Mathematical Excalibur

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Olympiad Corner

Below is the Bulgarian selection test for the 46th IMO given on May 18 – 19, 2005.

Problem 1. An acute triangle ABC is given. Find the locus of points M in the interior of the triangle such that $AB-FG = (MF\cdot AG+MG\cdot BF)/CM$, where F and G are the feet of perpendiculars from M to the lines BC and AC, respectively.

Problem 2. Find the number of subsets B of the set $\{1, 2, \ldots, 2005\}$ such that the sum of the elements of B is congruent to 2006 modulo 2048.

Problem 3. Let R_* be the set of non-zero real numbers. Find all functions $f: R_* \to R_*$ such that

$$f(x^2+y) = f^2(x) + \frac{f(xy)}{f(x)}$$

for all $x, y \in R_*$, $y \neq -x^2$.

Problem 4. Let $a_1, a_2, \ldots, a_{2005}, b_1, b_2, \cdots, b_{2005}$ be real numbers such that

$$(a_i x - b_i)^2 \geq \sum_{j=1, j\neq i}^{2005} (a_j x - b_j)$$

for any real number x and $i = 1, 2, \ldots, 2005$. What is the maximal number of positive a_i's and b_i's?

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Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***December 10, 2005***.

For individual subscription for the next five issues for the 03-04 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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The Method of Infinite Descent

Leung Tat-Wing

The technique of infinite descent (*descent infini*) was developed by the great amateur mathematician Pierre de Fermat (1601-1665). Besides using the technique to prove negative results such as the equation $x^4 + y^4 = z^2$ has no nontrivial integer solution, he also used the technique to prove positive results.

For instance, he knew that an odd prime p can be expressed as the sum of two integer squares if and only if p is of the form $4k + 1$. To show that a prime of the form $4k + 3$ is not a sum of two squares is not hard. In fact, every square equals 0 or 1 mod 4, thus no matter what possibilities, the sum of two squares cannot be of the form $4k + 3 \equiv 3 \pmod 4$. To prove a prime of the form $4k + 1$ is the sum of two squares, he assumed that if there is a prime of the form $4k + 1$ which is not the sum of two squares, then there will be another (smaller) prime of the same nature, and hence a third one, and so on. Eventually he would come to the number 5, which should not be the sum of two squares. But we know $5 = 1^2 + 2^2$ a sum of two squares, a contradiction!

The idea of infinite descent may be described as follows. Mainly it is because a finite subset of natural numbers must have a smallest member. So if A is a subset of the natural numbers N, and if we need to prove, for every $a \in A$, the statement $P(a)$ is valid. Suppose by contradiction, the statement is not valid for all $a \in A$, i.e. there exists a non-empty subset of A, denoted by B, and such that $P(x)$ is not true for any $x \in B$. Now because B is non-empty, there exists a smallest element of B, denoted by b and such that $P(b)$ is not valid. Using the given conditions, if we can find a still smaller $c \in A$ ($c < b$), and such that $P(c)$ is not valid, then this will contradict the assumption of b. The conclusion is that $P(a)$ must be valid for all $a \in A$.

There are variations of this scenario. For instance, suppose there is a positive integer a_1 such that $P(a_1)$ is valid, and from this, if we can find a smaller positive integer a_2 such that $P(a_2)$ is valid, then we can find a still smaller positive integer a_3 such that $P(a_3)$ is valid, and so on. Hence we can find an infinite and decreasing chain of positive integers (infinite descent) $a_1 > a_2 > a_3 > \cdots$. This is clearly impossible. So the initial hypothesis $P(a_1)$ cannot be valid.

So the method of descent is essentially another form of induction. Recall that in mathematical induction, we start from a smallest element a of a subset of natural numbers, (initial step), and prove the so-called inductive step. So we can go from $P(a)$ to $P(a + 1)$, then $P(a + 2)$ and so on.

Many problems in mathematics competition require the uses of the method of descent. We give a few examples. First we use the method of infinite descent to prove the well-known result that $\sqrt{2}$ is irrational. Of course the classical proof is essentially a descent argument.

__Example 1__: Show that $\sqrt{2}$ is irrational.

__**Solution.**__ We need to show that there do not exist positive integers x and y such that $x/y = \sqrt{2}$ or by taking squares, we need to show the equation $x^2 = 2y^2$ has no positive integer solution.

Suppose otherwise, let $x = m$, $y = n$ be a solution of the equation and such that m is the *smallest* possible value of x that satisfies the equation. Then $m^2 = 2n^2$ and this is possible only if m is even, hence $m=2m_1$. Thus, $4m_1^2 = (2m_1)^2 = 2n^2$, so $n^2 = 2m_1^2$. This implies n is also a possible value of x in the equation $x^2 = 2y^2$. However, $n < m$, contradicting the minimality of m.

Example 2 (Hungarian MO 2000): Find all positive primes p for which there exist positive integers x, y and n such that $p^n = x^3 + y^3$.

Solution. Observe $2^1 = 1^3 + 1^3$ and $3^2 = 2^3 + 1^3$. After many trials we found no more primes with this property. So we suspect the only answers are $p = 2$ or $p = 3$. Thus, we need to prove there exists no prime p ($p > 3$) satisfying $p^n = x^3 + y^3$. Clearly we need to prove by contradiction and one possibility is to make use of the descent method. (In this case we make descent on n and it works.)

So we assume $p^n = x^3 + y^3$ with x, y, n positive integers and n of the smallest possible value. Now $p \geq 5$. Hence at least one of x and y is greater than 1. Also

$$x^3 + y^3 = (x+y)(x^2 - xy + y^2),$$

with $x + y \geq 3$ and

$$x^2 - xy + y^2 = (x-y)^2 + xy \geq 2.$$

Hence both $x + y$ and $x^2 - xy + y^2$ are divisible by p. Therefore

$$(x+y)^2 - (x^2 - xy + y^2) = 3xy$$

is also divisible by p. However, 3 is not divisible by p, so at least one of x or y must be divisible by p. As $x + y$ is divisible by p, both x and y are divisible by p. Then $x^3 + y^3 \geq 2p^3$. So we must have $n > 3$ and

$$p^{n-3} = \frac{p^n}{p^3} = \frac{x^3}{p^3} + \frac{y^3}{p^3} = \left(\frac{x}{p}\right)^3 + \left(\frac{y}{p}\right)^3.$$

This contradicts the minimality of n.

Example 3 (Putnam Exam 1973): Let $a_1, a_2, \cdots, a_{2n+1}$ be a set of integers such that, if any one of them is removed, the remaining ones can be divided into two sets of n integers with equal sums. Prove $a_1 = a_2 = \cdots = a_{2n+1}$.

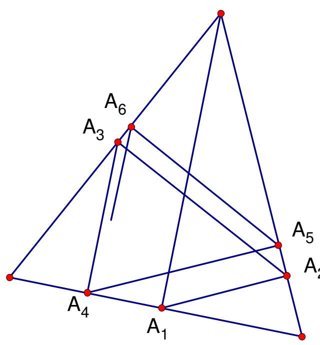
Solution. Assume $a_1 \leq a_2 \leq \cdots \leq a_{2n+1}$. By subtracting the smallest number from the sequence we observe the new sequence still maintain the property. So we may assume $a_1 = 0$. The sum of any $2n$ members equals 0 mod 2, so any two members must be of the same parity, (otherwise we may swap two members to form two groups of $2n$ elements which are of different parity). Therefore

$$0 = a_1 \equiv a_2 \equiv \cdots \equiv a_{2n+1} \pmod 2.$$

Dividing by 2, we note the new sequence will maintain the same property. Using the same reasoning we see that $0 = a_1 \equiv a_2 \equiv \cdots \equiv a_{2n+1} \pmod{2^2}$. We may descent to $0 = a_1 \equiv a_2 \equiv \cdots \equiv a_{2n+1} \pmod{2^m}$ for all $m \geq 1$. This is possible only if the initial numbers are all equal to others.

Example 4: Starting from a vertex of an acute triangle, the perpendicular is drawn, meeting the opposite side (side 1) at A_1. From A_1, a perpendicular is drawn to meet another side (side 2) at A_2. Starting from A_2, the perpendicular is drawn to meet the third side (side 3) at A_3. The perpendicular from A_3 is then drawn to meet side 1 at A_4 and then back to side 2, and so on.

Prove that the points A_1, A_2, … are all distinct.



Solution. First note that because the triangle is acute, all the points A_i, $i \geq 1$ lie on the sides of the triangle, instead of going outside or coincide with the vertices of the triangle. This implies A_i and A_{i+1} will not coincide because they lie on adjacent sides of the triangle. Suppose now A_i coincides with A_j ($i < j$), and i is the smallest index with this property. Then in fact $i = 1$. For otherwise A_{i-1} will coincide with A_{j-1}, contradicting the minimality of i. Finally suppose A_1 coincides with A_j, $j \geq 3$, this happens precisely when A_{j-1} is the vertex of the triangle facing side 1. But we know that no vertices of the triangle are in the list, so again impossible.

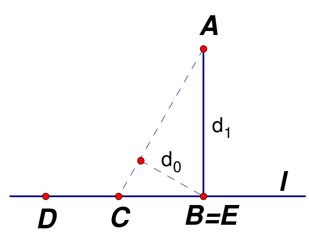
The following example was a problem of Sylvester (1814-1897). Accordingly Sylvester was annoyed to find that he was unable to tackle this deceptively simple problem. It was later solved by the technique of descent. The idea is to consider the smallest possible element with a certain property.

Example 5 (Sylvester's Problem): Given n ($n \geq 3$) points on the plane. If a line passing through any two points also passes through a third point of the set, then prove that all the points lie on the same line.

Solution. We prove an equivalent statement. Namely if there are n ($n \geq 3$) points on the plane and such that they are not on the same line, then there exists a line passing through exactly two points.

Now there are finitely many lines that may be formed by the points of the point set. Given such a line, there is at least one point of the set which does not lie on the line. We then consider the distance between the point and the line. Finally we list all such distances as $d_1 \leq d_2 \leq \cdots \leq d_m$, namely d_1 is the minimum distance between all possible points and all possible lines, say it is the distance between A and the line l. We now proceed to show that l contains exactly two points of the point set.

Suppose not, say points B, C and D of the point set also lie on l. From A, draw the line AE perpendicular to l, with E on l. If E is one of the B, C or D, say E and B coincide, we have the picture



Now $AB = d_1$. However if we draw a perpendicular line from B to AC, then we will get a distance d_0 less than d_1, contradicting its minimality. Similarly if E coincides with C or D, we can also obtain a smaller distance.

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for submitting solutions is ***December 10, 2005.***

Problem 236. Alice and Barbara order a pizza. They choose an arbitrary point P, different from the center of the pizza and they do three straight cuts through P, which pairwise intersect at 60° and divide the pizza into 6 pieces. The center of the pizza is not on the cuts. Alice chooses one piece and then the pieces are taken clockwise by Barbara, Alice, Barbara, Alice and Barbara. Which piece should Alice choose first in order to get more pizza than Barbara?
(*Source: 2002 Slovenian National Math Olympiad*)

Problem 237. Determine (with proof) all polynomials p with real coefficients such that $p(x)\,p(x+1) = p(x^2)$ holds for every real number x.
(*Source*: *2000 Bulgarian Math Olympiad*)

Problem 238. For which positive integers n, does there exist a permutation $(x_1, x_2, \ldots, x_n)$ of the numbers 1, 2, …, n such that the number $x_1 + x_2 + \cdots + x_k$ is divisible by k for every $k \in \{1,2, \ldots, n\}$?
(*Source: 1998 Nordic Mathematics Contest*)

Problem 239. (*Due to José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain*) In any acute triangle ABC, prove that

$$\cos\left(\frac{A-B}{2}\right)+\cos\left(\frac{B-C}{2}\right)+\cos\left(\frac{C-A}{2}\right)$$
$$\le \frac{\sqrt{2}}{2}\left(\frac{a+b}{\sqrt{a^2+b^2}}+\frac{b+c}{\sqrt{b^2+c^2}}+\frac{c+a}{\sqrt{c^2+a^2}}\right).$$

Problem 240. Nine judges independently award the ranks of 1 to 20 to twenty figure-skaters, with no ties. No two of the rankings awarded to any figure-skater differ by more than 3. The nine rankings of each are added. What is the maximum of the lowest of the sums? Prove your answer is correct.

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Solutions

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Problem 231. On each planet of a star system, there is an astronomer observing the nearest planet. The number of planets is odd, and pairwise distances between them are different. Prove that at least one planet is not under observation.
(*Source*: *1966 Soviet Union Math Olympiad*)

Solution. **CHAN Pak Woon** (HKU Math, Year 1), **LEE Kai Seng** (HKUST), **WONG Kwok Cheung** (Carmel Alison Lam Foundation Secondary School, Form 7) and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 5).

Let there be n planets. The case of $n = 1$ is clear. For $n \ge 3$, suppose the case $n-2$ is true. For the two closest planets, the astronomers on them observe each other. If any of the remaining $n-2$ astronomers observes one of these two planets, then we do not have enough astronomers to observe the $n-2$ remaining planets. Otherwise, we can discard these two closest planets and apply the case $n-2$.

Commended Solvers: **Roger CHAN** (Vancouver, Canada) and **Anna Ying PUN** (STFA Leung Kau Kui College).

Problem 232. B and C are points on the segment AD. If $AB = CD$, prove that $PA+PD \ge PB+PC$ for any point P.
(*Source*: *1966 Soviet Union Math Olympiad*)

Solution 1. **Anna Ying PUN** (STFA Leung Kau Kui College).

Suppose P is not on line AD. Let P' be such that $PAP'D$ is a parallelogram. Now AB=CD implies $PBP'C$ is a parallelogram. By interchanging B and C, we may assume B is between A and C. Let line PB intersect AP' at F. Then $PA+PD = PA+AP' = PA+AF+FP' > PF + FP' = PB + BF + FP' > PB+BP' = PB + PC$. The case P is on line AD is easy to check.

Solution 2. **LEE Kai Seng** (HKUST).

Consider the complex plane with line AD as the real axis and the origin at the midpoint O of segment AD. Let the complex numbers correspond to A, B, P be a, b, p, respectively. Since $|p \pm a|^2 = |p|^2 \pm 2\mathrm{Re}\, ap + a^2$, so $(PA + PD)^2 = 2(|p|^2 + |p^2-a^2| + a^2)$. Then

$$(PA+PD)^2-(PB+PC)^2$$
$$=2(|p^2-a^2|+a^2-b^2-|p^2-b^2|)\ge 0$$

by the triangle inequality. So $PA+PD \ge PB+PC$.

Also equality holds if and only if the ratio of p^2-a^2 and a^2-b^2 is a nonnegative number, which is the same as $p \ge a$ or $p \le -a$.

Commended Solvers: **CHAN Wai Hung** (Carmel Divine Grace Foundation Secondary School, Form 7), **WONG Kwok Cheung** (Carmel Alison Lam Foundation Secondary School, Form 7) and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 5).

Problem 233. Prove that every positive integer not exceeding $n!$ can be expressed as the sum of at most n distinct positive integers each of which is a divisor of $n!$.

Solution. **CHAN Ka Lok** (STFA Leung Kau Kui College, Form 6), **G.R.A. 20 Math Problem Group** (Roma, Italy), **LEE Kai Seng** (HKUST) and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 5).

We prove by induction on n. The case $n = 1$ is clear. Suppose case $n-1$ is true. For $n > 1$, let $1 \le k \le n!$ and let q and r be such that $k = qn + r$ with $0 \le r < n$. Then $0 \le q \le (n-1)!$. By the case $n-1$, q can be expressed as $d_1 + d_2 + \cdots + d_m$, where $m \le n-1$ and d_i is a divisor of $(n-1)!$ and d_i's are distinct. Omitting r if $r = 0$, we see $d_1n + d_2n + \cdots + d_mn + r$ is a desired expansion of k.

Problem 234. Determine all polynomials $P(x)$ of the smallest possible degree with the following properties:

a) The coefficient of the highest power is 200.
b) The coefficient of the lowest power for which it is not equal to zero is 2.
c) The sum of all its coefficients is 4.
d) $P(-1) = 0$, $P(2) = 6$ and $P(3) = 8$.

(*Source: 2002 Austrian National Competition*)

Solution. **CHAN Pak Woon** (HKU Math, Year 1), **G.R.A. 20 Math Problem Group** (Roma, Italy), **WONG Kwok Cheung** (Carmel Alison Lam Foundation Secondary School, Form 7) and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 5).

Note c) is the same as $P(1) = 4$. For

$$P(x)=200x(x+1)(x-1)(x-2)(x-3)+2x+2$$
$$=200x^5-1000x^4+1000x^3+1000x^2-1198x+2,$$

all conditions are satisfied. Assume R is another such polynomial with degree at most 5. Then P and R agree at –1, 1, 2, 3. So

$$P(x)–R(x) = (x+1)(x–1)(x–2)(x–3)S(x)$$

with degree of S at most 1. If S is constant, then b) implies $P(0)–R(0)$ is 0 or 2. Then $S(x) = –1/3$ and we get

$$R(x) = P(x)+(x+1)(x–1)(x–2)(x–3)/3 = 200x^5 + \cdots –1196\tfrac{1}{3}x,$$

which fails b). If S is of degree 1, then a) and b) imply $S(x)=200x–1/3$ and we will get $R(x) =$

$$P(x) – (x+1)(x–1)(x–2)(x–3)(200x–1/3) = x^4/3 + \cdots,$$

which fails a). So no such R exists and P is the unique answer.

Problem 235. Forty-nine students solve a set of three problems. The score for each problem is an integer from 0 to 7. Prove that there exist two students A and B such that, for each problem, A will score at least as many points as B.
(*Source: 29th IMO Unused Problem*)

Solution. **LEE Kai Seng** (HKUST) and **Anna Ying PUN** (STFA Leung Kau Kui College).

For n = 0,1,2,3, let S_n be the set of ordered pairs $(0,n),(1,n),\cdots,(7–n,n)$ and $(7 – n, n + 1),\cdots, (7 – n, 7)$. Let $S_4 = \{(x,y): x=2 \text{ or } 3; y= 4,5,6 \text{ or } 7\}$ and $S_5= \{(x,y): x=0 \text{ or } 1; y=4,5,6 \text{ or } 7\}$.

For each student, let his/her score on the first problem be x and on the second problem be y. Note if two students have both of their (x,y) pairs in one of S_0, S_1, S_2 or S_3, then one of them will score at least as many point as the other in each of the first two problems.

Of the 49 pairs (x,y), there are $[49/6]+1 = 9$ of them belong to the same S_n. If this S_n is S_4 or S_5, which has 8 elements, then two of the 9 pairs are the same and the two students will satisfy the desired condition. If the S_n is S_0, S_1, S_2 or S_3, then two of these 9 students will have the same score on the third problem and they will satisfy the desired condition by the note in the last paragraph.

Commended Solvers: **CHAN Pak Woon** (HKU Math, Year 1), **LAW Yan Pui** (Carmel Divine Grace Foundation Secondary School, Form 7) and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 5).

Olympiad Corner

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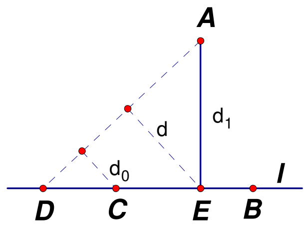
Problem 5. Let ABC be an acute triangle with orthocenter H, incenter I and $AC \neq BC$. The lines CH and CI meet the circumcircle of $\triangle ABC$ for the second time at points D and L, respectively. Prove that $\angle CIH = 90^\circ$ if and only if $\angle IDL = 90^\circ$.

Problem 6. In a group of nine people there are no four every two of which know each other. Prove that the group can be partitioned into four groups such that the people in every group do not know each other.

The Method of Infinite Descent

(continued from page 2)

Now if the perpendicular from A to l does not meet any of B, C or D, then by the pigeonhole principle, there are two points (say C and D) which lie on one side of the perpendicular. Again from the diagram



We draw perpendiculars from E and C to AD, and we observe the distances $d_0 < d < d_1$, again contradicting the minimality of d_1. From the above arguments, we conclude that l contains exactly two points.

From the above example, we have

__Example 6 (Polish MO 1967-68)__: Given n ($n \geq 3$) points on the plane and these points are not on the same line. From any two of these points a line is drawn and altogether k distinct lines are formed. Show that $k \geq n$.

__**Solution.**__ We proceed by induction. Clearly three distinct lines may be drawn from three points not on a line. Hence the statement is true for $n = 3$. Suppose the statement is valid for some $n \geq 3$. Now let $A_1, A_2, \ldots, A_n, A_{n+1}$ be $n + 1$ distinct points which are not on the same line. By Sylvester's "theorem", there exists a line containing exactly two points of the point set, say A_1A_{n+1}.

Let's consider the sets $Z_1 =\{A_1, A_2, \ldots, A_n\}$and $Z_2 =\{A_2, A_3, \ldots, A_n, A_{n+1}\}$ Clearly at least one of the point sets does not lie on a line. If $A_1, A_2, \ldots, A_n$ do not lie on a line, by the inductive hypothesis, we can form at least n lines using these points. As A_{n+1} is not one of the members of Z_1, so A_1A_{n+1} will form a new line, (A_1A_{n+1} contains no other points of the set) and we have at least $n + 1$ lines. If $A_2, A_3, \ldots, A_n, A_{n+1}$ do not lie on a line, then again we can form at least n lines using these points. As A_1 is not one of the members of Z_2, so A_1A_{n+1} will form a new line, (A_1A_{n+1} contains no other points of the set) and we have at least $n + 1$ lines.

The method of infinite descent was used to prove a hard IMO problem.

__Example 7 (IMO 1988)__: Prove that if positive integers a and b are such that $ab + 1$ divides $a^2 + b^2$, then $(a^2 + b^2)/(ab + 1)$ is a perfect square.

__**Solution.**__ Assume $(a^2 + b^2)/(ab + 1) = k$ and k is not a perfect square. After rearranging we have $a^2 – kab + b^2 = k$, with $a > 0$ and $b > 0$. Assume now (a_0, b_0) is a solution of the Diophantine equation and such that $a_0 + b_0$ is as *small* as possible. By symmetry we may assume $a_0 \geq b_0 > 0$. Fixing b_0 and k, we may assume a_0 is a solution of the quadratic equation

$$x^2 – kb_0x + b_0^2 – k = 0.$$

Now let the other root of the equation be a'. Using sum and product of roots, we have $a_0 + a' = kb_0$ and $a_0a' = b_0^2 – k$. The first equation implies a' is an integer. The second equation implies $a' \neq 0$, otherwise k is a perfect square, contradicting our hypothesis. Now a' also cannot be negative, otherwise

$$a'^2 – ka'b_0 + b_0^2 \geq a'^2 + k + b_0^2 > k.$$

Hence $a' > 0$. Finally

$$a' = \frac{b_0^2 - k}{a_0} \leq \frac{b_0^2 - 1}{a_0} \leq \frac{a_0^2 - 1}{a_0} < a_0.$$

This implies (a', b_0) is a positive integer solution of $a^2 – kab + b^2 = k$, and $a' + b_0 < a_0 + b_0$, contradicting the minimality of $a_0 + b_0$. Therefore k must be a perfect square.