inverse function is $f^{-1}(x) = \sqrt{x} - 1$.

2. c.

$$\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} = \lim_{x\to 4} \frac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x-4)(\sqrt{x}+2)} = \lim_{x\to 4} \frac{x-4}{(x-4)(\sqrt{x}+2)} = \lim_{x\to 4} \frac{1}{\sqrt{x}+2} = \frac{1}{4}.$$

3. a. Use the quadratic formula to solve $x^2 - x - 1 = 0$. We have

$$x_{\pm} = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

The root x_+ is choice (a).

Solutions to the Problems for Lecture 6

1. We have

$$e^x = \left(\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m\right)^x = \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{mx}.$$

Now, let $n = mx$, so that $1/m = x/n$. As m goes to infinity so does n, and we have

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n.$$

Solutions to the Problems for Lecture 7

1. Using $\ln x - \ln y = \ln (x/y)$, we have

$$\ln\left(\frac{B-C}{B}\right) - \ln\left(\frac{A-C}{A}\right) = \ln\left(\frac{A(B-C)}{B(A-C)}\right).$$

2.

a) Using $x^p = \exp(\ln x^p) = \exp(p \ln x)$, we have

$$x^p x^q = e^{p\ln x} e^{q \ln x} = e^{(p+q)\ln x} = e^{\ln x^{p+q}} = x^{p+q}.$$

b) Using $x^p = \exp(\ln x^p) = \exp(p \ln x)$, we have

$$(x^p)^q = \left(e^{p\ln x}\right)^q = e^{pq\ln x} = e^{\ln x^{pq}} = x^{pq}.$$

3. Let $y = \log_a x$, or $a^y = x$. To derive the change of base formula, we take the natural logarithm of both sides of the second equation: $\ln a^y = \ln x$, or $y \ln a = \ln x$, or $y = \ln x / \ln a$. We then equate the two expression for y to obtain

$$\log_a x = \frac{\ln x}{\ln a}.$$

Solutions to the Problems for Lecture 8

1. Using $360° = 2\pi$, we can determine that

$$45° = \frac{\pi}{4}, \quad 60° = \frac{\pi}{3}, \quad 90° = \frac{\pi}{2}, \quad 180° = \pi.$$

2. Using the additional formulas and

$$\cos(\pi/2) = 0, \quad \sin(\pi/2) = 1, \qquad \text{and} \qquad \cos(-x) = \cos x, \quad \sin(-x) = -\sin x,$$

we have

$$\cos\left(x - \frac{\pi}{2}\right) = \cos x \cos(-\pi/2) - \sin x \sin(-\pi/2) = \sin x;$$
$$\sin\left(x + \frac{\pi}{2}\right) = \sin x \cos(\pi/2) + \cos x \sin(\pi/2) = \cos x.$$

3. Using the addition formulas, we have

$$\sin(2x) = \sin(x + x) = \sin x \cos x + \cos x \sin x = 2 \sin x \cos x;$$
$$\cos(2x) = \cos(x + x) = \cos x \cos x - \sin x \sin x = \cos^2 x - \sin^2 x.$$

4. Using the double angle formulas and the Pythagorean theorem, we have

$$\cos(2x) = \cos^2 x - \sin^2 x = \cos^2 x - (1 - \cos^2 x) = 2\cos^2 x - 1.$$

Solving for $\cos^2 x$, we obtain

$$\cos^2 x = \frac{1 + \cos(2x)}{2}.$$

Using the power reduction formula for cosine and the Pythagorean theorem, we have

$$\sin^2 x = 1 - \cos^2 x = 1 - \frac{1 + \cos(2x)}{2} = \frac{1 - \cos(2x)}{2}.$$

5.

a) We start with the standard Pythagorean identity $\cos^2\theta + \sin^2\theta = 1$. We divide both sides by $\cos^2\theta$ and use the definitions of $\tan\theta$ and $\sec\theta$ to obtain $1 + \tan^2\theta = \sec^2\theta$.

b) We start with the standard Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$. We divide both sides by $\sin^2\theta$ and use the definitions of $\cot\theta$ and $\csc\theta$ to obtain $1 + \cot^2\theta = \csc^2\theta$.

6. Using

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

we obtain

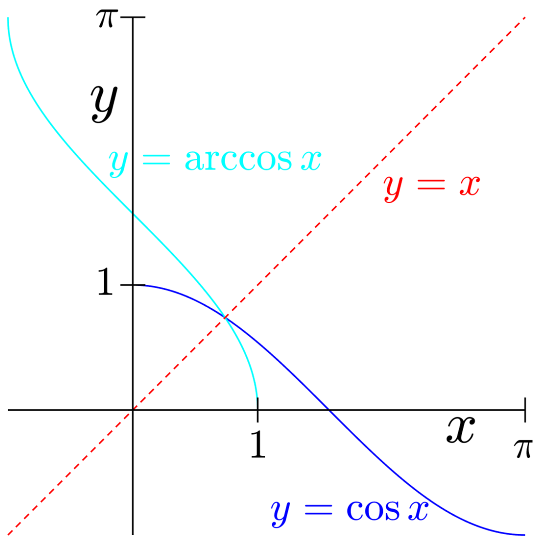
$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta + \sin^2\theta) = r^2,$$

and

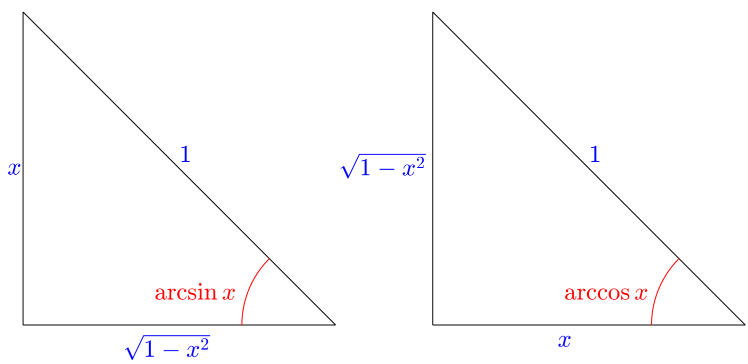
$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta.$$

Solutions to the Problems for Lecture 9

1. A graph of the cosine and arccosine functions is shown here.



2. The relevant triangles are shown here. By trigonometry on the drawn triangles, we



have

a) $\cos(\arcsin x) = \sqrt{1-x^2}$;

b) $\tan(\arcsin x) = \dfrac{x}{\sqrt{1-x^2}}$;

c) $\sin(\arccos x) = \sqrt{1-x^2}$;

d) $\tan(\arccos x) = \dfrac{\sqrt{1-x^2}}{x}$.

3. We can determine the values of the sine and cosine of $\arcsin x + \arccos x$ using the addition formulas $\sin(x+y) = \sin x \cos y + \cos x \sin y$ and $\cos(x+y) = \cos x \cos y -$

$\sin x \sin y$. We have

$$\begin{aligned}
&\sin(\arcsin x + \arccos x)\\
&\quad = \sin(\arcsin x)\cos(\arccos x) + \cos(\arcsin x)\sin(\arccos x) = x^2 + (\sqrt{1-x^2})^2 = 1;
\end{aligned}$$

and

$$\begin{aligned}
&\cos(\arcsin x + \arccos x)\\
&= \cos(\arcsin x)\cos(\arccos x) - \sin(\arcsin x)\sin(\arccos x) = x\sqrt{1-x^2} - x\sqrt{1-x^2} = 0.
\end{aligned}$$

With $-\pi/2 \leq \arcsin x \leq \pi/2$ and $0 \leq \arccos x \leq \pi$, the only angle satisfying $\sin\theta = 1$ and $\cos\theta = 0$ is $\theta = \pi/2$. Therefore,

$$\arcsin x + \arccos x = \frac{\pi}{2}.$$

Solutions to the Practice quiz: Transcendental functions

1. b.

$$e^{x^2} = \left(\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m\right)^{x^2} = \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{mx^2}$$

Let $n = m$, $n = mx$, or $n = mx^2$. With these substitutions, we have

$$e^{x^2} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{nx^2} = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{nx} = \lim_{n\to\infty}\left(1+\frac{x^2}{n}\right)^{n}.$$

2. b. The positive ln terms go in the numerator, and the negative ln terms go in the denominator:

$$\ln A - \ln B + \ln C - \ln D = \ln\left(\frac{AC}{BD}\right).$$

3. a.

$$\begin{aligned}
\frac{\sin(\arccos x)}{\cos(\arcsin x)} &= \frac{\sqrt{1-x^2}}{\sqrt{1-x^2}} = 1;\\
\frac{\sin(\arctan x)}{\cos(\arctan x)} &= \tan(\arctan x) = x;\\
\tan(\arcsin x)\cos(\arcsin x) &= \sin(\arcsin x) = x;\\
\cot(\arccos x)\sin(\arccos x) &= \cos(\arccos x) = x;
\end{aligned}$$

Solutions to the Problems for Lecture 10

1. We begin with our definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Let $x = a$. Our definition then becomes

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Now, let $x = a + h$, so that $h = x - a$. As $h \to 0$, we have $x \to a$. Therefore, we obtain

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

2.

$$y''' = \frac{d^3y}{dx^3}, \qquad y'''' = \frac{d^4y}{dx^4}.$$

3.

a) The function $y = \ln|x|$ only has a discontinuity at $x = 0$ and is therefore differentiable for all real values of x except $x = 0$.

b) The function $y = \arctan x$ has no discontinuities and is differentiable for all real values of x.

4. Because $\arctan x$ has a horizontal asymptote at $\pi/2$ as $x \to \infty$, and a constant function has zero slope, we must have

$$\lim_{x \to \infty} \frac{d}{dx} \arctan x = 0.$$

Solutions to the Problems for Lecture 11

1. With $f(x) = 1/x$, we have

$$f'(x) = \lim_{h \to 0} \frac{\dfrac{1}{x+h} - \dfrac{1}{x}}{h} = \lim_{h \to 0} \frac{x - (x+h)}{hx(x+h)} = \lim_{h \to 0} \frac{-h}{hx(x+h)} = \lim_{h \to 0} -\frac{1}{x(x+h)} = -\frac{1}{x^2}.$$

We can rewrite this result as

$$\frac{d}{dx}\left(x^{-1}\right) = -x^{-2},$$

and we see that the derivative follows the power rule even though the exponent is a negative integer.

2. With $f(x) = x^{3/2} = \sqrt{x^3}$, we have

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sqrt{(x+h)^3} - \sqrt{x^3}}{h} = \lim_{h\to 0} \frac{(\sqrt{(x+h)^3} - \sqrt{x^3})(\sqrt{(x+h)^3} + \sqrt{x^3})}{h(\sqrt{(x+h)^3} + \sqrt{x^3})} \\
&= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h(\sqrt{(x+h)^3} + \sqrt{x^3})} = \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h(\sqrt{(x+h)^3} + \sqrt{x^3})} \\
&= \lim_{h\to 0} \frac{h(3x^2 + 3xh + h^2)}{h(\sqrt{(x+h)^3} + \sqrt{x^3})} = \lim_{h\to 0} \frac{3x^2 + 3xh + h^2}{\sqrt{(x+h)^3} + \sqrt{x^3}} = \frac{3x^2}{2\sqrt{x^3}} = \frac{3}{2}x^{1/2}.
\end{aligned}$$

Therefore,

$$\frac{d}{dx}\left(x^{3/2}\right) = \frac{3}{2}x^{1/2},$$

and we see that the derivative follows the power rule.

Solutions to the Problems for Lecture 12

1. The function $f(x)$ and its first n derivatives are given by

$$\begin{aligned}
f(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n, \\
f'(x) &= a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1}, \\
f''(x) &= 2!a_2 + (3\cdot 2)a_3x + \cdots + n(n-1)a_nx^{n-2}, \\
f'''(x) &= 3!a_3 + \cdots + n(n-1)(n-2)a_nx^{n-3}, \\
\vdots &= \vdots, \\
f^{(n)}(x) &= n!a_n.
\end{aligned}$$

By evaluating the function and its derivatives at $x = 0$, and solving for $a_0, a_1, a_2, a_3, \ldots, a_n$, we find

$$a_0 = f(0), \quad a_1 = f'(0), \quad a_2 = f''(0)/2!, \quad a_3 = f'''(0)/3!, \quad \ldots, \quad a_n = f^{(n)}/n!.$$

Solutions to the Practice quiz: Differentiation rules I

1. c. The expression must represent the slope of tangent line to the curve at the point $(x, f(x))$, represented by $\frac{\Delta y}{\Delta x}$. For option (c), Δx should be equal to $2h$ and not h.

2. a.

$$\frac{d}{dx}\sqrt{x^3} = \frac{d}{dx}x^{3/2} = \frac{3}{2}x^{1/2} = \frac{3\sqrt{x}}{2}.$$

3. b.

$$\frac{d^2}{dx^2}\left(ax^3 + bx^2 + cx + d\right) = \frac{d}{dx}\left(3ax^2 + 2bx + c\right) = 6ax + 2b.$$

Solutions to the Problems for Lecture 13

1. Let $y = f(x)g(x)h(x)$ be the product of three functions. We have

$$\begin{aligned}\frac{dy}{dx} = \left(f(x)g(x)h(x)\right)' &= f'(x)\left(g(x)h(x)\right) + f(x)\left(g(x)h(x)\right)' \\ &= f'(x)\left(g(x)h(x)\right) + f(x)\left(g'(x)h(x) + g(x)h'(x)\right) \\ &= f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x).\end{aligned}$$

Note that this result generalizes to the product of any number of functions.

2. Let $y = x^2 = x \cdot x$. We have

$$\frac{dy}{dx} = \frac{d}{dx}(x \cdot x) = \frac{dx}{dx}x + x\frac{dx}{dx} = x + x = 2x.$$

Solutions to the Problems for Lecture 14

1. We start with

$$f(x) = g(x)\left[\frac{f(x)}{g(x)}\right].$$

We differentiate both sides using the product rule and obtain

$$f'(x) = g'(x)\left[\frac{f(x)}{g(x)}\right] + g(x)\left[\frac{f(x)}{g(x)}\right]'.$$

We rearrange terms to obtain

$$g(x)\left[\frac{f(x)}{g(x)}\right]' = f'(x) - g'(x)\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)}.$$

Dividing both sides by $g(x)$, we obtain the quotient rule,

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

2. We use the quotient rule to compute the derivative:

$$\frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right) = \frac{2x(x^2-1) - (x^2+1)2x}{(x^2-1)^2} = -\frac{4x}{(x^2-1)^2}.$$

3. Let $f(x) = x^{-n} = 1/x^n$, where n is a positive integer. We compute the derivative using the quotient rule and the power rule:

$$f'(x) = \frac{(1)'x^n - (1)(x^n)'}{(x^n)^2} = \frac{-nx^{n-1}}{x^{2n}} = \frac{-n}{x^{n+1}} = -nx^{-n-1}.$$

We have therefore determined that

$$\frac{d}{dx}(x^{-n}) = -nx^{-n-1},$$

so that the power rule is valid for all negative integer exponents.

Solutions to the Problems for Lecture 15

1. We have

$$\frac{d}{dx}\sqrt{1+x^2} = \frac{d}{dx}(1+x^2)^{1/2} = \frac{1}{2}(1+x^2)^{-1/2}2x = \frac{x}{\sqrt{1+x^2}}.$$

2. We have

$$\begin{aligned}\frac{d}{dx}\left[(x^2+1)^2(x^2-1)^2\right] &= 2(x^2+1)2x((x^2-1)^2 + (x^2+1)^2 2(x^2-1)2x \\ &= 4x(x^2+1)(x^2-1)[(x^2-1)+(x^2+1)] = 8x^3(x^2+1)(x^2-1)\end{aligned}$$

3. From the quotient rule and the chain rule, we have

$$\begin{aligned}\frac{d}{dx}\left[\frac{(x^2-1)^2}{(x^2+1)^2}\right] &= \frac{2(x^2-1)2x(x^2+1)^2 - (x^2-1)^2 2(x^2+1)2x}{(x^2+1)^4} \\ &= \frac{4x(x^2-1)(x^2+1)[(x^2+1)-(x^2-1)]}{(x^2+1)^4} = \frac{8x(x^2-1)}{(x^2+1)^3}\end{aligned}$$

4.

a) We are finding the derivative of the composition of three functions, which we first write as $z = z(y(x(t)))$. Using Leibniz notation, we have

$$\frac{dz}{dt} = \frac{dz}{dy}\frac{dy}{dx}\frac{dx}{dt}.$$

In the prime notation, this corresponds to

$$[f(g(h(x)))]' = f'(g(h(x)))g'(h(x))h'(x).$$

To take the derivative, we work our way from the outside to the inside.

b) We have

$$\frac{d}{dt}\left(1+(1+t^2)^2\right)^2 = 2\left(1+(1+t^2)^2\right)2(1+t^2)2t = 8t(1+t^2)\left(1+(1+t^2)^2\right).$$

Solutions to the Practice quiz: Differentiation rules II

1. d.

$$\frac{d}{dx}\sqrt{1-x^2} = \frac{d}{dx}(1-x^2)^{1/2} = \frac{1}{2}(1-x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{1-x^2}}.$$

2. a.

$$\frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right) = \frac{2x(x^2+1)-(x^2-1)2x}{(x^2+1)^2} = \frac{2x^3+2x-2x^3+2x}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}.$$

3. b.

$$\frac{d}{dx}\sqrt{\frac{1-x^2}{1+x^2}} = \frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)^{1/2} = \frac{1}{2}\left(\frac{1-x^2}{1+x^2}\right)^{-1/2}\left(\frac{-2x(1+x^2)-(1-x^2)2x}{(1+x^2)^2}\right)$$
$$= \left(\frac{1+x^2}{1-x^2}\right)^{1/2}\left(\frac{-x-x^3-x+x^3}{(1+x^2)^2}\right) = \frac{-2x}{(1-x^2)^{1/2}(1+x^2)^{3/2}} = \frac{-2x}{\sqrt{(1-x^2)(1+x^2)^3}}.$$

Solutions to the Problems for Lecture 16

1. The derivatives are

a) $\dfrac{d}{dx}\left(e^{-x}\right) = -e^{-x}$;

b) $\dfrac{d}{dx}\left(e^{-x^2}\right) = -2xe^{-x^2}$;

c) $\dfrac{d}{dx}\left(xe^{-x}\right) = e^{-x} - xe^{-x} = (1-x)e^{-x}$;

d) $\dfrac{d}{dx}\left(\dfrac{e^{-x}}{x}\right) = \dfrac{-xe^{-x}-e^{-x}}{x^2} = -\dfrac{(1+x)e^{-x}}{x^2}$.

2. We have

$$\frac{d}{dx}\exp(x) = \frac{d}{dx}\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \lim_{n\to\infty}\frac{d}{dx}\left(1+\frac{x}{n}\right)^n = \lim_{n\to\infty} n\left(1+\frac{x}{n}\right)^{n-1}\cdot\frac{1}{n}$$
$$= \lim_{n\to\infty}\frac{\left(1+\frac{x}{n}\right)^n}{\left(1+\frac{x}{n}\right)} = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \exp(x).$$

Solutions to the Problems for Lecture 17

1. On the one hand, using the chain rule, we have

$$\frac{d}{dx}\left[\ln(ax)\right] = \left(\frac{1}{ax}\right)(a) = \frac{1}{x}, \qquad \frac{d}{dx}\left(\ln x^r\right) = \left(\frac{1}{x^r}\right)\left(rx^{r-1}\right) = \frac{r}{x}.$$

On the other hand, using the properties of the logarithm, we have

$$\frac{d}{dx}\left[\ln(ax)\right] = \frac{d}{dx}\left[\ln a + \ln x\right] = \frac{1}{x}, \qquad \frac{d}{dx}\left(\ln x^r\right) = \frac{d}{dx}\left(r\ln x\right) = \frac{r}{x}.$$

2.

a) $$\frac{d}{dx}\left[\ln(\ln x)\right] = \frac{1}{x\ln x};$$

b) $$\frac{d}{dx}\left[\frac{x}{1+\ln x}\right] = \frac{(1+\ln x)-1}{(1+\ln x)^2} = \frac{\ln x}{(1+\ln x)^2}$$

c) $$\frac{d}{dx}\left[\ln\left(x+\sqrt{1+x^2}\right)\right] = \frac{1}{x+\sqrt{1+x^2}}\left(1+\frac{x}{\sqrt{1+x^2}}\right)$$
$$= \frac{x+\sqrt{1+x^2}}{\left(\sqrt{1+x^2}\right)\left(x+\sqrt{1+x^2}\right)} = \frac{1}{\sqrt{1+x^2}}.$$

Solutions to the Problems for Lecture 18

1.

$$\frac{d}{dx}\left(a^x\right) = \frac{d}{dx}\left(\exp\left(\ln a^x\right)\right) = \frac{d}{dx}\left(\exp\left(x\ln a\right)\right) = \exp\left(x\ln a\right)\ln a = a^x\ln a.$$

2.

$$\frac{d}{dx}\log_a x = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) = \frac{1}{x\ln a}.$$

3.

$$\frac{d}{dx}x^x = \frac{d}{dx}e^{\ln x^x} = \frac{d}{dx}e^{x\ln x} = e^{x\ln x}(\ln x+1) = (1+\ln x)x^x.$$

Solutions to the Practice quiz: Exponential and logarithm functions

1. d.

$$\frac{d}{dx}\left(xe^{-x^2}\right) = e^{-x^2} + xe^{-x^2}(-2x) = (1-2x^2)e^{-x^2}.$$

2. c.

$$\frac{d}{dx}\left(\frac{x}{1-\ln x}\right) = \frac{(1-\ln x)+1}{(1-\ln x)^2} = \frac{2-\ln x}{(1-\ln x)^2}.$$

3. b.

$$\frac{d}{dx}\left(x^{\ln x}\right) = \frac{d}{dx}\left(\exp\left(\ln\left(x^{\ln x}\right)\right)\right) = \frac{d}{dx}\left(\exp\left(\ln x\ln x\right)\right) = \frac{d}{dx}\left(\exp\left(\ln^2 x\right)\right)$$
$$= \exp\left(\ln^2 x\right)\frac{2\ln x}{x} = \frac{2x^{\ln x}\ln x}{x}.$$

Solutions to the Problems for Lecture 19

1.

$$\frac{d}{dx}\cos x = \frac{d}{dx}\sin\left(x+\pi/2\right) = \cos\left(x+\pi/2\right) = \cos x\cos\left(\pi/2\right) - \sin x\sin\left(\pi/2\right) = -\sin x.$$

2.

a) $\dfrac{d}{dx}\left[\ln\left(\sin x\right)\right] = \dfrac{1}{\sin x}\cos x = \cot x;$

b) $\dfrac{d}{dx}\left[\ln\left(\cos x\right)\right] = \dfrac{1}{\cos x}(-\sin x) = -\tan x;$

c) $\dfrac{d}{dx}\left[\sin\left(\ln x\right)\right] = \dfrac{\cos\left(\ln x\right)}{x};$

d) $\dfrac{d}{dx}\left[\cos\left(\ln x\right)\right] = -\dfrac{\sin\left(\ln x\right)}{x}.$

3. Let $y = A\cos x + B\sin x$. Then,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(A\cos x + B\sin x\right) = -A\sin x + B\cos x,\\ \frac{d^2y}{dx^2} &= \frac{d}{dx}\left(-A\sin x + B\cos x\right) = -A\cos x - B\sin x = -(A\cos x - B\sin x) = -y.\end{aligned}$$

4. Let $z = \cos x + i\sin x$. Then,

$$\frac{dz}{dx} = \frac{d}{dx}\left(\cos x + i\sin x\right) = -\sin x + i\cos x = i(\cos x + i\sin x) = iz.$$

Solutions to the Problems for Lecture 20

1. We differentiate

$$y = \frac{\tan x - 1}{\sec x}$$

in two ways.

a) In addition to the quotient rule, the differentiation rules for $\tan x$ and $\sec x$, and the definitions of $\tan x$ and $\sec x$, we will also make use of the Pythagorean relation $\tan^2 x + 1 = \sec^2 x$.

$$\begin{aligned}\frac{d}{dx}\left(\frac{\tan x - 1}{\sec x}\right) = \frac{\sec^3 x - (\tan x - 1)\sec x\tan x}{\sec^2 x} &= \frac{(\sec^2 x - \tan^2 x) + \tan x}{\sec x}\\ &= \frac{1+\tan x}{\sec x} = \cos x + \sin x.\end{aligned}$$

b) We first make use of the definitions of $\tan x$ and $\sec x$ and simplify the expression before differentiating.

$$\frac{d}{dx}\left(\frac{\tan x - 1}{\sec x}\right) = \frac{d}{dx}\left(\frac{\frac{\sin x}{\cos x} - 1}{\frac{1}{\cos x}}\right) = \frac{d}{dx}(\sin x - \cos x) = \cos x + \sin x.$$

Solutions to the Problems for Lecture 21

1. We take the derivative of both sides of $\arcsin x + \arccos x = \pi/2$. We obtain

$$\frac{d}{dx}\arcsin x + \frac{d}{dx}\arccos x = 0.$$

Thus,

$$\frac{d}{dx}\arccos x = -\frac{d}{dx}\arcsin x = -\frac{1}{\sqrt{1-x^2}}.$$

2.

$$\begin{aligned}\frac{d}{dx}\arcsin\left(\frac{1}{\sqrt{1+x^2}}\right) &= \left(\frac{1}{\sqrt{1-\frac{1}{1+x^2}}}\right)\frac{d}{dx}\left(1+x^2\right)^{-1/2} \\ &= \left(\frac{(1+x^2)^{1/2}}{x}\right)\left(\frac{-x}{(1+x^2)^{3/2}}\right) = -\frac{1}{1+x^2}.\end{aligned}$$

Solutions to the Practice quiz: Trigonometric functions

1. a.

$$\frac{d}{dx}\cos\left(x - \frac{\pi}{2}\right) = \frac{d}{dx}\left(\cos x \cos\left(-\frac{\pi}{2}\right) - \sin x \sin\left(-\frac{\pi}{2}\right)\right) = \frac{d}{dx}(\sin x) = \cos x.$$

2. b.

$$\frac{d^2}{dx^2}\tan x = \frac{d}{dx}\sec^2 x = 2\sec x \tan x \sec x = 2\tan x \sec^2 x.$$

3. d.

$$\frac{d^2}{dx^2}\arctan x = \frac{d}{dx}\left(\frac{1}{1+x^2}\right) = \frac{-2x}{(1+x^2)^2}.$$

Solutions to the Problems for Lecture 22

1. Notice that $\cos mx$ is an even function and $\sin nx$ is an odd function. Thus, the product $\cos mx \sin nx$ is an odd function. Integrals of odd functions over symmetric intervals around the origin are zero, so that

$$\int_{-\pi}^{\pi} \cos mx \sin nx \, dx = 0.$$

2. Notice that x is an odd function, x^2 is an even function, $\cos mx$ is an even function, and $\sin mx$ is an odd function. The product of two even functions or two odd functions is even, and the product of an even and an odd function is odd. Integrals of odd functions over symmetric intervals around the origin are zero, and integrals of even functions over symmetric intervals are twice the integral over the positive interval only. By identifying the symmetries of the integrands,

a) $\displaystyle\int_{-\pi}^{\pi} x \cos mx \, dx = 0;$

b) $\displaystyle\int_{-\pi}^{\pi} x \sin mx \, dx = 2\int_{0}^{\pi} x \sin mx \, dx;$

c) $\displaystyle\int_{-\pi}^{\pi} x^2 \cos mx \, dx = 2\int_{0}^{\pi} x^2 \cos mx \, dx;$

d) $\displaystyle\int_{-\pi}^{\pi} x^2 \sin mx \, dx = 0.$

Solutions to the Problems for Lecture 23

1. The integral $\displaystyle\int_a^b f(x)\, dx$ is a constant, independent of x. The x inside the integral is a dummy variable. Therefore,

$$\frac{d}{dx}\int_a^b f(x)\, dx = 0.$$

2.

a) We can reverse the limits of integration.

$$\frac{d}{dx}\int_x^a f(s)\, ds = -\frac{d}{dx}\int_a^x f(s)\, ds = -f(x).$$

b) We notice that $\displaystyle\int_a^{g(x)} f(s)\, ds$ is a composite function, $F(g(x))$, where $F(x) = \displaystyle\int_a^x f(s)\, ds$. Accordingly, we apply the chain rule,

$$\frac{d}{dx}\int_a^{g(x)} f(s)\, ds = f(g(x))g'(x).$$

c) Here, we make use of the previous two results.

$$\begin{aligned}\frac{d}{dx}\int_{g_1(x)}^{g_2(x)} f(s)\,ds &= \frac{d}{dx}\int_{g_1(x)}^{c} f(s)\,ds + \frac{d}{dx}\int_{c}^{g_2(x)} f(s)\,ds \\ &= -\frac{d}{dx}\int_{c}^{g_1(x)} f(s)\,ds + \frac{d}{dx}\int_{c}^{g_2(x)} f(s)\,ds = f(g_2(x))g_2'(x) - f(g_1(x))g_1'(x).\end{aligned}$$

3. Using the definition of the derivative,

$$\begin{aligned}\frac{d}{dx}\left[\int_a^x f(x)dy\right] &= \lim_{h\to 0}\frac{\int_a^{x+h} f(x+h)\,dy - \int_a^x f(x)\,dy}{h} \\ &= \lim_{h\to 0}\frac{\int_a^{x+h}\left[f(x)+hf'(x)\right]\,dy - \int_a^x f(x)\,dy}{h} \\ &= \lim_{h\to 0}\frac{\int_a^{x+h} f(x)\,dy - \int_a^x f(x)\,dy}{h} + \lim_{h\to 0}\frac{h\int_a^{x+h} f'(x)dy}{h} = f(x) + \int_a^x f'(x)dy.\end{aligned}$$

Solutions to the Problems for Lecture 24

1.

a) When n is a positive integer, the antiderivative of x^n is $x^{n+1}/(n+1)$. Therefore,

$$\int_a^b x^n\,dx = \left.\frac{x^{n+1}}{n+1}\right|_a^b = \frac{1}{n+1}\left(b^{n+1} - a^{n+1}\right).$$

b) For $x > 0$, the antiderivative of $1/x$ is $\ln x$. Therefore

$$\int_a^b \frac{1}{x}\,dx = \ln x|_a^b = \ln b - \ln a = \ln(b/a).$$

c) The antiderivative of $\cos x$ is $\sin x$. Therefore,

$$\int_0^{\pi/2} \cos x\,dx = \sin x|_0^{\pi/2} = \sin\frac{\pi}{2} - \sin 0 = 1.$$

d) The antiderivative of e^x is e^x. Therefore,

$$\int_{-\infty}^{1} e^x\,dx = e^x|_{-\infty}^{1} = e^1 - \lim_{x\to-\infty} e^x = e.$$

Solutions to the Problems for Lecture 25

1. The indefinite integral is given by

$$\int \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n\right) dx$$
$$= C + a_0 x + \frac{a_1 x^2}{2} + \frac{a_2 x^3}{3} + \frac{a_3 x^4}{4} + \cdots + \frac{a_n x^{n+1}}{n+1}.$$

2. The indefinite integral is given by

$$\int (\cos x - \sin x)\, dx = \sin x + \cos x + C.$$

3. The indefinite integral is given by

$$\int (\exp(x) + \exp(-x))\, dx = \exp(x) - \exp(-x) + C.$$

Solutions to the Practice quiz: Fundamental theorems

1. c. The functions e^{x^2} and $\cos x$ are even functions, while the function $\sin x$ is an odd function. Symmetrical integrals over even functions can be computed over the positive region only and multiplied by two, while symmetrical integrals over odd functions equal zero. Thus,

$$\int_{-\pi}^{\pi} e^{x^2} (\cos mx + \sin mx)\, dx = \int_{-\pi}^{\pi} e^{x^2} \cos mx\, dx + \int_{-\pi}^{\pi} e^{x^2} \sin mx\, dx$$
$$= 2\int_0^{\pi} e^{x^2} \cos mx\, dx + 0 = 2\int_0^{\pi} e^{x^2} \cos mx\, dx.$$

2. d. Using the first fundamental theorem of calculus, along with the chain rule and the power rule, we have

$$\frac{d}{dx}\int_0^{x^2} f(s)\, ds = 2x f(x^2).$$

3. b. Using the known formula for the derivative of x^3, we find

$$\int_0^1 x^2\, dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}.$$

4. d. Taking the indefinite integral of the polynomial function, we have

$$\int \left(ax^2 + bx + c\right) dx = \frac{ax^3}{3} + \frac{bx^2}{2} + cx + C.$$

Solutions to the Problems for Lecture 26

1.

a) To integrate $\int_0^t e^{rs}\, ds$, let $u = rs$, $du = rds$. Then,

$$\int_0^t e^{rs}\, ds = \frac{1}{r}\int_0^{rt} e^u du = \frac{1}{r}e^u\bigg|_0^{rt} = \frac{1}{r}\left(e^{rt} - 1\right).$$

b) To integrate $\int \cos(\omega t + \phi)\, dt$, let $u = \omega t + \phi$, $du = \omega dt$. Then,

$$\int \cos(\omega t + \phi)\, dt = \frac{1}{\omega}\int \cos u\, du = \frac{1}{\omega}\sin u + C = \frac{1}{\omega}\sin(\omega t + \phi) + C.$$

2.

a) To integrate $\int_0^x \frac{ds}{(1 + s/R)^2}$, let $u = 1 + s/R$, $du = ds/R$. Then,

$$\int_0^x \frac{ds}{(1+\frac{s}{R})^2} = \int_1^{1+x/R} \frac{Rdu}{u^2} = -\frac{R}{u}\bigg|_1^{1+x/R} = R - \frac{R}{1+\frac{x}{R}} = \frac{x}{1+\frac{x}{R}} = \frac{Rx}{R+x}.$$

b) To integrate $\int_0^1 x\sqrt{1-x^2}\, dx$, let $u = 1 - x^2$, $du = -2x\, dx$. Then,

$$\int_0^1 x\sqrt{1-x^2}\, dx = -\frac{1}{2}\int_1^0 u^{1/2}\, du = \frac{1}{2}\int_0^1 u^{1/2}\, du = \frac{1}{3}u^{3/2}\bigg|_0^1 = \frac{1}{3}.$$

c) To integrate $\int \frac{\ln x}{x}\, dx$, let $u = \ln x$, $du = dx/x$. Then,

$$\int \frac{\ln x}{x}\, dx = \int u\, du = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln x)^2 + C.$$

d) To integrate $\int \frac{dx}{x \ln x}$, let $u = \ln x$, $du = dx/x$. Then,

$$\int \frac{dx}{x\ln x} = \int \frac{du}{u} = \ln u + C = \ln(\ln x) + C.$$

Solutions to the Problems for Lecture 27

1.

a) We integrate $\int x \sin x \, dx$ by parts. Let

$$\begin{aligned} u &= x, & dv &= \sin x \, dx, \\ du &= dx, & v &= -\cos x. \end{aligned}$$

Then,

$$\int x \sin x \, dx = -x \cos x + \int \cos x \, dx = -x \cos x + \sin x + C = \sin x - x \cos x + C.$$

b) We integrate $\int \ln x \, dx$ by parts. Let

$$\begin{aligned} u &= \ln x, & dv &= dx, \\ du &= dx/x, & v &= x. \end{aligned}$$

Then,

$$\int \ln x \, dx = x \ln x - \int dx = x \ln x - x + C.$$

c) We integrate $\int e^x \sin x \, dx$ by parts. Let

$$\begin{aligned} u &= \sin x, & dv &= e^x \, dx, \\ du &= \cos x \, dx, & v &= e^x. \end{aligned}$$

Then,

$$\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx.$$

We integrate by parts again. Let

$$\begin{aligned} u &= \cos x, & dv &= e^x \, dx, \\ du &= -\sin x \, dx, & v &= e^x. \end{aligned}$$

Then,

$$\int e^x \sin x \, dx = e^x \sin x - \left(e^x \cos x + \int e^x \sin x \, dx \right).$$

Solving for the desired integral and adding an integration constant, we have

$$\int e^x \sin x \, dx = \frac{1}{2} e^x \left(\sin x - \cos x \right) + C.$$

d) We first integrate $\int \arcsin x \, dx$ by parts. Let

$$\begin{aligned} u &= \arcsin x, & dv &= dx, \\ du &= \tfrac{dx}{\sqrt{1-x^2}}, & v &= x. \end{aligned}$$

Then,

$$\int \arcsin x \, dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx.$$

The remaining integral can be done by substitution. Let $u = 1 - x^2$ so that $du = -2x \, dx$. Then we have

$$\int \frac{x}{\sqrt{1-x^2}} dx = -\frac{1}{2} \int u^{-1/2} \, du = -u^{1/2} + c = -\sqrt{1-x^2} + C',$$

where we have called the integration constant C'. Thus,

$$\int \arcsin x \, dx = x \arcsin x + \sqrt{1-x^2} + C.$$

2. Consider the definite integral given by

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t} \, dt.$$

a) We integrate by parts. Let

$$\begin{aligned} u &= t^x, & dv &= e^{-t} \, dt, \\ du &= x t^{x-1} \, dt, & v &= -e^{-t}. \end{aligned}$$

Then

$$\Gamma(x+1) = -t^x e^{-t} \Big|_0^\infty + x \int_0^\infty t^{x-1} e^{-t} \, dt = x\Gamma(x).$$

b) Now,

$$\Gamma(1) = \int_0^\infty e^{-t} \, dt = -e^{-t} \Big|_0^\infty = 1.$$

c) It n is a positive integer, then iterating $\Gamma(n+1) = n\Gamma(n)$, we have

$$\Gamma(n+1) = n \cdot \Gamma(n) = n \cdot (n-1) \cdot \Gamma(n-1) = \cdots = n \cdot (n-1) \cdots 2 \cdot 1 \cdot \Gamma(1).$$

With $\Gamma(1) = 1$, we have shown that $\Gamma(n+1) = n!$.

Solutions to the Practice quiz: Elementary techniques of integration

1. b. To integrate $\int_0^{\pi/2} \cos^3\theta \sin\theta\, d\theta$, let $u = \cos\theta$, $du = -\sin\theta\, d\theta$. Then,

$$\int_0^{\pi/2} \cos^3\theta \sin\theta\, d\theta = -\int_1^0 u^3\, du = \int_0^1 u^3\, du = \left.\frac{u^4}{4}\right|_0^1 = \frac{1}{4}.$$

2. d. To integrate $\int_0^{a/b} \frac{dx}{a+bx}$, let $u = a + bx$, $du = b\, dx$. Then,

$$\int_0^{a/b} \frac{dx}{a+bx} = \frac{1}{b}\int_a^{2a} \frac{du}{u} = \frac{1}{b}\ln u\Big|_a^{2a} = \frac{1}{b}\ln 2.$$

3. d. To integrate $\int_1^x s\ln s\, ds$, we use integration parts. Let

$$\begin{aligned} u &= \ln s, & dv &= s\, ds, \\ du &= \frac{1}{s}\, ds, & v &= \frac{1}{2}s^2. \end{aligned}$$

Then,

$$\int_1^x s\ln s\, ds = \frac{1}{2}s^2\ln s\Big|_1^x - \frac{1}{2}\int_1^x s\, ds = \frac{1}{2}x^2\ln x - \frac{1}{4}s^2\Big|_1^x = \frac{1}{2}x^2\ln x - \frac{1}{4}(x^2-1).$$

Solutions to the Problems for Lecture 28

1.

a) Use the reduction formula for $\sin^2\theta$:

$$\int_0^{2\pi} \sin^2\theta\, d\theta = \frac{1}{2}\int_0^{2\pi}(1-\cos 2\theta)\, d\theta = \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{2\pi} = \pi.$$

b) Use the Pythagorean theorem, $\sin^2\theta = 1 - \cos^2\theta$, and integrate by substitution:

$$\int_0^{\pi/2} \sin^3\theta\, d\theta = \int_0^{\pi/2}(1-\cos^2\theta)\sin\theta\, d\theta.$$

Let $u = \cos\theta$, $du = -\sin\theta\, d\theta$. Then since $\cos 0 = 1$ and $\cos\pi/2 = 0$, we have

$$\int_0^{\pi/2} \sin^3\theta\, d\theta = \int_0^1 (1-u^2)\, du = \left[u - \frac{1}{3}u^3\right]_0^1 = \frac{2}{3}.$$

c) Use the reduction formula for $\cos^2\theta$ as often as needed:

$$\int_0^{2\pi} \cos^4\theta\, d\theta = \frac{1}{4}\int_0^{2\pi} (1+\cos 2\theta)(1+\cos 2\theta)\, d\theta$$
$$= \frac{1}{4}\int_0^{2\pi} (1+2\cos 2\theta)\, d\theta + \frac{1}{4}\int_0^{2\pi} \cos^2 2\theta\, d\theta = \frac{1}{4}\left[\theta + \sin 2\theta\right]_0^{2\pi} + \frac{1}{8}\int_0^{2\pi} (1+\cos 4\theta)\, d\theta$$
$$= \frac{\pi}{2} + \frac{1}{8}\left[\theta + \frac{1}{4}\sin 4\theta\right]_0^{2\pi} = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}.$$

Solutions to the Problems for Lecture 29

1. In part (a) we verify the secant integral, and in part (b) we use it to solve a definite integral.

a)

$$\frac{d}{dx}\left[\ln(\sec\theta + \tan\theta) + C\right] = \frac{\frac{d}{dx}(\sec\theta + \tan\theta)}{\sec\theta + \tan\theta} = \frac{\tan\theta\sec\theta + \sec^2\theta}{\sec\theta + \tan\theta}$$
$$= \frac{\sec\theta(\sec\theta + \tan\theta)}{\sec\theta + \tan\theta} = \sec\theta.$$

We have thus verified that

$$\int \sec\theta\, d\theta = \ln(\sec\theta + \tan\theta) + C.$$

b) Let $x = \tan\theta$, $dx = \sec^2\theta\, d\theta$. Note that $x = 0$ when $\theta = 0$ and $x = 1$ when $\theta = \pi/4$. Then

$$\int_0^1 \frac{dx}{\sqrt{1+x^2}} = \int_0^{\pi/4} \frac{\sec^2\theta}{\sqrt{1+\tan^2\theta}} d\theta = \int_0^{\pi/4} \frac{\sec^2\theta}{\sqrt{\sec^2\theta}} d\theta = \int_0^{\pi/4} \sec\theta\, d\theta$$
$$= \ln(\sec\theta + \tan\theta)\Big|_0^{\pi/4} = \ln(\sec(\pi/4) + \tan(\pi/4)) - \ln(\sec 0 + \tan 0) = \ln(\sqrt{2}+1).$$

Solutions to the Problems for Lecture 30

1. To integrate $\displaystyle\int_0^x \frac{ds}{(a-s)(b-s)}$, with $x < a, b$ and $a \neq b$, we first perform a partial fraction decomposition,

$$\frac{1}{(a-s)(b-s)} = \frac{A}{a-s} + \frac{B}{b-s}.$$

To find A, we multiply both sides by $a-s$ and set $s = a$, yielding $A = 1/(b-a)$. To find B, we multiply both sides by $b-s$ and set $s = b$, yielding $B = 1/(a-b)$. Thus,

$$\frac{1}{(a-s)(b-s)} = \frac{1}{b-a}\left(\frac{1}{a-s} - \frac{1}{b-s}\right).$$

Now, we integrate,

$$\begin{aligned}
&\int_0^x \frac{ds}{(a-s)(b-s)} \\
&\quad = \frac{1}{b-a}\left(\int_0^x \frac{ds}{a-s} - \int_0^x \frac{ds}{b-s}\right) = \frac{1}{b-a}\left(-\ln(a-s)\big|_0^x + \ln(b-s)\big|_0^x\right) \\
&\qquad = \frac{1}{b-a}\left(-\ln(a-x) + \ln a + \ln(b-x) - \ln b\right) = \frac{1}{b-a}\ln\frac{a(b-x)}{b(a-x)}.
\end{aligned}$$

As a final check, the solution remains unchanged if a and b are interchanged, which is consistent with the symmetry of the integrand.

Solutions to the Practice quiz: Advanced techniques of integration

1. c. To integrate $\int_0^{2\pi} \sin^4\theta\, d\theta$, we use the reduction formula for $\sin^2\theta$ twice:

$$\int_0^{2\pi} \sin^4\theta\, d\theta = \frac{1}{4}\int_0^{2\pi} (1-\cos 2\theta)(1-\cos 2\theta)\, d\theta.$$

Expanding the integrand, and using the reduction formula for $\cos^2\theta$, we have

$$\begin{aligned}
&\int_0^{2\pi} \sin^4\theta\, d\theta = \frac{1}{4}\int_0^{2\pi} (1-2\cos 2\theta)\, d\theta + \frac{1}{4}\int_0^{2\pi} \cos^2 2\theta\, d\theta \\
&= \frac{1}{4}\left[\theta - \sin 2\theta\right]_0^{2\pi} + \frac{1}{8}\int_0^{2\pi} (1+\cos 4\theta)\, d\theta = \frac{\pi}{2} + \frac{1}{8}\left[\theta + \frac{1}{4}\sin 4\theta\right]_0^{2\pi} = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}.
\end{aligned}$$

2. c. To integrate $\int_0^1 \sqrt{1+x^2}\, dx$, let $x = \tan\theta$, $dx = \sec^2\theta\, d\theta$. Then,

$$\int_0^1 \sqrt{1+x^2}\, dx = \int_0^{\pi/4} \sqrt{1+\tan^2\theta}\,\sec^2\theta\, d\theta = \int_0^{\pi/4} \sec^3\theta\, d\theta.$$

3. d. The partial fraction decomposition, found using the cover-up method, is

$$\frac{1}{x(1-x)(1+x)} = \frac{1}{x} + \frac{1}{2(1-x)} - \frac{1}{2(1+x)}.$$

The integral then becomes

$$\int_a^b \frac{dx}{x(1-x)(1+x)} = \int_a^b \frac{dx}{x} + \frac{1}{2}\int_a^b \frac{dx}{1-x} - \frac{1}{2}\int_a^b \frac{dx}{1+x}.$$

Integrating and simplifying, we have

$$\begin{aligned}\int_a^b \frac{dx}{x(1-x)(1+x)} &= \ln\left(\frac{b}{a}\right) - \frac{1}{2}\ln\left(\frac{1-b}{1-a}\right) - \frac{1}{2}\ln\left(\frac{1+b}{1+a}\right)\\ &= \ln\left(\frac{b\sqrt{1-a}\sqrt{1+a}}{a\sqrt{1-b}\sqrt{1+b}}\right) = \ln\left(\frac{b\sqrt{1-a^2}}{a\sqrt{1-b^2}}\right)\end{aligned}$$

Solutions to the Problems for Lecture 31

1. To prove $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$, we start with the right-hand side and write

$$\begin{aligned}F_nF_{n+1} &= F_n(F_n + F_{n-1})\\ &= F_n^2 + F_{n-1}F_n\\ &= F_n^2 + F_{n-1}(F_{n-1} + F_{n-2})\\ &= F_n^2 + F_{n-1}^2 + F_{n-2}F_{n-1}\\ &= \dots\\ &= F_n^2 + F_{n-1}^2 + \cdots + F_2^2 + F_1F_2.\end{aligned}$$

Because $F_2 = F_1$, the identity is proved.

2. Let X be the value of the partial sum. We add two of the sums in reverse order:

$$\begin{array}{rcccccc} X = & 1 & + & 2 & + \dots + & n \\ X = & n & + & (n-1) & + \cdots + & 1 \\ \hline 2X = & (n+1) & + & (n+1) & + \cdots + & (n+1). \end{array}$$

We have found $2X = n(n+1)$, or

$$\sum_{k=1}^{n} k = \frac{1}{2}n(n+1).$$

3. We consider the identity

$$(k+1)^3 - k^3 = 3k^2 + 3k + 1,$$

and write it for each k from 1 to n and sum all of the equations. The left-hand side will be a telescoping series. We have

$$\begin{aligned} 2^3 - 1^3 &= 3 \cdot 1^2 + 3 \cdot 1 + 1 \\ 3^3 - 2^3 &= 3 \cdot 2^2 + 3 \cdot 2 + 1 \\ &\vdots \\ (n+1)^3 - n^3 &= 3 \cdot n^2 + 3 \cdot n + 1 \end{aligned}$$

$$\overline{(n+1)^3 - 1 = 3(1^2 + \cdots + n^2) + 3(1 + \cdots + n) + n}$$

The final equation can be solved for the sum of the squares:

$$\begin{aligned} \sum_{k=1}^{n} k^2 = \frac{1}{3}\left[(n+1)^3 - n - 1 - 3\sum_{k=1}^{n} k\right] &= \frac{1}{3}\left[n^3 + 3n^2 + 3n + 1 - n - 1 - \frac{3}{2}n^2 - \frac{3}{2}n\right] \\ &= \frac{1}{6}\left[2n^3 + 3n^2 + n\right] = \frac{1}{6}n(n+1)(2n+1). \end{aligned}$$

Solutions to the Problems for Lecture 32

1. We have

$$1 + \int_1^{\infty} \frac{1}{x^p}\,dx = 1 + \frac{x^{-p+1}}{-p+1} = 1 + \frac{1}{p-1} = \frac{p}{p-1}\,.$$

The inequalities are therefore

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.6449341 < \frac{2}{1} = 2 \qquad \sum_{n=1}^{\infty} \frac{1}{n^3} \approx 1.2020569 < \frac{3}{2} = 1.5$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \approx 1.0823232 < \frac{4}{3} = 1.3333333 \qquad \sum_{n=1}^{\infty} \frac{1}{n^5} \approx 1.0369278 < \frac{5}{4} = 1.25$$

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945} \approx 1.0173431 < \frac{6}{5} = 1.2 \qquad \sum_{n=1}^{\infty} \frac{1}{n^7} \approx 1.0083493 < \frac{7}{6} = 1.1666667\,.$$

Solutions to the Problems for Lecture 33

1. We divide pizzas among diners as follows:

a) Six pizzas among eight diners:

$$\frac{6}{8} = \frac{3}{4}, \qquad \frac{3}{4} - \frac{1}{2} = \frac{1}{4}.$$

Thus, $6/8 = 1/2 + 1/4$. Cut four pizzas into halves and give each diner one-half of a pizza, and cut two pizzas into quarters and give each diner a quarter of a pizza.

b) Seven pizzas among eight diners:

$$\frac{7}{8} - \frac{1}{2} = \frac{3}{8}, \qquad \frac{3}{8} - \frac{1}{4} = \frac{1}{8}.$$

Thus, $7/8 = 1/2 + 1/4 + 1/8$. Cut four pizzas into halves and give each diner one-half of a pizza, cut two pizzas into quarters and give each diner a quarter of a pizza, and cut the seventh pizza into eighths and give each diner one-eighth of a pizza.

2. The rearranged alternating harmonic series is

$$\begin{aligned}
1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \dots \\
&= (1 - \frac{1}{2}) - \frac{1}{4} + (\frac{1}{3} - \frac{1}{6}) - \frac{1}{8} + (\frac{1}{5} - \frac{1}{10}) - \frac{1}{12} + \dots \\
&= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \dots \\
&= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots\right).
\end{aligned}$$

This is half the value of the original alternating harmonic series.

Solutions to the Problems for Lecture 34

1.

$$0.333\dots = 0.3\sum_{n=0}^{\infty} 0.1^n = \frac{0.3}{1-0.1} = \frac{0.3}{0.9} = \frac{1}{3}.$$

2. Let B_n represent the drug concentration in the blood immediately after the n-th dose. The first few values are

$$\begin{aligned}
B_1 &= D, \\
B_2 &= \frac{1}{2}D + D = D\left(1 + \frac{1}{2}\right), \\
B_3 &= \frac{1}{2}\left(\frac{1}{2}D + D\right) + D = D\left(1 + \frac{1}{2} + \frac{1}{4}\right), \\
B_4 &= \frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2}D + D\right)\right) + D = D\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right).
\end{aligned}$$

Evidently, the general formula is

$$B_n = D\sum_{k=1}^{n} \frac{1}{2^k}.$$

At steady state, as $n \to \infty$, the geometric series converges and we obtain $B_\infty = 2D$. This is the maximum concentration in the blood, occurring immediately after taking the drug. The minimum concentration occurs 12 hours later, just before the next dose, and is half the maximum, or D.

To quickly reach the steady state, it is beneficial to take an initial dose of $2D$, followed by regular doses of D. This means doubling the first dose.

Solutions to the Practice quiz: Sequences and series

1. a. To find the sum, use the formula for the sum of squares and the formula for the sum of natural numbers:

$$\sum_{k=1}^{50} k^2 + 2\sum_{k=1}^{100} k = \frac{1}{6} \times 50 \times 51 \times 101 + 2 \times \frac{1}{2} \times 100 \times 101 = 53025.$$

2. c. As discussed in the video lecture, the p-Series converges only if $p > 1$.

3. b. For a dose D of a drug taken every H hours that also has a half-life of H hours, in the steady state the range of drug dosage in the body is between D and $2D$.

Solutions to the Problems for Lecture 35

1. We apply the ratio test to the series

$$\sum_{n=1}^{\infty} \frac{a^n n!}{n^n}.$$

We calculate the limit of the ratio of successive terms:

$$\begin{aligned}\lim_{n\to\infty} \frac{\left(\frac{a^{n+1}(n+1)!}{(n+1)^{n+1}}\right)}{\left(\frac{a^n n!}{n^n}\right)} &= \lim_{n\to\infty} \left(\frac{a^{n+1}}{a^n}\right)\left(\frac{(n+1)!}{n!}\right)\left(\frac{n^n}{(n+1)^{n+1}}\right)\\ &= \lim_{n\to\infty} a(n+1)\frac{n^n}{(n+1)^{n+1}} = a\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n.\end{aligned}$$

Now, we evaluate the remaining limit:

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \frac{1}{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}.$$

Therefore, the ratio test results in the value a/e, and the series converges if $0 < a < r$, with $r = e$.

Solutions to the Problems for Lecture 36

1. Let

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots .$$

a) The derivative of $f(x)$ is

$$f'(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = f(x).$$

b) Let

$$g(x) = e^{-x} f(x).$$

Taking the derivative using the product rule, we have

$$g'(x) = \left(e^{-x}\right)' f(x) + e^{-x} f'(x) = -e^{-x} f(x) + e^{-x} f(x) = 0.$$

Since $g'(x) = 0$ for all values of x, $g(x)$ is a constant function.

c) Evaluating $g(x)$ at $x = 0$, we have

$$g(0) = e^0 f(0) = 1.$$

Therefore, $g(x) = 1$ for all x.

d) We have shown that

$$e^{-x} f(x) = 1 \implies f(x) = e^x.$$

Thus,

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = e^x.$$

2. Let

$$f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots, \qquad g(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots.$$

a) The derivatives of $f(x)$ and $g(x)$ are

$$f'(x) = -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots = -g(x), \qquad g'(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = f(x).$$

b) Let

$$h(x) = (\cos x - f(x))^2 + (\sin x - g(x))^2.$$

Taking the derivative using the chain rule, we have

$$\begin{aligned} h'(x) &= 2(\cos x - f(x))\left(-\sin x - f'(x)\right) + 2(\sin x - g(x))\left(\cos x - g'(x)\right) \\ &= -2(\cos x - f(x))(\sin x - g(x)) + 2(\sin x - g(x))(\cos x - f(x)) = 0. \end{aligned}$$

Since $h'(x) = 0$ for all x, $h(x)$ is a constant function.

c) Evaluating $h(x)$ at $x = 0$, we have

$$h(0) = (1-1)^2 + (0-0)^2 = 0.$$

Therefore, $h(x) = 0$ for all x.

d) We have shown that

$$(\cos x - f(x))^2 + (\sin x - g(x))^2 = 0.$$

Since the sum of two squares is zero, each squared term must be zero. Therefore, $f(x) = \cos x$ and $g(x) = \sin x$, and we have shown that

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \cos x, \qquad x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sin x.$$

3. Define the power series function

$$f(x) = \sum_{n=0}^{\infty} c_n x^n.$$

If $f(x) = 0$ for all values of x, then all derivatives of $f(x)$ must also be zero. In particular, the value of the function and all its derivatives are zero at $x = 0$. We find

$$f(0) = c_0 = 0, \quad f'(0) = c_1 = 0, \quad f''(0) = 2c_2 = 0, \quad f'''(0) = 6c_3 = 0, \quad \text{and so on.}$$

By taking the nth derivative of $f(x)$ and setting $x = 0$, we find that $c_n = 0$. Therefore, if $f(x) = 0$ for all x, then $c_n = 0$ for all n.

Solutions to the Practice quiz: Power series

1. b. The correct statement is $g'(x) = -h(x)$.

2. b. If $\sum_{n=1}^{\infty} a_n$ converges, then the terms in the series must go to zero, that is, $\lim_{n\to\infty} a_n = 0$. Therefore, A must be true. Furthermore, the ratio test says that the series diverges if $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$. Therefore C must be true. For finite k, there is no particular requirement on the relative values of a_{k+1} and a_k. Therefore, B is not necessarily true.

3. d. The power series equation $\sum_{n=0}^{\infty} c_n x^n = 0$ results in $c_n = 0$ for all n. Therefore, if

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + a_n \right] x^n = 0,$$

then

$$(n+2)(n+1)a_{n+2} + a_n = 0,$$

or

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)}.$$

Solutions to the Problems for Lecture 37

1.

a) Using $\dfrac{1}{1+\epsilon} \approx 1 - \epsilon$, and replacing ϵ by $-\epsilon$, we obtain

$$\frac{1}{1-\epsilon} \approx 1 + \epsilon.$$

b) Let $f(x) = \sqrt{x}$. Then $f'(x) = 1/(2\sqrt{x})$. The linear approximation is

$$\sqrt{1+\epsilon} = f(1+\epsilon) \approx f(1) + f'(1)\epsilon = 1 + \frac{1}{2}\epsilon.$$

c) Let $f(x) = 1/\sqrt{x}$. Then $f'(x) = -1/(2x^{3/2})$. The linear approximation is

$$\frac{1}{\sqrt{1+\epsilon}} = f(1+\epsilon) \approx f(1) + f'(1)\epsilon = 1 - \frac{1}{2}\epsilon.$$

d) Let $f(x) = \ln x$. Then $f'(x) = 1/x$. The linear approximation is

$$\ln(1+\epsilon) = f(1+\epsilon) \approx f(1) + f'(1)\epsilon = \epsilon.$$

Solutions to the Problems for Lecture 38

1. Assume $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$, and that $\lim_{x\to a} f(x)/g(x) = L$ exists. By considering the reciprocals and applying L'Hospital's rule, we have

$$L = \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{\frac{1}{g(x)}}{\frac{1}{f(x)}} = \lim_{x\to a} \frac{\frac{-g'(x)}{g(x)^2}}{\frac{-f'(x)}{f(x)^2}} = \lim_{x\to a} \frac{g'(x)f(x)^2}{f'(x)g(x)^2} = L^2 \lim_{x\to a} \frac{g'(x)}{f'(x)}.$$

Solving for L, we find

$$L = \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

This confirms that L'Hospital's rule also applies to the indeterminate form ∞/∞.

2. We use L'Hospital's rule to compute the following limits:

a) $\lim_{x\to 0} \dfrac{x}{\tan x}$

Direct substitution of $x = 0$ results in $0/0$. Applying L'Hospital's rule,

$$\lim_{x\to 0} \frac{x}{\tan x} = \lim_{x\to 0} \frac{1}{1/\cos^2 x} = \lim_{x\to 0} \cos^2 x = 1;$$

b) $\lim_{x\to 1} \dfrac{\ln x}{x-1}$

Direct substitution of $x = 1$ results in $0/0$. Applying L'Hospital's rule,

$$\lim_{x\to 1} \frac{\ln x}{x-1} = \lim_{x\to 1} \frac{1/x}{1} = \lim_{x\to 1} \frac{1}{x} = 1;$$

c) $\lim_{x\to 0} x \ln x$

Direct substitution of $x = 0$ results in $0 \times \infty$. We turn this into an ∞/∞ and apply L'Hospital's rule,

$$\lim_{x\to 0} x \ln x = \lim_{x\to 0} \ln x/(1/x) = \lim_{x\to 0} \frac{1/x}{-1/x^2} = \lim_{x\to 0}(-x) = 0.$$

3. The limit

$$\lim_{x\to\infty} \frac{x^n}{e^x}$$

is of the indeterminate form ∞/∞. We can apply L'Hospital's rule repeatedly, assuming n is a positive integer. We have

$$\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{nx^{n-1}}{e^x} = \lim_{x\to\infty} \frac{n(n-1)x^{n-2}}{e^x} = \cdots = \lim_{x\to\infty} \frac{n!}{e^x}.$$

After differentiating the numerator and denominator n times, the numerator becomes a constant $n!$, while the denominator remains the exponential function. Now taking the limit, we find

$$\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{n!}{e^x} = 0.$$

This shows that the exponential function e^x grows faster than any power law x^n.

4. The limit

$$\lim_{x\to\infty} \frac{\ln x}{x^n}, \quad n > 0$$

is of the indeterminate form ∞/∞, so we can apply L'Hospital's rule. Differentiating the numerator and denominator, we have

$$\lim_{x\to\infty} \frac{\ln x}{x^n} = \lim_{x\to\infty} \frac{1/x}{nx^{n-1}} = \lim_{x\to\infty} \frac{1}{nx^n} = 0.$$

This proves that the logarithm function $\ln x$ grows slower than any power law x^n, where $n > 0$.

Solutions to the Problems for Lecture 39

1. We differentiate the Taylor series for $\exp(x)$, $\sin x$, and $\cos x$.

a)

$$\frac{d}{dx}\exp(x) = \frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \exp(x);$$

b)

$$\frac{d}{dx}\sin x = \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots = \cos x;$$

c)

$$\begin{aligned}\frac{d}{dx}\cos x = \frac{d}{dx}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right) &= -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \dots \\ &= -\left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right) = -\sin x;\end{aligned}$$

2.

a) $$\lim_{h\to 0}\frac{e^h - 1}{h} = \lim_{h\to 0}\frac{(1 + h + \frac{h^2}{2} + \dots) - 1}{h}$$
$$= \lim_{h\to 0}\frac{h(1 + \frac{h}{2} + \dots)}{h} = \lim_{h\to 0}(1 + \frac{h}{2} + \dots) = 1;$$

b) $$\lim_{h\to 0}\frac{\sin h}{h} = \lim_{h\to 0}\frac{(h - \frac{h^3}{3!} + \dots)}{h}$$
$$= \lim_{h\to 0}\frac{h(1 - \frac{h^2}{3!} + \dots)}{h} = \lim_{h\to 0}(1 - \frac{h^2}{3!} + \dots) = 1;$$

c) $$\lim_{h\to 0}\frac{\cos h - 1}{h} = \lim_{h\to 0}\frac{(1 - \frac{h^2}{2!} + \dots) - 1}{h}$$
$$= \lim_{h\to 0}\frac{-\frac{h^2}{2!} + \dots}{h} = \lim_{h\to 0}(-\frac{h}{2!} + \dots) = 0;$$

3. We verify the addition formulas for sine and cosine using the Taylor series.

a)

$$\begin{aligned}
&\sin x \cos y + \cos x \sin y \\
&= \left(x - \frac{x^3}{3!} + \dots\right)\left(1 - \frac{y^2}{2!} + \dots\right) + \left(1 - \frac{x^2}{2!} + \dots\right)\left(y - \frac{y^3}{3!} + \dots\right) \\
&= (x+y) - \left(\frac{x^3}{3!} + \frac{x^2 y}{2!} + \frac{x y^2}{2!} + \frac{y^3}{3!}\right) + \dots \\
&= (x+y) - \frac{1}{3!}\left(x^3 + 3x^2 y + 3xy^2 + y^3\right) + \dots \\
&= (x+y) - \frac{(x+y)^3}{3!} + \dots = \sin(x+y).
\end{aligned}$$

b)

$$\begin{aligned}
&\cos x \cos y - \sin x \sin y \\
&= \left(1 - \frac{x^2}{2!} + \dots\right)\left(1 - \frac{y^2}{2!} + \dots\right) - \left(x - \frac{x^3}{3!} + \dots\right)\left(y - \frac{y^3}{3!} + \dots\right) \\
&= 1 - \left(\frac{x^2}{2!} + \frac{y^2}{2!} + xy\right) + \dots = 1 - \frac{1}{2!}\left(x^2 + 2xy + y^2\right) + \dots \\
&= 1 - \frac{(x+y)^2}{2!} + \dots = \cos(x+y).
\end{aligned}$$

4. We use

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

and

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots .$$

a)

$$\begin{aligned}
&\cosh x = \frac{e^x + e^{-x}}{2} \\
&= \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots\right) + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \dots\right)\right] \\
&= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots ;
\end{aligned}$$

$$\begin{aligned}
&\sinh x = \frac{e^x - e^{-x}}{2} \\
&= \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \dots\right)\right] \\
&= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots .
\end{aligned}$$

b)

$$\frac{d}{dx}\sinh x = \frac{d}{dx}\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \cosh x;$$

$$\frac{d}{dx}\cosh x = \frac{d}{dx}\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots\right) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sinh x.$$

Solutions to the Problems for Lecture 40

1. We calculate the function and its derivatives at $x = 0$:

$$\begin{aligned}
f(x) &= (1+x)^n, & f(0) &= 1;\\
f'(x) &= n(1+x)^{n-1}, & f'(0) &= n;\\
f''(x) &= n(n-1)(1+x)^{n-2}, & f''(0) &= n(n-1);\\
f'''(x) &= n(n-1)(n-2)(1+x)^{n-3}, & f''(0) &= n(n-1)(n-2);\\
&\vdots & &\vdots\\
f^{(n)}(x) &= n!\,, & f^{(n)}(0) &= n!\,.
\end{aligned}$$

All higher-order derivatives are zero. The Taylor series expansion is then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + \frac{n!}{n!}x^n.$$

If we define the binomial coefficient as

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}\,,$$

then the binomial theorem is written succinctly as

$$(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k.$$

2. We will find the Taylor series for $\tan x$ using two methods.

First method

We calculate the value of $f(x) = \tan x$ and its derivatives at $x = 0$. We use the derivatives of the trigonometric functions found in Lecture 20.

$$\begin{aligned}
f(x) &= \tan x, & f(0) &= 0;\\
f'(x) &= \sec^2 x, & f'(0) &= 1;\\
f''(x) &= 2\tan x \sec^2 x, & f''(0) &= 0;\\
f'''(x) &= 4\tan^2 x \sec^2 x + 2\sec^4 x, & f'''(0) &= 2;\\
f^{(4)}(x) &= 8\tan^3 x \sec^2 x + 16\tan x \sec^4 x\,, & f^{(4)}(0) &= 0;\\
f^{(5)}(x) &= 24\tan^2 x \sec^4 x + 16\tan^4 x \sec^2 x + 16\sec^6 x + 64\tan^2 x \sec^4 x\,, & f^{(5)}(0) &= 16.
\end{aligned}$$

The Taylor series is then

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots .$$

Second method

We write $\tan x = \sin x / \cos x$, expand sine and cosine using their Taylor series, and then raise the denominator to the numerator using the series expansion for $1/(1-x)$. The goal is to keep all terms up to x^5.

$$\begin{aligned}
\tan x = \frac{\sin x}{\cos x} &= \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots}{1 - \left(\frac{x^2}{2!} - \frac{x^4}{4!} + \dots\right)} \\
&= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)\left(1 + \left(\frac{x^2}{2!} - \frac{x^4}{4!} + \dots\right) + \left(\frac{x^2}{2!} - \frac{x^4}{4!} + \dots\right)^2 + \dots\right) \\
&= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)\left(1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^4}{4} + \dots\right) \\
&= \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \dots\right)\left(1 + \frac{x^2}{2} + \frac{5x^4}{24} + \dots\right) \\
&= x + \left(\frac{1}{2} - \frac{1}{6}\right)x^3 + \left(\frac{5}{24} - \frac{1}{12} + \frac{1}{120}\right)x^5 + \dots = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots .
\end{aligned}$$

Both methods yield the same first three non-zero terms in the Taylor series for $\tan x$.

3. Starting with the Taylor series for the exponential function,

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots,$$

we substitute $(-x^2)$ for x to obtain

$$\exp(-x^2) = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \dots .$$

Now we integrate term-by-term from zero to one, to find

$$\begin{aligned}
\int_0^1 \exp(-x^2)dx &= \int_0^1 \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} + \dots\right) dx \\
&= \left[x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \frac{x^9}{9\cdot 4!} - \dots\right]_0^1 = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \dots .
\end{aligned}$$

These first five terms sum to 0.7475. A more accurate solution is 0.746824.

Solutions to the Practice quiz: Taylor series

1. c. We use the Taylor series approximation, $f(x+\epsilon) \approx f(x) + f'(x)\epsilon$, with $f(x) = \sqrt[3]{1+\epsilon} = x^{1/3}$ and $x = 1$. The derivative is $f'(x) = \frac{1}{3}x^{-2/3}$. Substituting these, we get for our linear approximation,

$$\sqrt[3]{1+\epsilon} \approx 1 + \frac{1}{3}\epsilon.$$

2. d. Direct substitution of $x = 0$ results in a $0/0$ indeterminate form. We can apply L'Hospital's rule until the form is no longer indeterminate. We have

$$\begin{aligned}\lim_{x\to 0} \frac{3\sin x - \sin 3x}{x - \sin x} = \lim_{x\to 0} \frac{3\cos x - 3\cos 3x}{1 - \cos x} &= \lim_{x\to 0} \frac{-3\sin x + 9\sin 3x}{\sin x} \\ &= \lim_{x\to 0} \frac{-3\cos x + 27\cos 3x}{\cos x} = -3 + 27 = 24.\end{aligned}$$

3. d. By differentiating the power series, we observe that all of the statements are true.

4. c. We compute the first three derivatives of $f(x) = (1+x)^{1/2}$:

$$\begin{aligned} f'(x) &= \frac{1}{2}(1+x)^{-1/2}, & f''(x) &= -\frac{1}{4}(1+x)^{-3/2}, & f'''(x) &= \frac{3}{8}(1+x)^{-5/2}; \\ f'(0) &= \frac{1}{2}, & f''(0) &= -\frac{1}{4}, & f'''(0) &= \frac{3}{8}. \end{aligned}$$

Thus, the first four terms in the Taylor series are

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{4}\frac{x^2}{2!} + \frac{3}{8}\frac{x^3}{3!} \dots .$$

The fourth term is $\frac{3}{8}\frac{x^3}{3!} = \frac{x^3}{16}$.

Solutions to the Problems for Lecture 41

1. Let

$$g(x) = e^{-ix}\left(\cos x + i\sin x\right).$$

a) Taking the derivative using the product rule and the chain rule, we have

$$\begin{aligned} g'(x) &= -ie^{-ix}\left(\cos x + i\sin x\right) + e^{-ix}\left(-\sin x + i\cos x\right) \\ &= e^{-ix}\left(-i\cos x + \sin x - \sin x + i\cos x\right) = 0. \end{aligned}$$

Since $g'(x) = 0$, $g(x)$ is a constant function.

b) Evaluating $g(x)$ at $x = 0$, we have

$$g(0) = e^0\left(\cos 0 + i\sin 0\right) = 1.$$

Therefore, $g(x) = 1$ for all values of x.

c) We have shown that

$$e^{-ix}(\cos x + i\sin x) = 1 \implies \cos x + i\sin x = e^{ix}.$$

2. To integrate

$$\int_0^{2\pi} \cos^4 x\, dx,$$

we first express $\cos x$ as

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

to obtain

$$\int_0^{2\pi} \cos^4 x\, dx = \frac{1}{16}\int_0^{2\pi} \left(e^{ix} + e^{-ix}\right)^4 dx.$$

Applying the binomial expansion, $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$, we have

$$\int_0^{2\pi} \cos^4 x\, dx = \frac{1}{16}\int_0^{2\pi} \left(e^{4ix} + 4e^{2ix} + 6 + 4e^{-2ix} + e^{-4ix}\right) dx\,.$$

All terms in the definite integral evaluate to zero except for the constant term. Thus,

$$\int_0^{2\pi} \cos^4 x\, dx = \frac{6}{16}\int_0^{2\pi} dx = \frac{12\pi}{16} = \frac{3\pi}{4}.$$

3. To prove the identity, $\cos^2 x + \sin^2 x = 1$, we calculate

$$\cos^2 x + \sin^2 x = \left(\frac{e^{ix} + e^{-ix}}{2}\right)^2 + \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^2.$$

Simplifying, we have

$$\begin{aligned}\cos^2 x + \sin^2 x &= \frac{1}{4}\left[e^{2ix} + 2 + e^{-2ix} + \frac{e^{2ix} + e^{-2ix} - 2}{i^2}\right] \\ &= \frac{1}{4}\left[e^{2ix} + 2 + e^{-2ix} - e^{2ix} - e^{-2ix} + 2\right] = 1.\end{aligned}$$

4.

a) We verify the relation for $\cosh x$:

$$\cos ix = \frac{e^{i(ix)} + e^{-i(ix)}}{2} = \frac{e^{-x} + e^{x}}{2} = \frac{e^{x} + e^{-x}}{2} = \cosh x.$$

We verify the relation for $\sinh x$:

$$-i\sin ix = -i\left(\frac{e^{i(ix)} - e^{-i(ix)}}{2i}\right) = -\left(\frac{e^{-x} - e^{x}}{2}\right) = \frac{e^{x} - e^{-x}}{2} = \sinh x.$$

b) We prove $e^x = \cosh x + \sinh x$:

$$\cosh x + \sinh x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = \frac{1}{2}\left[e^x + e^{-x} + e^x - e^{-x}\right] = e^x.$$

c) We prove $\cosh^2 x - \sinh^2 x = 1$:

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 + \left(\frac{e^x - e^{-x}}{2}\right)^2.$$

Simplifying, we get

$$\cosh^2 x - \sinh^2 x = \frac{1}{4}\left[\left(e^{2x} + e^{-2x} + 2\right) - \left(e^2 x + e^{-2x} - 2\right)\right] = 1.$$

Solutions to the Problems for Lecture 42

1. The polar form of $\alpha + i\beta$ is given by

$$\alpha + i\beta = \sqrt{\alpha^2 + \beta^2} e^{i\phi}, \quad \text{where} \quad \tan\phi = \beta/\alpha.$$

Then, the complex number z is given by

$$z = \frac{\alpha + i\beta}{\alpha^2 + \beta^2} e^{i\gamma} = \frac{\sqrt{\alpha^2 + \beta^2}}{\alpha^2 + \beta^2} e^{i\phi} e^{i\gamma} = \frac{1}{\sqrt{\alpha^2 + \beta^2}} e^{i(\gamma + \phi)}.$$

Therefore, the real part of z simplifies to

$$\operatorname{Re} z = \frac{1}{\sqrt{\alpha^2 + \beta^2}} \cos(\gamma + \phi), \quad \text{with} \quad \tan\phi = \beta/\alpha\,.$$

2.

a) Assuming k is an integer, and using the periodicity of sine and cosine with a period of 2π, we have

$$e^{i(\theta + 2\pi k)} = \cos(\theta + 2\pi k) + i\sin(\theta + 2\pi k) = \cos\theta + i\sin\theta = e^{i\theta}.$$

b) To find the three cube roots of unity, we write

$$1 = e^0, \qquad 1 = e^{i2\pi}, \qquad 1 = e^{-i2\pi}.$$

Raising to the one-third power, the three complex cube roots of unity are

$$1, \qquad e^{i2\pi/3} = \cos(2\pi/3) + i\sin(2\pi/3) = -\frac{1}{2} + i\frac{\sqrt{3}}{2},$$
$$e^{-i2\pi/3} = \cos(2\pi/3) - i\sin(2\pi/3) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

Solutions to the Problems for Lecture 43

1. The addition formulas that we need are

$$\cos\big((m+n)x\big) = \cos mx \cos nx - \sin mx \sin nx,$$
$$\cos\big((m-n)x\big) = \cos mx \cos nx + \sin mx \sin nx.$$

Using these formulas, we have

$$\cos mx \cos nx = \frac{1}{2}\left[\cos(m-n)x + \cos(m+n)x\right],$$
$$\sin mx \sin nx = \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right].$$

The integrals for $m \neq n$ become

$$\begin{aligned}\int_{-\pi}^{\pi} \cos mx \cos nx\, dx &= \frac{1}{2}\int_{-\pi}^{\pi} \left[\cos(m-n)x + \cos(m+n)x\right]\, dx \\ &= \frac{1}{2}\left[\frac{1}{m-n}\sin\big((m-n)x\big) + \frac{1}{m+n}\sin\big((m+n)x\big)\right]_{-\pi}^{\pi} = 0;\end{aligned}$$

$$\begin{aligned}\int_{-\pi}^{\pi} \sin mx \sin nx\, dx &= \frac{1}{2}\int_{-\pi}^{\pi} \left[\cos(m-n)x - \cos(m+n)x\right]\, dx \\ &= \frac{1}{2}\left[\frac{1}{m-n}\sin\big((m-n)x\big) - \frac{1}{m+n}\sin\big((m+n)x\big)\right]_{-\pi}^{\pi} = 0.\end{aligned}$$

For $m = n$, however, we have

$$\int_{-\pi}^{\pi} \cos^2(nx)dx = \frac{1}{2}\int_{-\pi}^{\pi}\big(1 + \cos(2nx)\big)dx = \frac{1}{2}\left[x + \frac{1}{2n}\sin 2nx\right]_{-\pi}^{\pi} = \pi;$$

$$\int_{-\pi}^{\pi} \sin^2(nx)dx = \frac{1}{2}\int_{-\pi}^{\pi}\big(1 - \cos(2nx)\big)dx = \frac{1}{2}\left[x - \frac{1}{2n}\sin 2nx\right]_{-\pi}^{\pi} = \pi.$$

2. To prove these hyperbolic trigonometric identities, we will use the relationships between the hyperbolic trigonometric and the trigonometric functions,

$$\cosh x = \cos ix, \qquad \sinh x = -i \sin ix,$$

and the trigonometric identities

$$\begin{aligned}
\cos(x+y) &= \cos x \cos y - \sin x \sin y, \\
\sin(x+y) &= \sin x \cos y + \cos x \sin y, \\
\cos 2x &= \cos^2 x - \sin^2 x, \\
\sin 2x &= 2 \cos x \sin x.
\end{aligned}$$

a) We prove $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$:

$$\begin{aligned}
\cosh(x+y) &= \cos(ix+iy) = \cos ix \cos iy - \sin ix \sin iy \\
&= \cos ix \cos iy + (-i \sin ix)(-i \sin iy) = \cosh x \cosh y + \sinh x \sinh y.
\end{aligned}$$

b) We prove $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$:

$$\begin{aligned}
\sinh(x+y) &= -i \sin(ix+iy) = -i(\sin ix \cos iy + \cos ix \sin iy) \\
&= (-i \sin ix) \cos iy + \cos ix(-i \sin iy) \\
&= \sinh x \cosh y + \cosh x \sinh y.
\end{aligned}$$

c) We prove $\cosh 2x = \cosh^2 x + \sinh^2 x$:

$$\cosh 2x = \cos 2ix = \cos^2 ix - \sin^2 ix = \cos^2 ix + (-i \sin ix)^2 = \cosh^2 x + \sinh^2 x.$$

d) We prove $\sinh 2x = 2 \cosh x \sinh x$:

$$\sinh 2x = -i \sin 2ix = -i(2 \cos ix \sin ix) = 2 \cos ix(-i \sin ix) = 2 \cosh x \sinh x.$$

Solutions to the Practice quiz: Complex exponential function

1. d. We use the complex exponential forms of $\sin x$ and $\cos x$ to compute

$$\sin x + i \cos x = \frac{e^{ix} - e^{-ix}}{2i} + i\left(\frac{e^{ix} + e^{-ix}}{2}\right) = \frac{i}{2}\left(-e^{ix} + e^{-ix} + e^{ix} + e^{-ix}\right) = ie^{-ix}.$$

2. b. We simplify by converting $(1+i)/2$ to polar form:

$$\left(\frac{1+i}{2}\right) e^{i\pi/4} = \frac{\sqrt{2}}{2} e^{i\pi/4} e^{i\pi/4} = \frac{\sqrt{2}}{2} e^{i\pi/2} = \frac{i\sqrt{2}}{2}.$$

3. b. Recall the definitions of the hyperbolic functions:

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}.$$

Thus,

$$\cosh x + \sinh x = \frac{1}{2}\left(e^x + e^{-x} + e^x - e^{-x}\right) = e^x.$$

It follows that

$$\cosh(x+y) + \sinh(x+y) = e^{x+y}.$$

Solutions to the Problems for Lecture 44

1. To compute the circumference, we calculate the arc length of the circle in the first quadrant and multiply by four. From the equation for the circle, we have

$$y = \sqrt{R^2 - x^2}, \qquad \frac{dy}{dx} = -\frac{x}{\sqrt{R^2 - x^2}}.$$

Then

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (dy/dx)^2}\, dx,$$

where

$$1 + (dy/dx)^2 = 1 + \frac{x^2}{R^2 - x^2} = \frac{R^2}{R^2 - x^2}.$$

Thus,

$$ds = \frac{R}{\sqrt{R^2 - x^2}}\, dx,$$

and the circumference of the circle is given by

$$C = 4\int_0^R \frac{R}{\sqrt{R^2 - x^2}}\, dx = 4R\int_0^1 \frac{1}{\sqrt{1 - \xi^2}}\, d\xi,$$

where to obtain the last integral we have used the substitution $\xi = x/R$. The last integral can be solved using a trigonometric substitution. Let

$$\xi = \sin\theta, \qquad d\xi = \cos\theta\, d\theta.$$

Then $1 - \xi^2 = 1 - \sin^2\theta = \cos^2\theta$. When $\xi = 0$, $\theta = 0$, and when $\xi = 1$, $\theta = \pi/2$. Therefore,

$$C = 4R\int_0^1 \frac{1}{\sqrt{1 - \xi^2}}\, d\xi = 4R\int_0^{\pi/2} d\theta = 2\pi R.$$

Solutions to the Problems for Lecture 45

1. To compute the area of an ellipse, we find the area in the first quadrant and multiply by four. In this first quadrant, y is given by

$$y = b\sqrt{1 - \frac{x^2}{a^2}}, \qquad 0 \le x \le a.$$

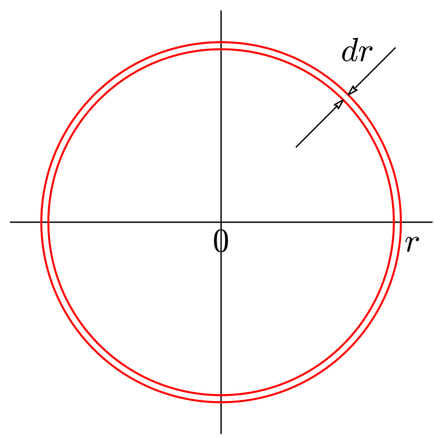
We have

$$A = 4\int_0^a y\,dx = 4b\int_0^a \sqrt{1 - \frac{x^2}{a^2}}\,dx.$$

We let $x = a\sin\theta$ so $dx = a\cos\theta d\theta$. When $x = 0$, $\theta = 0$, and when $x = a$, $\theta = \pi/2$. Therefore,

$$\begin{aligned}
A &= 4b\int_0^a \sqrt{1 - \frac{x^2}{a^2}}\,dx = 4ab\int_0^{\pi/2} \sqrt{1 - \sin^2\theta}\cos\theta\,d\theta \\
&= 4ab\int_0^{\pi/2} \cos^2\theta\,d\theta = 2ab\int_0^{\pi/2} (1 + \cos 2\theta)\,d\theta \\
&= 2ab\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \pi ab.
\end{aligned}$$

2. Consider the infinitesimal area dA of a ring formed by two concentric circles with radius r separated by a distance dr, as shown in the figure. Let the circumference of a circle of radius r be denoted by $C = C(r)$. Then the area of the infinitesimal ring is given by



$$dA = Cdr.$$

The circumference of the circle is then given by

$$C = \frac{dA}{dr} = \frac{d}{dr}\left(\pi r^2\right) = 2\pi r.$$

Solutions to the Problems for Lecture 46

1. The equation for a cone is

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = \left(\frac{z}{h}\right)^2.$$

We draw a circle on the cone parallel to the x-y plane, with radius ξ, centered at a value of z between zero and h. Comparing the ratio of the sides of two similar triangles, we have

$$\frac{z}{h} = \frac{\xi}{r},$$

so the radius of the circle is

$$\xi = \frac{rz}{h},$$

and the circumference is

$$C = \frac{2\pi rz}{h}.$$

We draw a second circle on the cone an infinitesimal distance dz above the first circle. The two circles are separated a constant distance ds along the surface of the cone. Where $y = 0$, the infinitesimals dx, dz, and ds form a right triangle with ds as the hypotenuse. Therefore,

$$ds = \sqrt{dz^2 + dx^2} = \sqrt{1 + (dx/dz)}\, dz.$$

Also at $y = 0$,

$$x = \pm\frac{rz}{h}, \quad \text{and} \quad \frac{dx}{dz} = \pm\frac{r}{h}.$$

Thus,

$$ds = \sqrt{1 + (dx/dz)}\, dz = \sqrt{1 + \frac{r^2}{h^2}}\, dz\,,$$

and the surface area of this infinitesimal ring on the surface of the cone is given by

$$dS = Cds = \frac{2\pi rz}{h}\sqrt{1 + \frac{r^2}{h^2}}\, dz\,.$$

Integrating from $z = 0$ to h, we find for the lateral surface area of the cone with base radius r and height h,

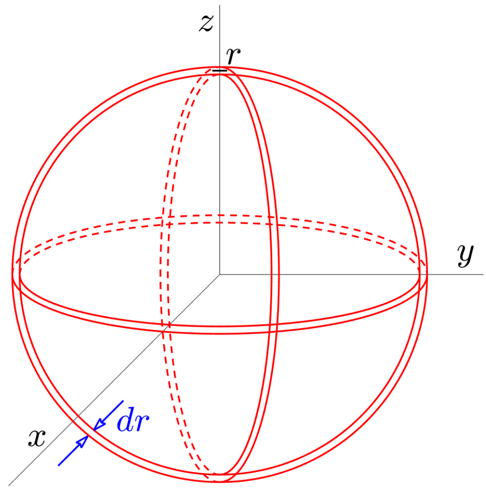
$$S_{\text{lat}} = \frac{2\pi r}{h}\sqrt{1 + \frac{r^2}{h^2}}\int_0^h z\, dz = \pi rh\sqrt{1 + \frac{r^2}{h^2}} = \pi r\sqrt{h^2 + r^2}.$$

If we also include the surface area of the base, given by πr^2, then the surface area of the entire cone is

$$S = \pi r\left(r + \sqrt{h^2 + r^2}\right).$$

Solutions to the Problems for Lecture 47

1. Consider the infinitesimal volume dV contained between two spherical shells with radius r and separated by a distance dr, as shown in the figure. Let the surface area of a sphere with radius r be denoted by $S = S(r)$. Then the volume contained within the two concentric shells is given by



$$dV = S dr.$$

The surface area of the sphere is then

$$S = \frac{dV}{dr} = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2.$$

2. The equation for a cone is

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = \left(\frac{z}{h}\right)^2.$$

We draw a circle on the cone parallel to the x-y plane, with radius ξ, centered at a height z between zero and h. Comparing the ratio of the sides of two similar triangles, we have

$$\frac{z}{h} = \frac{\xi}{r},$$

so the radius of the circle is

$$\xi = \frac{rz}{h},$$

and the area of the disk bordered by the circle is

$$A = \frac{\pi r^2 z^2}{h^2}.$$

We draw a second circle on the cone an infinitesimal distance dz above the first circle. The infinitesimal volume contained within the two disks bordered by the two circles is

$$dV = \frac{\pi r^2 z^2}{h^2}\, dz.$$

Integrating dV from $z = 0$ to h, we obtain

$$V = \int dV = \frac{\pi r^2}{h^2}\int_0^h z^2\, dz = \frac{\pi r^2 h}{3}.$$

Solutions to the Practice quiz: Areas and volumes

1. d. The arc length infinitesimal is given by

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + (dy/dx)^2}\, dx = \sqrt{1 + (2x)^2}\, dx.$$

Thus, the arc length is

$$\int_0^1 \sqrt{1 + 4x^2}\, dx.$$

2. b. To differentiate $A = 4\int_0^r \sqrt{r^2 - x^2}\, dx$ with respect to r, we could first perform the integral with respect to x, but instead we choose to differentiate directly. We make use of the following result from the problems of Lecture 23:

$$\frac{d}{dx}\left[\int_a^x f(x)\, dy\right] = f(x) + \int_a^x f'(x)\, dy.$$

We have

$$\frac{d}{dr}\left[4\int_0^r \sqrt{r^2 - x^2}\, dx\right] = 0 + 4\int_0^r \frac{d}{dr}\left[\sqrt{r^2 + x^2}\right] dx = 4\int_0^r \frac{r}{\sqrt{r^2 - x^2}} dx.$$

The final integral is done with the substitution $x = r\sin\theta$, $dx = r\cos\theta\, d\theta$; and the limits become 0 to $\pi/2$. Thus,

$$\frac{d}{dr}\left[4\int_0^r \sqrt{r^2 - x^2}\, dx\right] = 4\int_0^{\pi/2} r d\theta = 2\pi r.$$

Note that the integral represents the area of a circle of radius r.

3. d. The sphere satisfies the equation $x^2 + y^2 + z^2 = R^2$. The radius of the circle at fixed z is $\sqrt{x^2 + y^2} = \sqrt{R^2 - z^2}$. Thus the circumference is $C = 2\pi\sqrt{R^2 - z^2}$.

4. c. For a sphere of radius r, the surface area is $S(r) = 4\pi r^2$ and the volume is $V(r) = \frac{4}{3}\pi r^3$. The surface area is the derivative of the volume, and the volume is the integral of the surface area.

Solutions to the Problems for Lecture 48

1. We solve $f(x) = 0$, where $f(x) = x^2 - 3$. The derivative is $f'(x) = 2x$. Therefore, Newton's method iterates

$$x_{n+1} = x_n - \frac{x_n^2 - 3}{2x_n} = \frac{x_n^2 + 3}{2x_n}.$$

Starting with the initial guess $x_0 = 1$, we have

$$x_1 = \frac{(1)^2+3}{2\times 1} = 2,$$
$$x_2 = \frac{2^2+3}{2\times 2} = \frac{7}{4} = 1.75,$$
$$x_3 = \frac{(\frac{7}{4})^2+3}{2\times(\frac{7}{4})} = \frac{97}{56} = 1.73214,$$

which is already accurate to four significant digits.

Solutions to the Problems for Lecture 49

1. We take $x_0 = 1$, $x_{10} = 2$, and $x_1, x_2, \ldots, x_9 = 1.1, 1.2, \ldots, 1.9$. Applying the trapezoidal rule with $h = 0.1$, we have

$$\int_1^2 \frac{dx}{x} \approx \frac{0.1}{2}(\frac{1}{1}+\frac{1}{2}) + 0.1(\frac{1}{1.1}+\frac{1}{1.2}+\cdots+\frac{1}{1.9}) = 0.075 + 0.6187714 = 0.6937714,$$

which is accurate to three decimal places.

2. Apply the trapezoidal rule to each pair of neighboring points. The rule approximates the integral as

$$\int_a^b f(x)dx \approx \frac{1}{2}\sum_{k=1}^{n}(x_k - x_{k-1})(f_{k-1}+f_k).$$

To identify a pattern, expand and simplify the first few terms. We have

$$\begin{aligned}\int_a^b f(x)dx &\approx \frac{1}{2}\Big[(x_1-x_0)(f_0+f_1) + (x_2-x_1)(f_1+f_2) + (x_3-x_2)(f_2+f_3) + \ldots\Big]\\ &= \frac{1}{2}\Big[(x_1-x_0)f_0 + (x_1-x_0+x_2-x_1)f_1 + (x_2-x_1+x_3-x_2)f_2 + \ldots\Big]\\ &= \frac{1}{2}\Big[(x_1-x_0)f_0 + (x_2-x_0)f_1 + (x_3-x_1)f_2 + \ldots\Big].\end{aligned}$$

The pattern continues until the last term f_n, where there is no further cancellation. Extending the sum to include the last two terms gives the trapezoidal rule approximation,

$$\int_a^b f(x)dx \approx \frac{1}{2}\Big[(x_1-x_0)f_0 + (x_2-x_0)f_1 + \cdots + (x_n-x_{n-2})f_{n-1} + (x_n-x_{n-1})f_n\Big].$$

Solutions to the Problems for Lecture 50

1. Divide the domain $0 \le x \le 1$ into a grid with n intervals and $n+1$ grid points, with a grid spacing $h = 1/n$. The grid points are

$$x_k = kh, \qquad n = 0, 1, 2, \ldots, n.$$

The centered-difference formula for the second derivative of y at an internal grid point is

$$y_k'' = \frac{y_{k+1} - 2y_k + y_{k-1}}{h^2}.$$

Substituting into the differential equation $y'' = 0$, we get

$$y_{k+1} - 2y_k + y_{k-1} = 0 \qquad \text{or} \qquad y_k = \frac{y_{k+1} + y_{k-1}}{2}.$$

Hence, the value of y at any internal grid point is equal to the average of y at its two neighboring grid points

Solutions to the Practice quiz: Numerical methods

1. c. To find $\sqrt{5}$, solve $f(x) = 0$ where $f(x) = x^2 - 5$. The derivative is $f'(x) = 2x$. Using Newton's method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 5}{2x_n} = \frac{x_n^2 + 5}{2x_n}.$$

Starting with $x_0 = 2$, we have

$$x_1 = \frac{4+5}{4} = \frac{9}{4}, \qquad x_2 = \frac{\frac{81}{16} + 5}{\frac{9}{2}} = \frac{161}{72}.$$

2. b. The trapezoidal rule with $h = 0.5$ uses points $x_0 = 0$, $x_1 = 0.5$, and $x_2 = 1$. The approximation is

$$\int_0^1 \exp\left(-x^2\right) dx \approx (0.25)(1 + e^{-1}) + (0.5)(e^{-0.25}) \approx 0.34197 + 0.38940 \approx 0.7314\,.$$

3. c. We need only test the monomials 1, x, x^2, and x^3 and apply the addition and constant multiple rule.

- $f(x) = 1$:

$$f'(x) = \frac{1-1}{2h} = 0.$$

- $f(x) = x$:

$$f'(x) = \frac{(x+h) - (x-h)}{2h} = \frac{2h}{2h} = 1.$$

- $f(x) = x^2$:

$$f'(x) = \frac{(x+h)^2 - (x-h)^2}{2h} = \frac{(x^2 + 2hx + h^2) - (x^2 - 2hx + h^2)}{2h} = \frac{4hx}{2h} = 2x.$$

- $f(x) = x^3$:

$$
\begin{aligned}
f'(x) = \frac{(x+h)^3 - (x-h)^3}{2h} &= \frac{(x^3 + 3hx^2 + 3h^2x + h^3) - (x^3 - 3hx^2 + 3h^2x - h^3)}{2h} \\
&= \frac{6hx^2 + 2h^3}{2h} = 3x^2 + h^2.
\end{aligned}
$$

The approximation is exact for polynomials of degree two or lower. It is not exact when $f(x) = ax^3 + bx^2 + cx + d$.

Solutions to the Problems for Lecture 51

1. A cubic polynomial extends to positive infinity at one end, and negative infinity at the other. It can either have both a local maximum and a local minimum, or have none. To find the critical points of a general cubic polynomial, we take its derivative and set it to zero:

$$\frac{d}{dx}\left(ax^3 + bx^2 + cx + d\right) = 3ax^2 + 2bx + c = 0.$$

The roots of this quadratic equation are given by

$$x_{\pm} = \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}.$$

There are no real solutions, indicating no local maxima or minima, when

$$b^2 - 3ac < 0.$$

In the boundary case where $b^2 - 3ac = 0$, there is one critical point at $x = -b/3a$. This point cannot be an isolated local maximum or minimum. At this critical point, the second derivative of the cubic polynomial is zero and changes sign, indicating an inflection point of the function.

2. We are considering the cubic polynomial given by

$$f(x) = x^3 - px + q, \qquad \text{with} \quad p > 0.$$

a) The critical points are found by solving $f'(x) = 0$. Taking the derivative, we have

$$3x^2 - p = 0.$$

This gives us the critical points

$$x = \pm\sqrt{p/3}.$$

b) The values of $f(x)$ at the critical points are

$$f(\pm\sqrt{p/3}) = (\pm\sqrt{p/3})^3 \mp p\sqrt{p/3} + q = \pm\frac{2p^{3/2}}{3\sqrt{3}} + q.$$

c) The polynomial will have three distinct real roots if the function values at the local maximum and minimum are on opposite sides of the x-axis. We can express this condition as

$$f(-\sqrt{p/3})f(\sqrt{p/3}) < 0.$$

Substituting the values we found, this becomes

$$\left(\frac{2p^{3/2}}{3\sqrt{3}} + q\right)\left(-\frac{2p^{3/2}}{3\sqrt{3}} + q\right) < 0;$$

Multiplying this out, we have

$$-\frac{4p^3}{27} + q^2 < 0.$$

Finally, we obtain the condition for three distinct real roots:

$$4p^3 - 27q^2 > 0.$$

Solutions to the Problems for Lecture 52

1. Let the side of the rectangle parallel to the barn have length y, and the other two sides have length x. The formula for the total length of fencing L is

$$L = 2x + y;$$

and the formula for the area of the enclosure is

$$A = xy.$$

To maximize the area A while keeping the total length of fencing L fixed, we solve the L equation for y and substitute it into the area equation to express A as a function of x alone:

$$A(x) = x\,(L - 2x) = Lx - 2x^2.$$

To find the maximum area, we take the derivative of A with respect to x, and set it equal to zero:

$$A'(x) = L - 4x = 0.$$

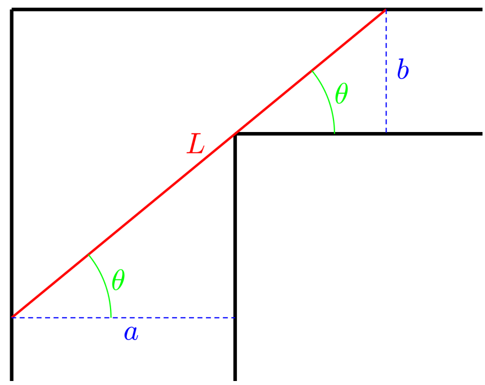
Solving for x, and then for y, we find

$$x = \frac{1}{4}L, \qquad y = \frac{1}{2}L.$$

We observe that $y = 2x$, so that the single fenced side is twice as long as each of the parallel fenced sides.

Solutions to the Problems for Lecture 53

1. The figure shows a top view of the corner with the pipe wedged into the hallway. Using trigonometry, we express the pipe length, L, as a function of the angle θ:



$$L(\theta) = \frac{a}{\cos\theta} + \frac{b}{\sin\theta}.$$

To find the longest pipe that can be carried around the corner, we need to find the shortest pipe that can touch both outer walls. Accordingly, we minimize L with respect to θ. Taking the derivative, we obtain

$$\frac{dL}{d\theta} = \frac{a\sin\theta}{\cos^2\theta} - \frac{b\cos\theta}{\sin^2\theta} = 0.$$

Solving this equation results in

$$\tan^3\theta = \frac{b}{a} \quad\Longrightarrow\quad \tan\theta = \left(\frac{b}{a}\right)^{1/3} \quad\Longrightarrow\quad \theta = \arctan\left(\frac{b}{a}\right)^{1/3}.$$

Using the identities,

$$\cos(\arctan\xi) = \frac{1}{\sqrt{1+\xi^2}}, \qquad \sin(\arctan\xi) = \frac{\xi}{\sqrt{1+\xi^2}},$$

with $\xi = (b/a)^{1/3}$, we find

$$\begin{aligned} L &= a\sqrt{1+(b/a)^{2/3}} + \frac{b\sqrt{1+(b/a)^{2/3}}}{(b/a)^{1/3}} \\ &= a^{2/3}\sqrt{a^{2/3}+b^{2/3}} + b^{2/3}\sqrt{a^{2/3}+b^{2/3}} = \left(a^{2/3}+b^{2/3}\right)^{3/2}. \end{aligned}$$

We see that this solution is symmetric in the hallway widths, a and b, and has units of length. In the special case where both hallways are the same width, $a = b$, the maximum pipe length is $L = 2\sqrt{2}a \approx 2.8a$.

Solutions to the Problems for Lecture 54

1. Using time equal to distance divided by velocity, the total time T taken by the lifeguard to reach the swimmer is given by

$$T = \frac{\sqrt{x_1^2+y_1^2}}{u_1} + \frac{\sqrt{x_2^2+y_2^2}}{u_2},$$

where $x_2 = x - x_1$, and x is given. To minimize T, we first write T as a function of x_1,

$$T(x_1) = \frac{\sqrt{x_1^2 + y_1^2}}{u_1} + \frac{\sqrt{(x - x_1)^2 + y_2^2}}{u_2},$$

and then differentiate T with respect to x_1, setting the derivative to zero. We find

$$\frac{dT}{dx_1} = \frac{x_1}{u_1\sqrt{x_1^2 + y_1^2}} - \frac{x - x_1}{u_2\sqrt{(x - x_1)^2 + y_2^2}} = 0.$$

This implies that the minimum rescue time occurs when

$$\frac{x_1}{u_1\sqrt{x_1^2 + y_1^2}} = \frac{x_2}{u_2\sqrt{x_2^2 + y_2^2}}.$$

Examining the figure and the definitions of the angles θ_1 and θ_2, we find this is equivalent to

$$\frac{\sin\theta_1}{u_1} = \frac{\sin\theta_2}{u_2},$$

which resembles Snell's law in optics. Note that this equation only partially determines the entry point x_1. The lifeguard must also reach the swimmer, requiring a numerical root-finding method, such as Newton's method, to completely solve the problem.

Solutions to the Practice quiz: Local extrema

1. b. To find local maxima and minima, take the derivative and set it to zero. For $f(x) = x^3 - 3x + 2$, we have

$$f'(x) = 3x^2 - 3 = 0,$$

which has the solutions $x = \pm 1$. Since the cubic polynomial goes to negative infinity as $x \to -\infty$, the local maximum corresponds to the first root, that is, $x = -1$.

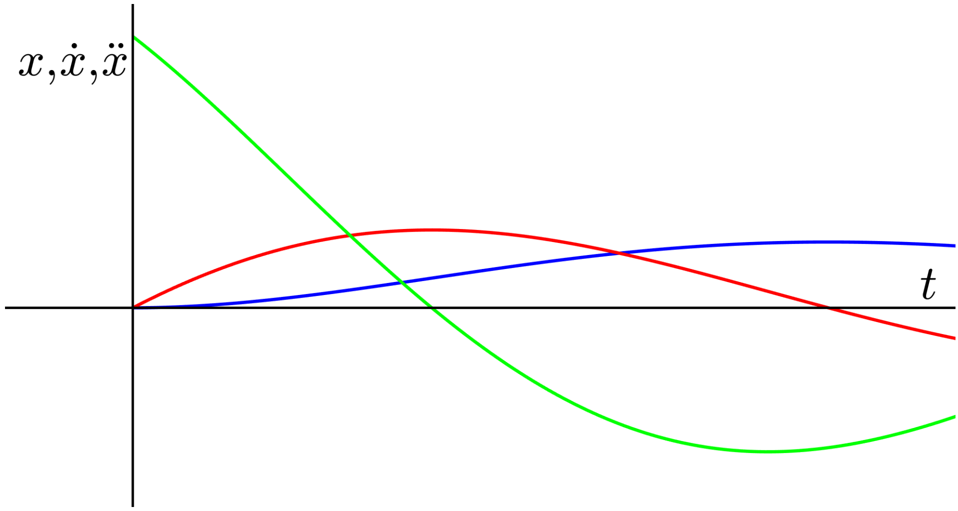
2. c. The fencing parallel to the barn should be twice as long as each of the sides perpendicular to the barn. Let x be the length of the fenced side perpendicular to the barn. Then the sum of the three side lengths is given by $x + x + 2x = 100\,\text{m}$, or $x = 25\,\text{m}$. So the length of the side parallel to the barn is $50\,\text{m}$.

3. a. The shortest path occurs when the angles between the river and the two walking paths are equal, making the distance along the river to the water collection point $L/2$ from each village. Thus, the shortest path is $2\sqrt{D^2 + \frac{1}{4}L^2}$.

4. c. Assuming the lifeguard runs four times faster than they can swim, they should run approximately $L - L/4 = 3L/4$, or three-quarters of the way to the point on the shore closest to the swimmer.

Solutions to the Problems for Lecture 55

1. We know that the derivative of a function at a local maxima or minima is zero. The

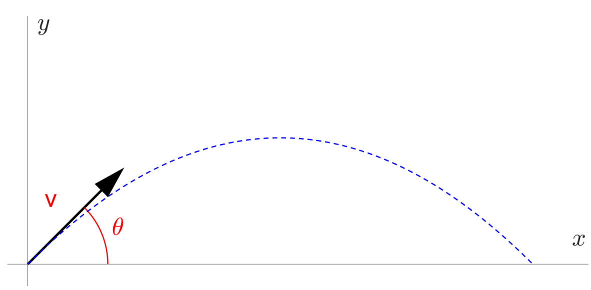


blue curve has a local maximum at which the red curve is zero. The red curve has a local maximum at which the green curve is zero. And the green curve has a local minimum at which no other curve is zero. So the blue curve can be the position, the red curve can be the velocity, and the green curve can be the acceleration.

Solutions to the Problems for Lecture 56

1. The trajectory of the cannonball is shown in the figure. Using trigonometry, the initial horizontal and vertical velocities of the cannonball are

$$v_x = v\cos\alpha, \qquad v_y = v\sin\alpha.$$



With constant horizontal velocity and a vertical acceleration $-g$, the horizontal and vertical positions of the cannonball at time t after firing is given by

$$x(t) = v_x t, \qquad y(t) = v_y t - \frac{1}{2}gt^2.$$

a) To determine the shape of the cannonball's trajectory, we eliminate t in favor of x in the equation for y. Substituting $t = x/v_x$, we find

$$y = \left(-\frac{g}{2v_x^2}\right)x^2 + \left(\frac{v_y}{v_x}\right)x,$$

which demonstrates that the path of the cannonball is a parabola.

b) To find the angle α that maximizes the horizontal distance traveled before the cannonball lands, we first determine the time T when this occurs, specifically when $y(T) = 0$. The equation is

$$v_y T - \frac{1}{2} g T^2 = 0.$$

Factoring out the solution $T = 0$ corresponding to the initial position, the remaining root is

$$T = \frac{2v_y}{g}.$$

The horizontal position of the cannonball at time T is then found to be

$$v_x T = \frac{2v_x v_y}{g} = \frac{2v^2 \cos\theta \sin\theta}{g}.$$

To maximize the horizontal position at time T with respect to the angle α, we set the derivative equal to zero:

$$\frac{d}{d\alpha}\left(\frac{2v^2 \cos\theta \sin\theta}{g}\right) = \frac{2v^2}{g}\left(-\sin^2\alpha + \cos^2\alpha\right) = 0.$$

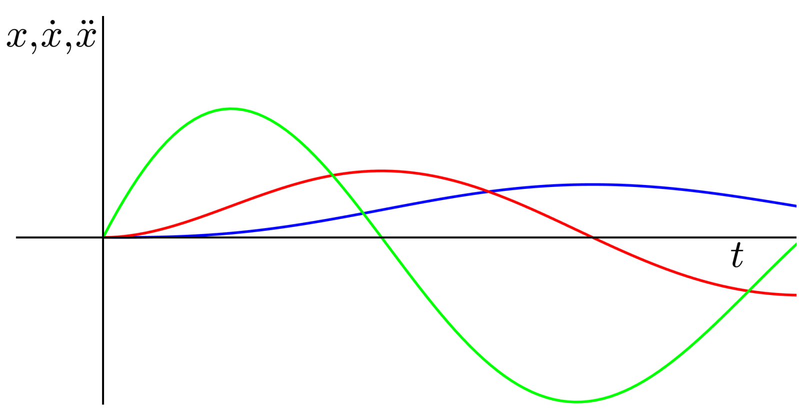
Since $0 < \alpha < \pi/2$, both $\sin\alpha$ and $\cos\alpha$ are positive. Thus the derivative is zero when $\cos\alpha = \sin\alpha$, or when the angle takes the value

$$\alpha = \frac{\pi}{4}.$$

It is evident that the horizontal distance traveled by the cannonball attains its maximum value when the cannon is fired at this angle.

Solutions to the Practice quiz: Newton's second law

1. c. We know that the derivative of a function at a local maxima or minima is zero. The



blue curve has a local maximum at which the red curve is zero. The red curve has a local maximum at which the green curve is zero. And the green curve has a local maximum and a local minimum at which no other curve is zero. So the blue curve can be the position, the red curve can be the velocity, and the green curve can be the acceleration.

2. d. A is the definition of the acceleration, and B follows from $v = dx/dt$. D follows from the chain rule if we write $v = v(x(t))$. C is false.

3. c. Place the origin of the x-axis at the top of the well, pointing downward. The equation of motion is

$$x(t) = x_0 + v_0 t + \frac{1}{2} g t^2.$$

Given parameters are $x_0 = 0$, $t = 4.5\,\text{s}$, $v_0 = 0$, $g = 9.8\,\text{m}/\text{s}^2$. The depth of the well is

$$x(t) = \frac{1}{2} g t^2 = \frac{1}{2}(9.8)(4.5)^2\,\text{m} = 99.2\,\text{m} \approx 100\,\text{m}.$$

Solutions to the Problems for Lecture 57

1. We start with the complex differential equation, $y' = iy$.

a) The general solution is $y = Ce^{ix}$, where C is a complex constant. Using the initial condition, $y(0) = 1$, we find $C = 1$. The solution is therefore

$$y(x) = e^{ix} = \cos x + i \sin x.$$

b) Write $y = u + iv$, where $u = u(x)$ and $v = v(x)$ are real functions. Then

$$u(x) + iv(x) = \cos x + i \sin x.$$

Equating real and imaginary parts, we have

$$u(x) = \cos x, \qquad v(x) = \sin x.$$

c) Substituting $y = u + iv$ into $y' = iy$ yields

$$u' + iv' = i(u + iv) = -v + iu.$$

Equating real and imaginary parts, we find

$$u' = -v, \qquad v' = u.$$

The initial condition, $y(0) = 1$ implies $u(0) = 1$ and $v(0) = 0$.

d) To derive a second-order equation for u, differentiate $u' = -v$ and use $v' = u$ to eliminate v:

$$u'' = -v' = -u \qquad \Longrightarrow \qquad u'' + u = 0.$$

Similarly, differentiate $v' = u$ and use $u' = -v$ to eliminate u:

$$v'' = u' = -v \qquad \Longrightarrow v'' + v = 0.$$

The initial condition $y(0) = 1$ gives $u(0) = 1$ and $v(0) = 0$. Since $u' = -v$, $v(0) = 0$ implies $u'(0) = 0$. Since $v' = u$, $u(0) = 1$ implies $v'(0) = 1$.

Therefore, the initial conditions for $u'' + u = 0$ are $u(0) = 1$, $u'(0) = 0$, and for $v'' + v = 0$, they are $v(0) = 0$ and $v'(0) = 1$. These second-order differential equations and initial conditions are standard for the cosine and sine functions.

Solutions to the Problems for Lecture 58

1. An initial amount S_0 is invested at an annual compound interest rate r. The investment grows exponentially as

$$S(t) = S_0 e^{rt}.$$

a) If the investment doubles in value after a time T, then $S(T) = 2S_0$, resulting in the equation

$$e^{rT} = 2 \quad \Longrightarrow \quad T = \frac{\ln 2}{r}.$$

b) Using the approximation $100 \ln 2 \approx 72$, we write the doubling time T as

$$T = \frac{\ln 2}{r} = \frac{100 \ln 2}{100r} \approx \frac{72}{100r}.$$

Here, $100r$ represents the annual interest rate expressed as a percentage. Using $72 = 2 \times 2 \times 2 \times 3 \times 3$, we can easily complete the table:

interest rate (%)	years to double
2	36
4	18
6	12
8	9
9	8
12	6

This calculation of the doubling time is known as the Rule of 72, which is useful for quick mental calculations.

2. Newton's law of cooling is given by

$$\frac{dT}{dt} = -k(T - T_s).$$

a) When the object's temperature is higher than that of its surroundings, $T - T_s > 0$. In this case, $dT/dt < 0$ indicating that the temperature of the object decreases over time. Conversely, when the object's temperature is lower than that of its surroundings, $T - T_s < 0$, and in this case, $dT/dt > 0$ and the object's temperature increases over time. If $T = T_s$, then $dT/dt = 0$, and the object's temperature remains unchanged.

b) Let $x = T - T_s$. Then $dT/dt = dx/dt$, and Newton's law of cooling becomes

$$\frac{dx}{dt} = -kx.$$

The solution to this differential equation is

$$x(t) = Ae^{-kt}.$$

Substituting back for x we find

$$T(t) - T_s = Ae^{-kt}.$$

Using the initial condition $T(0) = T_0$, we obtain $A = T_0 - T_s$. Thus, we have

$$T(t) = T_s + (T_0 - T_s)e^{-kt}.$$

Solutions to the Problems for Lecture 59

1. The velocity of the mass is given by $\dot{s} = l\dot{\theta}$. Therefore, the initial conditions are

$$\theta(0) = 0, \qquad \dot{\theta}(0) = \frac{u_0}{l}.$$

With $\omega = \sqrt{g/l}$, the general solution to the differential equation is

$$\theta(t) = A\cos\omega t + B\sin\omega t.$$

Applying the initial conditions, we have

$$\theta(0) = A = 0, \qquad \dot{\theta}(0) = \omega B = \frac{u_0}{l}.$$

Thus,

$$A = 0, \qquad B = \frac{u_0}{l\omega} = \frac{u_0}{\sqrt{gl}}.$$

The solution is

$$\theta(t) = \frac{u_0}{\sqrt{gl}}\sin\omega t.$$

2. The forces on the pendulum are

- Gravity: $F_g = -mg\sin\theta$;
- Frictional force opposing the velocity: $F_f = -cl\dot{\theta}$;
- External force: F_e.

The governing equation for the pendulum is therefore

$$ml\ddot{\theta} = -mg\sin\theta - cl\dot{\theta} + F_e.$$

Simplifying, we obtain

$$\ddot{\theta} + \frac{c}{m}\dot{\theta} + \frac{g}{l}\theta = F_e.$$

Note that c/m has units of inverse time, and g/l has units of inverse time squared.

Solutions to the Practice quiz: Growth, decay, and oscillations

1. a. Both the hyperbolic sine and cosine functions satisfy $\ddot{x} = x$. Additionally, $\cosh 0 = 1$ and $\sinh 0 = 0$.

2. c. The rule of 72 estimates the doubling time as $72/6 = 12$ years.

3. a. A simple pendulum launched from the bottom with an initial velocity u_0 to the right has the solution

$$\theta(t) = \frac{u_0}{\sqrt{gl}}\sin\omega t.$$

The maximum angle occurs when $\sin\omega t = 1$. If at this time, $\theta(t) = \pi/4$, then solving for u_0 gives

$$u_0 = \frac{\pi\sqrt{gl}}{4}.$$