Inclusion-Exclusion Principle and Turan's Theorem

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For a finite set A, the cardinality $|A|$ denote its number of elements. If there are two finite sets A and B, let $A \cup B$ denote the union of A and B. (It includes the elements in A or B.) Let $A \cap B$ denote the intersection of A and B. (It includes the elements in both A and B.) Everybody knows that if A and B do not have any common element, then $|A \cup B| = |A| + |B|$. However, if A and B have a common element x, then in counting the elements of A, x will be counted once, but in counting the elements of B, x will be counted once more. In order to avoid such repetition, in computing $|A \cup B|$, we have to subtract the number of repetitions, namely $|A \cap B|$. So $|A \cup B| = |A| + |B| - |A \cap B|$.

For the union $A \cup B \cup C$ of three sets, we can first compute the cardinalities of A, B and C. Adding them, we find it is too big. So we have to subtract the cardinalities of some intersections. Now the intersection of any two of A, B and C can be $A \cap B, A \cap C$ or $B \cap C$. When we subtract the number of elements in these intersections, we find it becomes too small. Finally we have to add the number of elements in the intersection of the three sets. At the end $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.

In general, if we have n finite sets $A_1, A_2, \ldots, A_n$, then $|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1} |A_1 \cap A_2 \cap \cdots \cap A_n|$, where the first sum on the right side is the total of the cardinalities of A_1 to A_n, the second sum is the total of the cardinalities of the intersection of every two sets and so on until we get to the intersection of $A_1, A_2, \ldots, A_n$.

The equation above is generally called the Inclusion-Exclusion Principle, whose name is obvious. It can be proved by mathematical induction. Moreover it can also be proved by binomial theorem like the following. For x belongs to $A_1 \cup A_2 \cup \cdots \cup A_n$, let x belongs to k A_i $(k \ge 1)$, say for convenience, x belongs to $A_1, A_2, \ldots, A_k$, but does not belong to $A_{k+1}, \ldots, A_n$. Then the "contribution" of x in $A_1 \cup A_2 \cdots \cup A_n$ is 1. In the first sum on right side, the "contribution" of x is $k = C_1^k$. In the second sum, as x appears in $A_1, A_2, \ldots, A_k$, x will appear in the intersesection of every two of them. So the "contribution" in the second sum is C_2^k. Analyzing these further, we will find the sum of all "contributions" of x on the right side is $C_1^k - C_2^k + C_3^k - \cdots + (-1)^{k+1} C_k^k = 1 - (1-1)^k = 1$. Note we have used the binomial theorem. As the contribution of x on both sides are equal, we have obtained a proof of the Inclusion-Exclusion Principle.

Furthermore the binomial coefficients have the following properties. When $m \le \frac{k}{2}$, C_m^k increases. When $k \ge \frac{k}{2}$, it decreases. (For example, when $k = 5$, we have $C_0^5 < C_1^5 < C_2^5 = C_3^5 > C_4^5 > C_5^5$, C_m^5 reaches maximum when $k = 2, 3$. When $k = 6$, $C_0^6 < C_1^6 < C_2^6 < C_3^6 > C_4^6 > C_5^6 > C_6^6$, C_m^6 reaches maximum when $k = 3$.) Using this relation, the reader can prove that if in the right side of the inclusion-exclusion formula, deleting a positive term and all other terms that follow it, the left side will become greater than the right side. This is because the contribution by x on the right will become nonpositive. Or the contribution in the deleted terms is nonnegative. Similarly, if in the right side of the inclusion-exclusion formula, deleting a negative term and all other terms that follow it, the left side will become less than the right side. This is a useful estimate.

The use of the Inclusion-Exclusion Principle in computing the sizes of sets appears often, with a wide range of applications.

Example 1: This is a classical problem. Take a rearrangement of the numbers $1, 2, \ldots, n$. If no number occupied the same position as before, then we say it is a derangement. (For example, 4321 is a derangement, but 4213 is not.) Now, how many derangements are there?

Solution: Obviously, there are $n! = n \times (n-1) \times \cdots \times 1$ rearrangements. If we try to find the number of derangement directly, this is not easy. So for $1 \le i \le n$, we define A_i to be the set of rearrangements having i in the correct position. It is easy to see that $|A_i| = (n-1)!$, similarly, $|A_i \cap A_j| = (n-2)!$, here $i \ne j$, and so on. Hence

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

$$= n(n-1)! - C_2^n(n-2)! + C_3^n(n-3)! - \cdots + (-1)^{n-1}1 = n! - \frac{n!}{2!} + \frac{n!}{3!} - \cdots + (-1)^{n-1}\frac{n!}{n!}.$$

Finally, the number of derangements is $n! - |A_1 \cup A_2 \cup \cdots \cup A_n| = n!(\frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!})$.

__Example 2__: (IMO 1991) Let $S = \{1, 2, \ldots, 280\}$. Find the smallest natural number n such that every n element subset of S contains 5 pairwise relatively prime numbers.

__Solution__: First we use the inclusion-exclusion principle to get $n \geq 217$. Let A_1, A_2, A_3, A_4 be the subsets of S containing multiples of $2, 3, 5, 7$, respectively. Then $|A_1| = 140, |A_2| = 93, |A_3| = 56, |A_4| = 40, |A_1 \cap A_2| = 46, |A_1 \cap A_3| = 28, |A_1 \cap A_4| = 20, |A_2 \cap A_3| = 18, |A_2 \cap A_4| = 13, |A_3 \cap A_4| = 8, |A_1 \cap A_2 \cap A_3| = 9, |A_1 \cap A_2 \cap A_4| = 6, |A_1 \cap A_3 \cap A_4| = 4, |A_2 \cap A_3 \cap A_4| = 2, |A_1 \cap A_2 \cap A_3 \cap A_4| = 1$. So $|A_1 \cap A_2 \cap A_3 \cap A_4| = 140 + 93 + 56 + 40 - 46 - 28 - 20 - 18 - 13 - 8 + 9 + 6 + 4 + 2 - 1 = 216$. For this 216 element set, among any 5 numbers, there must be two both belong to A_1, A_2, A_3 or A_4, hence not relatively prime. According to the problem, we must have $n \geq 217$.

Now we prove that every 217 element subset of S must have 5 pairwise relatively prime numbers. The idea is to construct proper "pigeonholes". Here is an elegant construction. Let A be a subset of S with $|A| \geq 217$. Define $B_1 = \{1 \text{ or prime numbers in } S\}, |B_1| = 60, B_2 = \{2^2, 3^2, 5^2, 7^2, 11^2, 13^2\}, |B_2| = 6, B_3 = \{2\times 131, 3\times 89, 5\times 53, 7\times 37, 11\times 23, 13\times 19\}, |B_3| = 6, B_4 = \{2\times 127, 3\times 87, 5\times 47, 7\times 31, 11\times 19, 13\times 17\}, |B_4| = 6, B_5 = \{2\times 113, 3\times 79, 5\times 43, 7\times 27, 11\times 17\}, |B_5| = 5, B_6 = \{2\times 109, 3\times 73, 5\times 41, 7\times 23, 11\times 13\}, |B_6| = 5$. It is easy to see B_1 and B_6 are disjoint. Also $|B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5 \cup B_6| = 88$. Removing these 88 numbers, S still has $280 - 88 = 192$ numbers. Now A has at least 217 elements, $217 - 192 = 25$, that is, there are at least 25 elements in A that belong to B_1 to B_6. Obviously it cannot be that every B_i only contains 4 or less elements of A. That is, there are at least 5 elements of A belong to the same B_i, hence are relatively prime. Notwe we have used another principle: pigeonhole principle.

__Example 3__: (1989 IMO) Let n be a positive integer. We say a permutation $(x_1, x_2, \ldots, x_{2n})$ of $\{1, 2, \ldots, 2n\}$ has property P if and only if there is at least one i in $\{1, 2, \ldots, 2n-1\}$ such that $|x_i - x_{i+1}| = n$ holds. Prove that there are more permutations with property P than those permutations without property P.

__Solution__: Note if $|x_i - x_{i+1}| = n$, then one of x_i or x_{i+1} must be less than $n + 1$. For $k = 1, 2, \ldots, n$, define A_k to be the set of all permutations having k and $k + n$ next to each other. It is easy to see that $|A_k| = 2 \times (2n-1)!$. (This is because k and $k+n$ are grouped together, their positions may be interchanged, think of them as one "number", there are $2n-2$ others, and so $(2n-2)+1$ positions for any number.) Also $|A_k \cap A_h| = 2^2 \times (2n-2)!, 1 \leq k < h \leq n$, ($k$ and $k + n$ are grouped as one "number" and h and $h + n$ are grouped as one "number".) So the number of permutations with property P is

$$|A_1 \cup A_2 \cup \cdots \cup A_n| \geq \sum_{k=1}^{n} |A_k| - \sum_{1 \leq k < h \leq n} |A_k \cap A_h| = 2 \times (2n-1)! \times n - C_2^n \times 2^2 \times (2n-2)!$$
$$= 2n \times (2n-2)! \times n = (2n)! \times \frac{n}{2n-1} > (2n)! \times \frac{1}{2}.$$

This number is more than half of $(2n)!$. So the permutations with property P is more than those permutations without property P. (Years ago this problem was considered difficult, but with inclusion-exclusion relations, this becomes easy.)

__Example 4__: Let n and k be positive integers, $n > 3$, $\frac{n}{2} < k < n$. There are n points on the plane, every three of them are not collinear. If every point is connected to at least k other points by segments, then there are three segments forming a triangle.

__Solution__: Since $n > 3$, $k > \frac{n}{2}$, so $k \geq 2$. Hence among the n points, there are two points v_1 and v_2 that are connected by a segments. Consider the remaining points. Let A be the set of points connected to v_1 and B be the set of points connected to v_2, then $|A| \geq k - 1, |B| \geq k - 1$. Also,

$$n - 2 \geq |A \cup B| = |A| + |B| - |A \cap B| \geq 2k - 2 - |A \cap B|,$$

that is $|A \cap B| \geq 2k - n > 0$. So there exists a point v_3 connected to v_1 and v_2 forming a triangle.

Example 5: There was 1990 mathematicians participated in a meeting. Every one of them has collaborated with at least 1327 others. Prove that we can find 4 mathematicians, every pair of them collaborated with each other.

Solution: Consider the mathematicians as points of a set. Connect pairs that collaborated with an edge to yield a graph. As the above example, for v_1 and v_2 that collaborated, they are connected. For the remaining points, let A be the set of collaborators of v_1 and B be the set of collaborators of v_2. Then $|A| \geq 1326, |B| \geq 1326$. Similarly,

$$|A \cap B| = |A| + |B| - |A \cup B| \geq 2 \times 1326 - 1998 = 664 > 0,$$

that is, we can find a mathematician v_3 that collaborated with v_1 and v_2. Let C be the set of mathematicians that collaborated with v_3 excluding v_1 and v_3. That is $|C| \geq 1325$. Also

$$1998 \geq |(A \cap B) \cup C| = |A \cap B| + |C| - |A \cap B \cap C|$$

that is $|A \cap B \cap C| \geq |A \cap B| + |C| - 1988 \geq 664 + 1325 - 1998 = 1 > 0$. So $A \cap B \cap C$ is nonempty. Take $v_4 \in A \cap B \cap C$. Then v_1, v_2, v_3, v_4 collaborated.

In graph theory terminologies, examples 4 and 5 can be interpreted as giving a graph of n vertices, to determine the least number of edges that will gaurantee the existence of a triangle (K_3) or a K_4 (a subgraph with four vertices, every two vertices are connected by an edge). Putting it in another way, for a graph of n vertices with no triangle, to determine the maximum number of its edges. This area of graph theory is called extremal graph theory. The first result is the following:

Mantel's Theorem (1907): For a simple graph with n vertices containing no K_3, the maximum number of edges is $[\frac{n^2}{4}]$.

(Here $[x]$ is the greatest integer less than or equal to x. In example four, the number of edges is greater than $\left(\frac{n}{2}\right) \times n \times \frac{1}{2} > \left[\frac{n^2}{4}\right]$, then the result follows immediately.)

A more delicate result is the following:

Theorem: If a graph with n vertices has q edges, then the graph has at least $\dfrac{4q(q-\frac{n^2}{4})}{3n}$ triangles.

Example 6: There are 21 points on a circle. Among the angles formed by extending pairs of points to the center, there are at most 110 of these are greater than 120°.

Solution: If the angle formed by extending two points to the center is greater than 120°, then connect these two points by an edge. This yields a graph. The graph has no triangles. So the number of edges is at most $\left[\frac{21^2}{4}\right] = \left[\frac{441}{4}\right] = 110$, or there can be at most 110 such angles greater than 120°.

As above, define K_p to be a p vertices complete graph, that is any two of the p vertices are connected by an edge. For a graph G with n vertices, if it does not contain any K_p, then what is the maximum number of edges G can have?

Turan's Theorem (1941): If a graph G with n vertices does not contain any K_p, then that graph has at most $\dfrac{p-2}{2(p-1)}n^2 - \dfrac{r(p-1-r}{2(p-1)}$ edges, where r is defined by $n = k(p-1) + r, 0 \leq r < p-1$. As in the situation of Mantel's theorem, this is a starting point of extremal graph theory.

Paul Turan (1910-1976) was a Jewish Hungarian. At the time he was considering these kinds of problems, he was in a concentration camp!