Mathematical Excalibur

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Olympiad Corner

Below is the Czech-Polish-Slovak Match held in Zwardon on June 20-21, 2005.

Problem 1. Let n be a given positive integer. Solve the system of equations

$$x_1 + x_2^2 + x_3^3 + \cdots + x_n^n = n,$$
$$x_1 + 2x_2 + 3x_3 + \cdots + nx_n = \frac{n(n+1)}{2}$$

in the set of nonnegative real numbers $x_1, x_2, \ldots, x_n$.

Problem 2. Let a convex quadrilateral $ABCD$ be inscribed in a circle with center O and circumscribed to a circle with center I, and let its diagonals AC and BD meet at a point P. Prove that the points O, I and P are collinear.

Problem 3. Determine all integers $n \geq 3$ such that the polynomial $W(x) = x^n - 3x^{n-1} + 2x^{n-2} + 6$ can be expressed as a product of two polynomials with positive degrees and integer coefficients.

Problem 4. We distribute $n \geq 1$ labelled balls among nine persons $A, B, C, D, E, F, G, H, I$. Determine in how many ways

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***February 12, 2006***.

For individual subscription for the next five issues for the 03-04 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Using Tangent Lines to Prove Inequalities

Kin-Yin Li

For students who know calculus, sometimes they become frustrated in solving inequality problems when they do not see any way of using calculus. Below we will give some examples, where finding the equation of a tangent line is the critical step to solving the problems.

Example 1. Let a,b,c,d be positive real numbers such that $a + b + c + d = 1$. Prove that

$$6(a^3+b^3+c^3+d^3) \geq (a^2+b^2+c^2+d^2) + 1/8.$$

Solution. We have $0 < a, b, c, d < 1$. Let $f(x) = 6x^3 - x^2$. (Note: Since there is equality when $a = b = c = d = 1/4$, we consider the graph of $f(x)$ and its tangent line at $x = 1/4$. By a simple sketch, it seems the tangent line is below the graph of $f(x)$ on the interval $(0,1)$. Now the equation of the tangent line at $x = 1/4$ is $y = (5x - 1)/8$.) So we claim that for $0 < x < 1$, $f(x) = 6x^3 - x^2 \geq (5x - 1)/8$. This is equivalent to $48x^3 - 8x^2 - 5x + 1 \geq 0$. (Note: Since the graphs intersect at $x = 1/4$, we expect $4x - 1$ is a factor.) Indeed, $48x^3 - 8x^2 - 5x + 1 = (4x - 1)^2(3x + 1) \geq 0$ for $0 < x < 1$. So the claim is true. Then $f(a) + f(b) + f(c) + f(d) \geq 5(a + b + c + d)/8 - 4/8 = 1/8$, which is equivalent to the required inequality.

Example 2. (*2003 USA Math Olympiad*) Let a,b,c be positive real numbers. Prove that

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2}+\frac{(2b+c+a)^2}{2b^2+(c+a)^2}+\frac{(2c+a+b)^2}{2c^2+(a+b)^2}\leq 8.$$

Solution. Setting $a' = a/(a + b + c)$, $b' = b/(a + b + c)$, $c' = c/(a + b + c)$ if necessary, we may assume $0 < a, b, c < 1$ and $a + b + c = 1$. Then the first term on the left side of the inequality is equal to

$$f(a) = \frac{(a+1)^2}{2a^2+(1-a)^2} = \frac{a^2+2a+1}{3a^2-2a+1}.$$

(Note: When $a = b = c = 1/3$, there is equality. A simple sketch of $f(x)$ on $[0,1]$ shows the curve is below the tangent line at $x = 1/3$, which has the equation $y = (12x + 4)/3$.) So we claim that

$$\frac{a^2+2a+1}{3a^2-2a+1} \leq \frac{12a+4}{3}$$

for $0 < a < 1$. Multiplying out, we see this is equivalent to $36a^3 - 15a^2 - 2a + 1 \geq 0$ for $0 < a < 1$. (Note: Since the curve and the line intersect at $a = 1/3$, we expect $3a-1$ is a factor.) Indeed, $36a^3 - 15a^2 - 2a + 1 = (3a - 1)^2(4a + 1) \geq 0$ for $0 < a < 1$. Finally adding the similar inequality for b and c, we get the desired inequality.

The next example looks like the last example. However, it is much more sophisticated, especially without using tangent lines. The solution below is due to Titu Andreescu and Gabriel Dospinescu.

Example 3. (*1997 Japanese Math Olympiad*) Let a,b,c be positive real numbers. Prove that

$$\frac{(b+c-a)^2}{(b+c)^2+a^2}+\frac{(c+a-b)^2}{(c+a)^2+b^2}+\frac{(a+b-c)^2}{(a+b)^2+c^2}\geq\frac{3}{5}.$$

Solution. As in the last example, we may assume $0 < a, b, c < 1$ and $a + b + c = 1$. Then the first term on the left become $\frac{(1-2a)^2}{(1-a)^2+a^2} = 2 - \frac{2}{1+(1-2a)^2}$.

Next, let $x_1 = 1 - 2a$, $x_2 = 1 - 2b$, $x_3 = 1 - 2c$, then $x_1 + x_2 + x_3 = 1$, but $-1 < x_1, x_2, x_3 < 1$. In terms of x_1, x_2, x_3, the desired inequality is

$$\frac{1}{1+x_1^2}+\frac{1}{1+x_2^2}+\frac{1}{1+x_3^2}\leq\frac{27}{10}.$$

(Note: As in the last example, we consider the equation of the tangent line to $f(x) = 1/(1 + x^2)$ at $x = 1/3$, which is $y = 27(-x + 2)/50$.) So we claim that $f(x) \leq 27(-x + 2)/50$ for $-1 < x < 1$. This is equivalent to $(3x - 1)^2(4 - 3x) \geq 0$. Hence the claim is true for $-1 < x < 1$. Then $f(x_1) + f(x_2) + f(x_3) \leq 27/10$ and the desired inequality follows.

Schur's Inequality

Kin Yin Li

Sometimes in proving an inequality, we do not see any easy way. It will be good to know some brute force methods in such situation. In this article, we introduce a simple inequality that turns out to be very critical in proving inequalities by brute force.

Schur's Inequality. For any $x, y, z \ge 0$ and any real number r,

$$x^r(x-y)(x-z) + y^r(y-x)(y-z) + z^r(z-x)(z-y) \ge 0.$$

Equality holds if and only if $x = y = z$ or two of x, y, z are equal and the third is zero.

Proof. Observe that the inequality is symmetric in x, y, z. So without loss of generality, we may assume $x \ge y \ge z$. We can rewrite the left hand side as $x^r(x-y)^2+(x^r-y^r+z^r)(x-y)(y-z)+z^r(y-z)^2$. The first and third terms are clearly nonnegative. For the second term, if $r \ge 0$, then $x^r \ge y^r$. If $r < 0$, then $z^r \ge y^r$. Hence, $x^r-y^r+z^r \ge 0$ and the second term is nonnegative. So the sum of all three terms is nonnegative. In case $x \ge y \ge z$, equality holds if and only if $x = y$ first and z equals to them or zero.

In using the Schur's inequality, we often expand out expressions. So to simplify writing, we introduce the *symmetric sum* notation $\sum_{sym} f(x,y,z)$ to denote the sum of the six terms $f(x,y,z)$, $f(x,z,y)$, $f(y,z,x)$, $f(y,x,z)$, $f(z,x,y)$ and $f(z,y,x)$. In particular,

$$\sum_{sym} x^3 = 2x^3 + 2y^3 + 2z^3,$$

$$\sum_{sym} x^2y = x^2y+x^2z+y^2z+y^2x+z^2x+z^2y \text{ and}$$

$$\sum_{sym} xyz = 6xyz.$$

Similarly, for a function of n variables, the symmetric sum is the sum of all $n!$ terms, where we take all possible permutations of the n variables.

The $r = 1$ case of Schur's inequality is $x(x-y)(x-z) + y(y-x)(y-z) + z(z-x)(z-y) = x^3 + y^3 + z^3 - (x^2y + x^2z + y^2x + y^2z + z^2x + z^2y) + 3xyz \ge 0$. In symmetric sum notation, it is

$$\sum_{sym}(x^3 - 2x^2y + xyz) \ge 0.$$

By expanding both sides and rearranging terms, each of the following inequalities is equivalent to the $r = 1$ case of Schur's inequality. These are common disguises.

a) $x^3+y^3+z^3+3xyz \ge xy(x+y)+yz(y+z)+zx(z+x)$,

b) $xyz \ge (x+y-z)(y+z-x)(z+x-y)$,

c) $4(x+y+z)(xy+yz+zx) \le (x+y+z)^3+9xyz$.

Example 1. (*2000 IMO*) Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$(a-1+\frac{1}{b})(b-1+\frac{1}{c})(c-1+\frac{1}{a}) \le 1.$$

Solution. Let $x = a$, $y = 1$, $z = 1/b = ac$. Then $a = x/y$, $b = y/z$ and $c = z/x$. Substituting these into the desired inequality, we get

$$\frac{(x-y+z)}{y}\frac{(y-z+x)}{z}\frac{(z-x+y)}{x} \le 1,$$

which is disguise *b*) of the $r = 1$ case of Schur's inequality.

Example 2. (*1984 IMO*) Prove that

$$0 \le yz + zx + xy - 2xyz \le 7/27,$$

where x, y, z are nonnegative real numbers such that $x + y + z = 1$.

Solution. In Schur's inequality, all terms are of the same degree. So we first change the desired inequality to one where all terms are of the same degree. Since $x + y + z = 1$, the desired inequality is the same as

$$0 \le (x+y+z)(yz+zx+xy) - 2xyz \le \frac{7(x+y+z)^3}{27}.$$

Expanding the middle expression, we get $xyz + \sum_{sym} x^2y$, which is clearly nonnegative and the left inequality is proved. Expanding the rightmost expression and subtracting the middle expression, we get

$$\frac{7}{54}\sum_{sym}(x^3 - \frac{12}{7}x^2y + \frac{5}{7}xyz). \quad (1)$$

By Schur's inequality, we have

$$\sum_{sym}(x^3 - 2x^2y + xyz) \ge 0. \quad (2)$$

By the AM-GM inequality, we have

$$\sum_{sym} x^2y \ge 6(x^6y^6z^6)^{1/6} = \sum_{sym} xyz,$$

which is the same as

$$\sum_{sym}(x^2y - xyz) \ge 0. \quad (3)$$

Multiplying (3) by 2/7 and adding it to (2), we see the symmetric sum in (1) is nonnegative. So the right inequality is proved.

Example 3. (*2004 APMO*) Prove that

$$(a^2+2)(b^2+2)(c^2+2) \ge 9(ab+bc+ca)$$

for any positive real numbers a,b,c.

Solution. Expanding and expressing in symmetric sum notation, the desired inequality is

$$(abc)^2 + \sum_{sym}(a^2b^2+2a^2)+8 \ge \frac{9}{2}\sum_{sym} ab.$$

As $a^2+b^2 \ge 2ab$, we get $\sum_{sym} a^2 \ge \sum_{sym} ab$.

As $a^2b^2 + 1 \ge 2ab$, we get

$$\sum_{sym} a^2b^2 + 6 \ge 2\sum_{sym} ab.$$

Using these, the problem is reduced to showing

$$(abc)^2 + 2 \ge \sum_{sym}(ab - \frac{1}{2}a^2).$$

To prove this, we apply the AM-GM inequality twice and disguise c) of the $r = 1$ case of Schur's inequality as follow:

$$\begin{aligned}(abc)^2+2 &\ge 3(abc)^{2/3} \\ &\ge 9abc/(a+b+c) \\ &\ge 4(ab+bc+ca) - (a+b+c)^2 \\ &= 2(ab+bc+ca) - (a^2+b^2+c^2) \\ &= \sum_{sym}(ab - \frac{1}{2}a^2).\end{aligned}$$

Example 4. (*2000 USA Team Selection Test*) Prove that for any positive real numbers a, b, c, the following inequality holds

$$\frac{a+b+c}{3} - \sqrt[3]{abc} \le \max\{(\sqrt{a}-\sqrt{b})^2, (\sqrt{b}-\sqrt{c})^2, (\sqrt{c}-\sqrt{a})^2\}.$$

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***February 12, 2006.***

Problem 241. Determine the smallest possible value of

$$S = a_1 \cdot a_2 \cdot a_3 + b_1 \cdot b_2 \cdot b_3 + c_1 \cdot c_2 \cdot c_3,$$

if a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 is a permutation of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9. (*Source: 2002 Belarussian Math. Olympiad*)

Problem 242. Prove that for every positive integer n, 7 is a divisor of $3^n + n^3$ if and only if 7 is a divisor of $3^n n^3 + 1$. (*Source: 1995 Bulgarian Winter Math Competition*)

Problem 243. Let R^+ be the set of all positive real numbers. Prove that there is no function $f: R^+ \to R^+$ such that

$$(f(x))^2 \geq f(x+y)(f(x)+y)$$

for arbitrary positive real numbers x and y. (*Source: 1998 Bulgarian Math Olympiad*)

Problem 244. An infinite set S of coplanar points is given, such that every three of them are not collinear and every two of them are not nearer than 1cm from each other. Does there exist any division of S into two disjoint infinite subsets R and B such that inside every triangle with vertices in R is at least one point of B and inside every triangle with vertices in B is at least one point of R? Give a proof to your answer. (*Source: 2002 Albanian Math Olympiad*)

Problem 245. $ABCD$ is a concave quadrilateral such that $\angle BAD = \angle ABC = \angle CDA = 45^\circ$. Prove that $AC = BD$.

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Solutions

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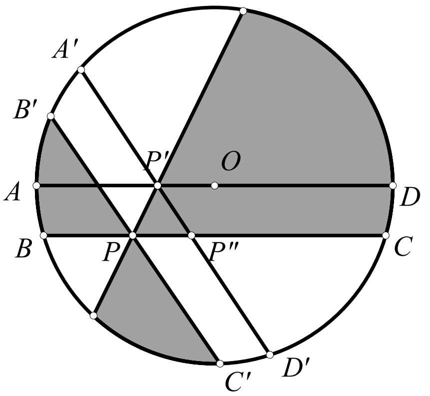
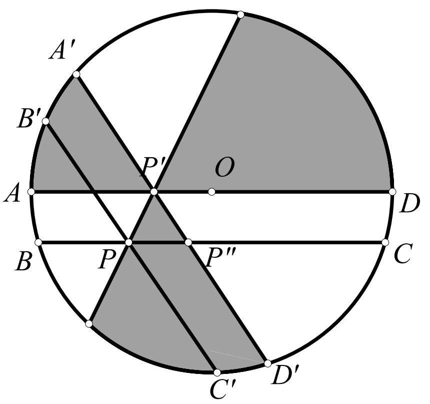
Problem 236. Alice and Barbara order a pizza. They choose an arbitrary point P, different from the center of the pizza and they do three straight cuts through P, which pairwise intersect at 60° and divide the pizza into 6 pieces. The center of the pizza is not on the cuts. Alice chooses one piece and then the pieces are taken clockwise by Barbara, Alice, Barbara, Alice and Barbara. Which piece should Alice choose first in order to get more pizza than Barbara? (*Source: 2002 Slovenian National Math Olympiad*)

Solution. (Official Solution)

Let Alice choose the piece that contains the center of the pizza first. We claim that the total area of the shaded regions below is greater than half of the area of the pizza.



Without loss of generality, we can assume the center of the pizza is at the origin O and one of the cuts is parallel to the x-axis (that is, BC is parallel to AD in the picture). Let P' be the intersection of the x-axis and the 60°-cut. Let $A'D'$ be parallel to the 120°-cut $B'C'$. Let P'' be the intersection of BC and $A'D'$. Then $\Delta PP'P''$ is equilateral. This implies the belts $ABCD$ and $A'B'C'D'$ have equal width. Since $AD > A'D'$, the area of the belt $ABCD$ is greater than the area of the belt $A'B'C'D'$. Now when the area of the belt $ABCD$ is subtracted from the total area of the shaded regions and the area of $A'B'C'D'$ is then added,



we get exactly half the area of the pizza. Therefore, the claim follows.

Problem 237. Determine (with proof) all polynomials p with real coefficients such that $p(x)\,p(x+1) = p(x^2)$ holds for every real number x. (*Source: 2000 Bulgarian Math Olympiad*)

Solution. **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 5).

Let $p(x)$ be such a polynomial. In case $p(x)$ is a constant polynomial, $p(x)$ must be 0 or 1. For the case $p(x)$ is nonconstant, let r be a root of $p(x)$. Then setting $x = r$ and $x + 1 = r$ in the equation, we see r^2 and $(r-1)^2$ are also roots of $p(x)$. Also, r^2 is a root implies $(r^2-1)^2$ is also a root. If $0 < |r| < 1$ or $|r| > 1$, then $p(x)$ will have infinitely many roots r, r^2, r^4, …, a contradiction. So $|r| = 0$ or 1 for every root r.

The case $|r| = 1$ and $|r-1| = 1$ lead to $r = (1 \pm i\sqrt{3})/2$, but then $|r^2 - 1| \neq 0$ or 1, a contradiction. Hence, either $|r| = 0$ or $|r-1| = 0$, that is, $r = 0$ or 1.

So $p(x) = x^m(x-1)^n$ for some nonnegative integers m, n. Putting this into the equation, we find $m = n$. Conversely, $p(x) = x^m(x-1)^m$ is easily checked to be a solution for every nonnegative integer m.

Problem 238. For which positive integers n, does there exist a permutation $(x_1, x_2, \ldots, x_n)$ of the numbers 1, 2, …, n such that the number $x_1 + x_2 + \cdots + x_k$ is divisible by k for every $k \in \{1,2, \ldots, n\}$? (*Source: 1998 Nordic Mathematics Contest*)

Solution. **G.R.A. 20 Math Problem Group** (Roma, Italy), **LEE Kai Seng** (HKUST), **LO Ka Wai** (Carmel Divine Grace Foundation Secondary School, Form 7), **Anna Ying PUN** (STFA Leung Kau Kui College, Form 7) and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 5).

For a solution n, since $x_1 + x_2 + \cdots + x_n = n(n+1)/2$ is divisible by n, n must be odd. The cases $n = 1$ and $n = 3$ (with permutation (1,3,2)) are solutions.

Assume $n \geq 5$. Then $x_1 + x_2 + \cdots + x_{n-1} = n(n+1)/2 - x_n \equiv 0 \pmod{n-1}$ implies $x_n \equiv (n+1)/2 \pmod{n-1}$. Since $1 \leq x_n \leq n$ and $3 \leq (n+1)/2 \leq n-2$, we get $x_n = (n+1)/2$. Similarly, $x_1 + x_2 + \cdots + x_{n-2} = n(n+1)/2 - x_n - x_{n-1} \equiv 0 \pmod{n-2}$ implies $x_{n-1} \equiv (n+1)/2 \pmod{n-2}$. Then also $x_{n-1} = (n+1)/2$, which leads to $x_n = x_{n-1}$, a contradiction. Therefore, $n = 1$ and 3 are the only solutions.

Problem 239. (*Due to José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain*) In any acute triangle *ABC*, prove that

$$\cos\left(\frac{A-B}{2}\right)+\cos\left(\frac{B-C}{2}\right)+\cos\left(\frac{C-A}{2}\right)$$

$$\le\frac{\sqrt{2}}{2}\left(\frac{a+b}{\sqrt{a^2+b^2}}+\frac{b+c}{\sqrt{b^2+c^2}}+\frac{c+a}{\sqrt{c^2+a^2}}\right).$$

Solution. (Proposer's Solution)

By cosine law and the *AM-GM* inequality,

$$1-2\sin^2\frac{A}{2}=\cos A=\frac{b^2+c^2-a^2}{2bc}$$

$$\ge\frac{b^2+c^2-a^2}{b^2+c^2}=1-\frac{a^2}{b^2+c^2}.$$

So $\sin\frac{A}{2}\le\frac{a}{\sqrt{2(b^2+c^2)}}$.

By sine law and $\cos(A/2) = \sin((B+C)/2)$, we get

$$\frac{a}{b+c}=\frac{\sin A}{\sin B+\sin C}=$$

$$\frac{2\sin(A/2)\cos(A/2)}{2\sin(\frac{B+C}{2})\cos(\frac{B-C}{2})}=\frac{\sin(A/2)}{\cos(\frac{B-C}{2})}.$$

Then

$$\cos(\frac{B-C}{2})=\frac{b+c}{a}\sin\frac{A}{2}\le\frac{\sqrt{2}}{2}\frac{b+c}{\sqrt{b^2+c^2}}.$$

Adding two similar inequalities, we get the desired inequality.

Commended solvers: **Anna Ying PUN** (STFA Leung Kau Kui College, Form 7) and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 5).

Problem 240. Nine judges independently award the ranks of 1 to 20 to twenty figure-skaters, with no ties. No two of the rankings awarded to any figure-skater differ by more than 3. The nine rankings of each are added. What is the maximum of the lowest of the sums? Prove your answer is correct. (*Source: 1968 All Soviet Union Math Competitions*)

Solution. **WONG Kwok Kit** (Carmel Divine Grace Foundation Secondary School, Form 7) and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 5).

Suppose the 9 first places go to the same figure skater. Then 9 is the lowest sum.

Suppose the 9 first places are shared by two figure skaters. Then one of them gets at least 5 first places and that skater's other rankings are no worse than fourth places. So the lowest sum is at most $5 \times 1 + 4 \times 4 = 21$.

Suppose the 9 first places are shared by three figure skaters. Then the other 18 rankings of these figure skaters are no worse than 9 third and 9 fourth places. Then the lowest sum is at most $9(1 + 3 + 4)/3 = 24$.

Suppose the 9 first places are shared by four figure skaters. Then their rankings must be all the first, second, third and fourth places. So the lowest sum is at most $9(1 + 2 + 3 + 4)/4 < 24$.

Suppose the 9 first places are shared by $k > 4$ figure skaters. On one hand, these k skaters have a total of $9k > 36$ rankings. On the other hand, these k skaters can only be awarded first to fourth places, so they can have at most $4 \times 9 = 36$ rankings all together, a contradiction.

Now 24 is possible if skaters *A, B, C* all received 3 first, 3 third and 3 fourth places; skater *D* received 5 second and 4 fifth places; skater *E* received 4 second and 5 fifth places; and skater *F* received 9 sixth places, …, skater *T* received 9 twentieth places. Therefore, 24 is the answer.

Olympiad Corner

(continued from page 1)

Problem 4. (Cont.) it is possible to distribute the balls under the condition that *A* gets the same number of balls as the persons *B, C, D* and *E* together.

Problem 5. Let *ABCD* be a given convex quadrilateral. Determine the locus of the point *P* lying inside the quadrilateral *ABCD* and satisfying

$$[PAB]\cdot[PCD] = [PBC]\cdot[PDA],$$

where $[XYZ]$ denotes the area of triangle *XYZ*.

Problem 6. Determine all pairs of integers (x,y) satisfying the equation

$$y(x + y) = x^3 - 7x^2 + 11x - 3.$$

Schur's Inequality

(continued from page 2)

Solution. From the last part of the solution of example 3, we get

$$3(xyz)^{2/3} \ge 2(xy + yz + zx) - (x^2 + y^2 + z^2)$$

for any $x, y, z > 0$. (*Note*: this used Schur's inequality.) Setting

$$x = \sqrt{a},\ y = \sqrt{b} \text{ and } z = \sqrt{c}$$

and arranging terms, we get

$$a + b + c - 3\sqrt[3]{abc}$$

$$\le 2(a+b+c-\sqrt{ab}-\sqrt{bc}-\sqrt{ca})$$

$$= (\sqrt{a}-\sqrt{b})^2+(\sqrt{b}-\sqrt{c})^2+(\sqrt{c}-\sqrt{a})^2$$

$$\le 3\max\{(\sqrt{a}-\sqrt{b})^2,(\sqrt{b}-\sqrt{c})^2,(\sqrt{c}-\sqrt{a})^2\}.$$

Dividing by 3, we get the desired inequality.

Example 5. (*2003 USA Team Selection Test*) Let *a,b,c* be real numbers in the interval $(0, \pi/2)$. Prove that

$$\frac{\sin a\sin(a-b)\sin(a-c)}{\sin(b+c)}+\frac{\sin b\sin(b-c)\sin(b-a)}{\sin(c+a)}$$

$$+\frac{\sin c\sin(c-a)\sin(c-b)}{\sin(a+b)}\ge 0.$$

Solution. Observe that

$$\sin(u - v)\sin(u + v) = (\cos 2v - \cos 2u)/2$$

$$= \sin^2 u - \sin^2 v.$$

Setting $x = \sin^2 a$, $y = \sin^2 b$, $z = \sin^2 c$, in adding up the terms, the left side of the inequality becomes

$$\frac{\sqrt{x}(x-y)(x-z)+\sqrt{y}(y-z)(y-x)+\sqrt{z}(z-x)(z-y)}{\sin(b+c)\sin(c+a)\sin(a+b)}.$$

This is nonnegative by the $r = 1/2$ case of Schur's inequality.

For many more examples on Schur's and other inequalities, we highly recommend the following book.

Titu Andreescu, Vasile Cîrtoaje, Gabriel Dospinescu and Mircea Lascu, Old and New Inequalities, GIL Publishing House, 2004.

Anyone interested may contact the publisher by post to GIL Publishing House, P. O. Box 44, Post Office 3, 450200, Zalau, Romania or by email to gil1993@zalau.astral.ro.