Construction of Logarithm Table

LEE Wing Lung

In this modern age of computers, to find a logarithmic number like $\ln 2$ requires only the touch of a finger. Have you ever wondered, before the appearance of calculators, how the thick logarithm tables were constructed with precisions? Of course, historically there appeared many methods of constructions, each has its advantages, each has its limits. Below we will discuss one such method systematically. It only uses a bit of basic calculus skills and yields an efficient way of constructing a logarithm table to any precision.

First note that since $\ln(xy) = \ln x + \ln y$, we only need to find the logarithm of prime numbers p to get the logarithm of the other positive integers. From the differentiation of $\ln(1+t)$ and the geometric series formula, we get

$$\frac{d}{dt}\ln(1+t) = \frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots + (-1)^{n-1}t^{n-1} + \frac{(-1)^n t^n}{1+t}.$$

By the fundamental theorem of calculus (that is the inverse relations of differentiation and integration), we get

$$\ln(1+x) = \int_0^x \frac{1}{1+t}dt = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1}\frac{x^n}{n} + \int_0^x \frac{(-1)^n t^n}{1+t}dt$$

for all positive integer n. Let us now estimate the integral remainder term. For $|x| < 1$, we have

$$\Big|\int_0^x \frac{(-1)^n t^n}{1+t}dt\Big| \le \Big|\int_0^x \Big|\frac{t^n}{1+t}\Big||dt|\Big| \le \Big|\int_0^x \frac{t^n}{1-|x|}dt\Big| = \frac{|x|^{n+1}}{(n+1)(1-|x|)}.$$

From this we can see that as n increases, the absolute value of the remainder term will go to 0. In other words, for n sufficiently large, the difference between $\ln(1+x)$ and $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1}\frac{x^n}{n}$ will become arbitrarily small. Hence we may represent this in the following way:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad (|x| < 1).$$

Thus by a choice of n we may ignore the difference. Replacing x by $-x$ and subtracting the two equation, we have the following equation

$$\ln\Big(\frac{1+x}{1-x}\Big) = \ln(1+x) - \ln(1-x) = 2\Big(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\Big). \qquad (*)$$

Unfortunately, if we substitute $x = \frac{p-1}{p+1}$ to get $\frac{1+x}{1-x} = p$, equation (*) will not be able to produce $\ln p$ efficiently. For example, take $p = 29$, then $x = \frac{29-1}{29+1} = \frac{14}{15}$, and compute even to the 100th term $\frac{2x^{199}}{199} \approx 1.1 \times 10^{-8}$, the value of $\ln p$ is still not determined precisely to the 8th decimal place (more rigorously, we should have used the remainder term of the equation (*) to estimate the difference, but here we just want to know the difference approximately); for another example, take $p = 113$, then $x = \frac{56}{57}$ and $\frac{2x^{199}}{199} \approx 3 \times 10^{-4}$, the precision of $\ln p$ is even worse. However, if we use $x = \frac{1}{2p^2-1}$, then

$$\ln\Big(\frac{1+x}{1-x}\Big) = \ln\frac{2p^2-1+1}{2p^2-1-1} = \ln\frac{p^2}{(p+1)(p-1)} = 2\ln p - \ln(p+1)(p-1).$$

For prime $p > 2$, the common prime divisors of $(p+1)$ and $(p-1)$ must be less than p. So if we have computed the logarithm of all prime numbers less than p, then we can use the above equation to compute the value of $\ln p$:

$$2\ln p = \ln\left(\frac{1+x}{1-x}\right) + \ln(p+1) + \ln(p-1).$$

Now the term $\ln\left(\frac{1+x}{1-x}\right)$ can be computed efficiently because the absolute value of x chosen will be small. For example, when $p = 29$, $x = \frac{1}{2 \cdot 29^2 - 1} = \frac{1}{1681}$, so just compute to $\frac{2x^5}{5} \approx 3 \times 10^{-17}$ we can obtain ten decimal place accuracy.

From the discussion above, suppose we now want to construct a 8 decimal place logarithm table. Then we can compute the logarithm of $2, 3, 5, 7, 11, 13, \ldots$ in order and the logarithm of the numbers can be obtained from the logarithm of the numbers preceding them. From this we see that in the beginning we need to compute $\ln 2$ to a high precision:

$$\ln 2 = \ln\left(\frac{1-\frac{1}{3}}{1+\frac{1}{3}}\right) \approx 2\left(\frac{1}{3} + \frac{(\frac{1}{3})^3}{3} + \frac{(\frac{1}{3})^5}{5} + \cdots + \frac{(\frac{1}{3})^{21}}{21}\right) = 0.6931471805589\ldots.$$

This and the actual answer $\ln 2 = 0.693147180559945\ldots$ agree to 11 decimal places. Then we compute $\ln 3$. Taking $x = \frac{1}{2 \cdot 3^2 - 1} = \frac{1}{17}$, we have

$$\ln\left(\frac{1-\frac{1}{17}}{1+\frac{1}{17}}\right) \approx 2\left(\frac{1}{17} + \frac{(\frac{1}{17})^3}{3} + \frac{(\frac{1}{17})^5}{5} + \frac{(\frac{1}{17})^7}{7}\right) = 0.117783035654504\ldots.$$

Note that since $\frac{2(\frac{1}{17})^9}{9} \approx 1.9 \times 10^{-12}$, we may ignore this if only 8 decimal place accuracy is desired. So

$$\ln 3 \approx \frac{1}{2}(0.11778303565\ldots + \ln 4 + \ln 2) = 1.098612288635\ldots.$$

This and the actual answer $\ln 3 = 1.09861228866811\ldots$ agree to 10 decimal places. We will leave the computations of $\ln 5, \ln 7$ and so on to the reader and let them compare with the results given by calculators.

Looking back at the above very clever computation method, everyone should respect the mathematicians in the past for their creativity and instinct in the relations of numbers and computations!

References:

Chapter 3 of Professor Wu-Yi Hsiang's Notes for his Lectures on Analysis available at

http://ihome.ust.hk/~malung/391.html