Mathematical Excalibur

Volume 11, Number 4 *November 2006 – December 2006*

Olympiad Corner

The 9th China Hong Kong Math Olympiad was held on Dec. 2, 2006. The following were the problems.

Problem 1. Let M be a subset of $\{1, 2, \ldots, 2006\}$ with the following property: For any three elements x, y and z ($x < y < z$) of M, $x + y$ does not divide z. Determine the largest possible size of M. Justify your claim.

Problem 2. For a positive integer k, let $f_1(k)$ be the square of the sum of the digits of k. (For example $f_1(123) = (1+2+3)^2 = 36$.) Let $f_{n+1}(k) = f_1(f_n(k))$. Determine the value of $f_{2007}(2^{2006})$. Justify your claim.

Problem 3. A convex quadrilateral $ABCD$ with $AC \neq BD$ is inscribed in a circle with center O. Let E be the intersection of diagonals AC and BD. If P is a point inside $ABCD$ such that

$$\angle PAB + \angle PCB = \angle PBC + \angle PDC = 90^\circ,$$

prove that O, P and E are collinear.

Problem 4. Let a_1, a_2, a_3,… be a sequence of positive numbers. If there exists a positive number M such that for every $n = 1, 2, 3, \ldots$,

$$a_1^2 + a_2^2 + \ldots + a_n^2 < M a_{n+1}^2,$$

then prove that there exists a positive number M' such that for every $n = 1, 2, 3, \ldots$,

$$a_1 + a_2 + \ldots + a_n < M' a_{n+1}.$$

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

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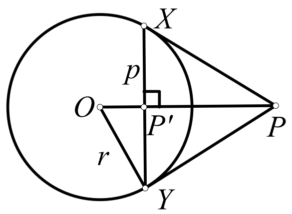
Pole and Polar

Kin Y. Li

Let C be a circle with center O and radius r. Recall the inversion with respect to C (see <u>*Mathematical Excalibur*</u>, *vol*. 9, *no*. 2, *p*.1) sends every point $P \neq O$ in the same plane as C to the image point P' on the ray $\overrightarrow{OP}$ such that $OP \cdot OP' = r^2$. The <u>*polar*</u> of P is the line p perpendicular to the line OP at P'. Conversely, for any line p not passing through O, the <u>*pole*</u> of p is the point P whose polar is p. The function sending P to p is called the <u>*pole-polar transformation*</u> (or <u>*reciprocation*</u>) with respect to O and r (or with respect to C).



Following are some useful facts:

(1) If P is outside C, then recall P' is found by drawing tangents from P to C, say tangent at X and Y. Then $P' = OP \cap XY$, where $\cap$ denotes intersection. By symmetry, $OP \perp XY$. So the polar p of P is the line XY.

Conversely, for distinct points X, Y on C, the pole of the line XY is the intersection of the tangents at X and Y. Also, it is the point P on the perpendicular bisector of XY such that O, X, P, Y are concyclic since $\angle OXP = 90^\circ = \angle OYP$.

(2) (**<u>*La Hire's Theorem*</u>**) Let x and y be the polars of X and Y, respectively. Then X is on line $y \Leftrightarrow Y$ is on line x.

<u>*Proof.*</u> Let X', Y' be the images of X, Y for the inversion with respect to C. Then $OX \cdot OX' = r^2 = OY \cdot OY'$ implies X, X', Y, Y' are concyclic. Now

$$X \text{ is on } y \Leftrightarrow \angle XY'Y = 90^\circ$$
$$\Leftrightarrow \angle XX'Y = 90^\circ$$
$$\Leftrightarrow Y \text{ is on } x.$$

(3) Let x,y,z be the polars of distinct points X,Y,Z respectively. Then $Z = x \cap y \Leftrightarrow z = XY$.

<u>*Proof.*</u> By La Hire's theorem, Z on $x \cap y$ $\Leftrightarrow$ X on z and Y on $z \Leftrightarrow z = XY$.

(4) Let W, X, Y, Z be on C. The polar p of $P = XY \cap WZ$ is the line through $Q = WX \cap ZY$ and $R = XZ \cap YW$.

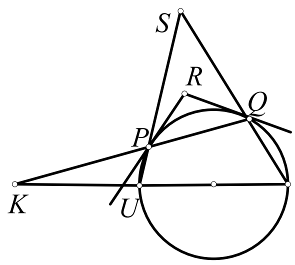
<u>*Proof.*</u> Let S, T be the poles of $s = XY$, $t = WZ$ respectively. Then $P = s \cap t$. By fact (3), $S = x \cap y$, $T = w \cap z$ and $p = ST$. For hexagon $WXXZYY$, we have

$$Q = WX \cap ZY, \quad S = XX \cap YY, \quad R = XZ \cap YW,$$

where XX denotes the tangent line at X. By Pascal's theorem (see <u>*Mathematical Excalibur*</u>, *vol*. 10, *no*. 3, *p*.1), Q,S,R are collinear. Similarly, considering the hexagon $XWWYZZ$, we see Q,T,R are collinear. Therefore, $p = ST = QR$.

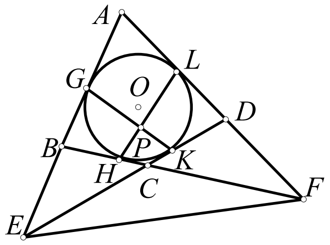
Next we will present some examples using the pole-polar transformation.

<u>Example 1.</u> Let UV be a diameter of a semicircle. P,Q are two points on the semicircle with $UP < UQ$. The tangents to the semicircle at P and Q meet at R. If $S = UP \cap VQ$, then prove that $RS \perp UV$.



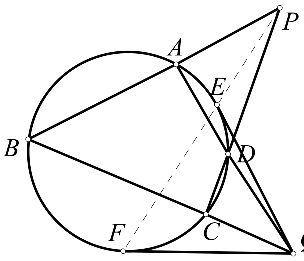
<u>Solution (due to CHENG Kei Tsi).</u> Let $K = PQ \cap UV$. With respect to the circle, by fact (4), the polar of K passes through $UP \cap VQ = S$. Since the tangents to the semicircle at P and Q meet at R, by fact (1), the polar of R is PQ. Since K is on line PQ, which is the polar of R, by La Hire's theorem, R is on the polar of K. So the polar of K is the line RS. As K is on the diameter UV extended, by the definition of polar, we get $RS \perp UV$.

Example 2. Quadrilateral *ABCD* has an inscribed circle *Γ* with sides *AB, BC, CD, DA* tangent to *Γ* at *G, H, K, L* respectively. Let $AB\cap CD=E$, $AD\cap BC=F$ and $GK\cap HL=P$. If *O* is the center of *Γ*, then prove that $OP\perp EF$.



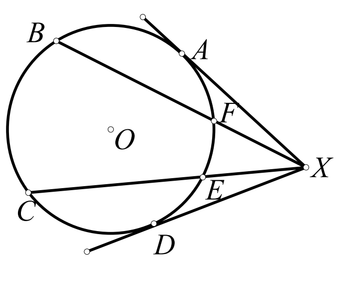
Solution. Consider the pole-polar transformation with respect to the inscribed circle. By fact (1), the polars of *E, F* are lines *GK, HL* respectively. Since $GK\cap HL=P$, by fact (3), the polar of *P* is line *EF*. By the definition of polar, we get $OP\perp EF$.

Example 3. (*1997 Chinese Math Olympiad*) Let *ABCD* be a cyclic quadrilateral. Let $AB\cap CD=P$ and $AD\cap BC=Q$. Let the tangents from *Q* meet the circumcircle of *ABCD* at *E* and *F*. Prove that *P, E, F* are collinear.



Solution. Consider the pole-polar transformation with respect to the circumcircle of *ABCD*. Since $P=AB\cap CD$, by fact (4), the polar of *P* passes through $AD\cap BC=Q$. By La Hire's theorem, *P* is on the polar of *Q*, which by fact (1), is the line *EF*.

Example 4. (*1998 Austrian-Polish Math Olympiad*) Distinct points *A, B, C, D, E, F* lie on a circle in that order. The tangents to the circle at the points *A* and *D*, the lines *BF* and *CE* are concurrent. Prove that the lines *AD, BC, EF* are either parallel or concurrent.

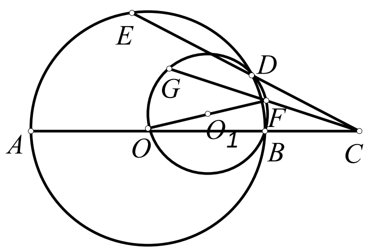


Solution. Let *O* be the center of the circle and $X=AA\cap DD\cap BF\cap CE$.

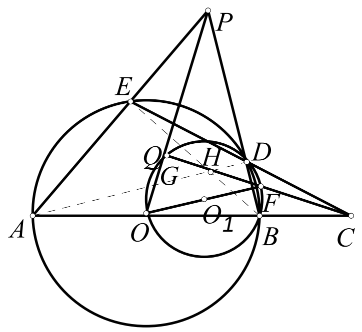
If $BC\|EF$, then by symmetry, lines *BC* and *EF* are perpendicular to line *OX*. Since $AD\perp OX$, we get $BC\|EF\|AD$.

If lines *BC, EF* intersect, then by fact (4), the polar of $X=CE\cap BF$ passes through $BC\cap EF$. Since the tangents at *A* and *D* intersect at *X*, by fact (1), the polar of *X* is line *AD*. Therefore, *AD*, *BC* and *EF* are concurrent in this case.

Example 5. (*2006 China Western Math Olympiad*) As in the figure below, *AB* is a diameter of a circle with center *O*. *C* is a point on *AB* extended. A line through *C* cuts the circle with center *O* at *D, E*. *OF* is a diameter of the circumcircle of ΔBOD with center O_1. Line *CF* intersect the circumcircle again at *G*. Prove that *O,A,E,G* are concyclic.



Solution (due to WONG Chiu Wai). Let $AE\cap BD=P$. By fact (4), the polar of *P* with respect to the circle having center *O* is the line through $BA\cap DE=C$ and $AD\cap EB=H$. Then $OP\perp CH$. Let $Q=OP\cap CH$.



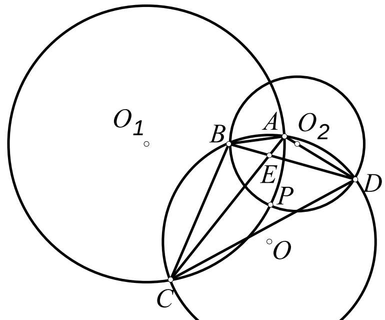
We claim $Q=G$. Once this shown, we will have $P=BD\cap OG$. Then $PE\cdot PA=PD\cdot PB=PG\cdot PO$, which implies *O, A, E, G* are concyclic.

To show $Q=G$, note that $\angle PQH$, $\angle PDH$ and $\angle PEH$ are 90°, which implies *P, E, Q, H, D* are concyclic. Then $\angle PQD=\angle PED=\angle DBO$, which implies *Q, D, B, O* are concyclic. Therefore, $Q=G$ since they are both the point of intersection (other than *O*) of the circumcircle of ΔBOD and the circle with diameter *OC*.

Example 6. (*2006 China Hong Kong Math Olympiad*) A convex quadrilateral *ABCD* with $AC\neq BD$ is inscribed in a circle with center *O*. Let *E* be the intersection of diagonals *AC* and *BD*. If *P* is a point inside *ABCD* such that

$$\angle PAB+\angle PCB=\angle PBC+\angle PDC=90^\circ,$$

prove that *O*, *P* and *E* are collinear.



Solution (due to WONG Chiu Wai). Let Γ, Γ_1, Γ_2 be the circumcircles of quadrilateral *ABCD*, ΔPAC, ΔPBD with centers *O*, O_1, O_2 respectively. We first show that the polar of O_1 with respect to Γ is line *AC*. Since OO_1 is the perpendicular bisector of *AC*, by fact (1), all we need to show is that

$$\angle AOC+\angle AO_1C=180^\circ.$$

For this, note

$$\begin{aligned}&\angle APC\\&=360^\circ-(\angle PAB+\angle PCB+\angle ABC)\\&=270^\circ-\angle ABC\\&=90^\circ+\angle ADC\end{aligned}$$

and so

$$\begin{aligned}\angle AO_1C&=2(180^\circ-\angle APC)\\&=2(90^\circ-\angle ADC)\\&=180^\circ-2\angle ADC\\&=180^\circ-\angle AOC.\end{aligned}$$

Similarly, the polar of O_2 with respect to Γ is line *BD*. By fact (3), since $E=AC\cap BD$, the polar of *E* with respect to Γ is line O_1O_2. So $OE\perp O_1O_2$.

(Next we will consider radical axis and radical center, see Mathematical Excalibur, *vol.* 4, *no.* 3, *p.* 2.) Among Γ, Γ_1, Γ_2, two of the pairwise radical axes are lines *AC* and *BD*. This implies *E* is the radical center. Since Γ_1, Γ_2 intersect at *P*, so *PE* is the radical axis of Γ_1, Γ_2, which implies $PE\perp O_1O_2$. Combining with $OE\perp O_1O_2$ proved above, we see *O*, *P* and *E* are collinear.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***January 25, 2007.***

Problem 261. Prove that among any 13 consecutive positive integers, one of them has sum of its digits (in base 10) divisible by 7.

Problem 262. Let O be the center of the circumcircle of ΔABC and let AD be a diameter. Let the tangent at D to the circumcircle intersect line BC at P. Let line PO intersect lines AC, AB at M, N respectively. Prove that $OM = ON$.

Problem 263. For positive integers m, n, consider a $(2m+1)\times(2n+1)$ table, where in each cell, there is exactly one ant. At a certain moment, every ant moves to a horizontal or vertical neighboring cell. Prove that after that moment, there exists a cell with no ant.

Problem 264. For a prime number $p > 3$ and arbitrary integers a, b, prove that $ab^p - ba^p$ is divisible by $6p$.

Problem 265. Determine (with proof) the maximum of

$$\sum_{j=1}^{n}(x_j^4 - x_j^5),$$

where $x_1, x_2, \ldots, x_n$ are nonnegative real numbers whose sum is 1.

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Solutions

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Problem 256. Show that there is a rational number q such that

$$\sin 1^\circ \sin 2^\circ \cdots \sin 89^\circ \sin 90^\circ = q\sqrt{10}.$$

Solution 1. **Jeff CHEN** (Virginia, USA), **Koyrtis G. CHRYSSOSTOMOS** (Larissa, Greece, teacher), **G.R.A. 20 Math Problem Group** (Roma, Italy) and **D. Kipp JOHNSON** (Valley Catholic School, Teacher, Beaverton, Oregon, USA).

Let $\omega = e^{2\pi i/180}$. Then

$$P(z) = \sum_{n=0}^{179} z^n = \prod_{k=1}^{179}(z - \omega^k).$$

Using $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i} = \dfrac{e^{2ix} - 1}{2ie^{ix}}$, we have

$$\prod_{k=1}^{90} \sin k^\circ = \prod_{k=1}^{90} \frac{\omega^k - 1}{2i\omega^{k/2}}.$$

Also, $\displaystyle\prod_{k=1}^{90} \sin k^\circ = \prod_{k=91}^{179} \sin k^\circ = \prod_{k=91}^{179} \frac{\omega^k - 1}{2i\omega^{k/2}}.$

Then

$$\left|\prod_{k=1}^{90} \sin k^\circ\right|^2 = \prod_{k=1}^{179} \frac{|\omega^k - 1|}{2} = \frac{|P(1)|}{2^{179}} = \frac{90}{2^{178}}.$$

Therefore, $\displaystyle\prod_{k=1}^{90} \sin k^\circ = \frac{3}{2^{89}}\sqrt{10}.$

Solution 2. **Jeff CHEN** (Virginia, USA), **Koyrtis G. CHRYSSOSTOMOS** (Larissa, Greece, teacher).

Let S be the left-handed side. Note

$$\begin{aligned}\sin 3\theta &= \sin\theta\cos 2\theta + \cos\theta\sin 2\theta\\ &= \sin\theta(\cos^2\theta - \sin^2\theta + 2\cos^2\theta)\\ &= 4\sin\theta\left(\frac{3}{4}\cos^2\theta - \frac{1}{4}\sin^2\theta\right)\\ &= 4\sin\theta\sin(60^\circ - \theta)\sin(60^\circ + \theta).\end{aligned}$$

So, $\sin\theta\sin(60^\circ - \theta)\sin(60^\circ + \theta) = \dfrac{\sin 3\theta}{4}.$

Using this, we have

$$\begin{aligned}S &= \sin 30^\circ \sin 60^\circ \prod_{n=1}^{29} \sin n^\circ \sin(60^\circ - n^\circ)\sin(60^\circ + n^\circ)\\ &= \frac{\sqrt{3}}{4^{30}} \sin 3^\circ \sin 6^\circ \sin 9^\circ \cdots \sin 87^\circ\\ &= \frac{\sqrt{3}}{4^{30}} \sin 30^\circ \sin 60^\circ \prod_{m=1}^{9} \sin 3m^\circ \sin(60^\circ - 3m^\circ)\sin(60^\circ + 3m^\circ)\\ &= \frac{3}{4^{40}} \sin 9^\circ \sin 18^\circ \sin 27^\circ \cdots \sin 81^\circ\\ &= \frac{3}{4^{40}} \sin 9^\circ \cos 9^\circ \sin 18^\circ \cos 18^\circ \sin 27^\circ \cos 27^\circ \sin 36^\circ \cos 36^\circ \sin 45^\circ\\ &= \frac{3\sqrt{2}}{2^{85}} \sin 18^\circ \sin 36^\circ \sin 54^\circ \sin 72^\circ\\ &= \frac{3\sqrt{2}}{2^{85}} \sin 18^\circ \cos 18^\circ \sin 36^\circ \cos 36^\circ\\ &= \frac{3\sqrt{2}}{2^{87}} \sin 36^\circ \sin 72^\circ\\ &= \frac{3\sqrt{2}}{2^{87}} \frac{\sqrt{10-2\sqrt{5}}}{4} \frac{\sqrt{10+2\sqrt{5}}}{4} = \frac{3}{2^{89}}\sqrt{10}.\end{aligned}$$

Problem 257. Let $n > 1$ be an integer. Prove that there is a unique positive integer $A < n^2$ such that $[n^2/A] + 1$ is divisible by n, where $[x]$ denotes the greatest integer less than or equal to x. (*Source: 1993 Jiangsu Math Contest*)

Solution. **Jeff CHEN** (Virginia, USA), **G.R.A. 20 Math Problem Group** (Roma, Italy) and **Fai YUNG**.

We claim the unique number is $A = n+1$. If $n = 2$, then $1 \le A < n^2 = 4$ and only $A = 3$ works. If $n > 2$, then $[n^2/A]+1$ divisible by n implies $\dfrac{n^2}{A} + 1 \ge \left[\dfrac{n^2}{A}\right] + 1 \ge n.$ This leads to

$A \le \dfrac{n^2}{n-1} = n + 1 + \dfrac{1}{n-1}.$ So $A \le n + 1$.

The case $A = n + 1$ works because

$$\left[\frac{n^2}{n+1}\right] + 1 = (n-1) + 1 = n.$$

The case $A = n$ does not work because $[n^2/n] + 1 = n + 1$ is not divisible by n when $n > 1$.

For $0 < A < n$, assume $[n^2/A]+1 = kn$ for some positive integer k. This leads to

$$kn - 1 = \left[\frac{n^2}{A}\right] \le \frac{n^2}{A} < \left[\frac{n^2}{A}\right] + 1 = kn,$$

which implies $n < kA \le (n^2+A)/n < n+1$. This is a contradiction as kA is an integer and cannot be strictly between n and $n + 1$.

Problem 258. (*Due to Mihaiela Vizental and Alfred Eckstein, Arad, Romaina*) Show that if A, B, C are in the interval $(0, \pi/2)$, then

$$f(A,B,C) + f(B,C,A) + f(C,A,B) \ge 3,$$

where

$$f(x,y,z) = \frac{4\sin x + 3\sin y + 2\sin z}{2\sin x + 3\sin y + 4\sin z}.$$

Solution. **Samuel Liló Abdalla** (Brazil), **Koyrtis G. CHRYSSOSTOMOS** (Larissa, Greece, teacher) and **Fai YUNG**.

Note

$$f(x,y,z) + 1 = \frac{6\sin x + 6\sin y + 6\sin z}{2\sin x + 3\sin y + 4\sin z}.$$

For a, b, $c > 0$, by the *AM-HM* inequality, we have

$$(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \ge 9.$$

Multiplying by $\frac{2}{3}$ on both sides, we get

$$(a + b + c)\frac{2}{3}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \ge 6. \quad (*)$$

Let $r = \sin A$, $s = \sin B$, $t = \sin C$, $a = 1/(2r + 3s + 4t)$, $b = 1/(2s + 3t + 4r)$ and

$c = 1/(2t + 3r + 4s)$. Then

$$\frac{2}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=6r+6s+6t.$$

Using (*), we get

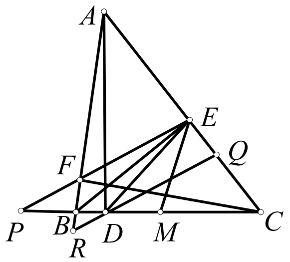
$$f(A,B,C)+f(B,C,A)+f(C,A,B)+3$$
$$=\frac{6r+6s+6t}{2r+3s+4t}+\frac{6r+6s+6t}{2s+3t+4r}+\frac{6r+6s+6t}{2t+3r+4s}$$
$$\geq 6.$$

The result follows.

Problem 259. Let AD, BE, CF be the altitudes of acute triangle ABC. Through D, draw a line parallel to line EF intersecting line AB at R and line AC at Q. Let P be the intersection of lines EF and CB. Prove that the circumcircle of ΔPQR passes through the midpoint M of side BC.

(*Source: 1994 Hubei Math Contest*)

Solution. **Jeff CHEN** (Virginia, USA).



Observe that

(1) $\angle BFC = 90^\circ = \angle BEC$ implies B, F, E, C concyclic;

(2) $\angle AEB = 90^\circ = \angle ADB$ implies A,B,D,E concyclic.

By (1), we have $\angle ACB = \angle AFE$. From $EF||QR$, we get $\angle AFE = \angle ARQ$. So $\angle ACB = \angle ARQ$. Then B, Q, R, C are concyclic. By the intersecting chord theorem,

$$RD\cdot QD=BD\cdot CD \qquad (*)$$

Since $\angle BEC = 90^\circ$ and M is the midpoint of BC, we get $MB = ME$ and $\angle EBM = \angle BEM$. Now

$$\angle EBM = \angle EPM + \angle BEP$$
$$\angle BEM = \angle DEM + \angle BED.$$

By (1) and (2), $\angle BEP = \angle BCF = 90^\circ - \angle ABC = \angle BAD = \angle BED$. So $\angle EPM = \angle DEM$. Then right triangles EPM and DEM are similar. We have $ME/MP = MD/ME$ and so

$$MB^2=ME^2=MD\cdot MP=MD(MD+PD)$$
$$=MD^2+MD\cdot PD.$$

Then
$$MD\cdot PD = MB^2 - MD^2$$
$$= (MB-MD)(MB+MD)$$
$$= BD(MC+MD)$$
$$= BD\cdot CD.$$

Using (*), we get $RD\cdot QD=MD\cdot PD$. By the converse of the intersecting chord theorem, P, Q, R, M are concyclic.

Commended solvers: **Koyrtis G. CHRYSSOSTOMOS** (Larissa, Greece, teacher).

Problem 260. In a class of 30 students, number the students 1, 2, …, 30 from best to worst ability (no two with the same ability). Every student has the same number of friends in the class, where friendships are mutual. Call a student <u>*good*</u> if his ability is better than more than half of his friends. Determine the maximum possible number of good students in this class.
(*Source: 1998 Hubei Math Contest*)

Solution. **Jeff CHEN** (Virginia, USA) and **Fai YUNG**.

Suppose each student has m friends and n is the maximum number of good students. There are $15m$ pairs of friendship.

For m odd, $m = 2k - 1$ for some positive integer k. For $j = 1, 2, …, k$, student j has at least $(2k-j) \geq k > m/2$ worse friends, hence student j is good. For the other $n - k$ good students, every one of them has at least k worse friends. Then

$$\sum_{j=1}^{k}(2k-j)+(n-k)k \leq 15(2k-1).$$

Solving for n, we get

$$n \leq 30.5-\left(\frac{15}{k}+\frac{k}{2}\right)\leq 30.5-\sqrt{30}<26.$$

For m even, $m = 2k$ for some positive integer k. For $j = 1, 2, …, k$, student j has at least $(2k + 1 - j) > k = m/2$ worse friends, hence student j is good. For the other $n - k$ good students, every one of them has at least $k + 1$ worse friends. Then

$$\sum_{j=1}^{k}(2k+1-j)+(n-k)(k+1)\leq 15\cdot 2k.$$

Solving for n, we get

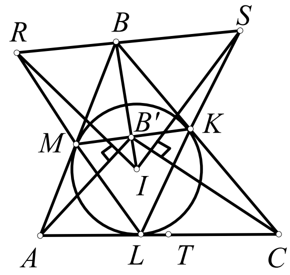
$$n \leq 31.5-\left(\frac{31}{k+1}+\frac{k+1}{2}\right)\leq 31.5-\sqrt{62}<24.$$

Therefore, $n \leq 25$. For an example of $n = 25$, in the odd case, we need to take $k = 5$ (so $m = 9$). Consider the 6×5 matrix M with $M_{ij}= 5(i-1)+j$. For M_{1j}, let his friends be M_{6j}, M_{1k} and M_{2k} for all $k \neq j$. For M_{ij} with $1 < i < 6$, let his friends be M_{6j}, $M_{(i-1)k}$ and $M_{(i+1)k}$ for all $k \neq j$. For M_{6j}, let his friends be M_{ij} and M_{5k} for all $i < 6$ and $k \neq j$. It is easy to check 1 to 25 are good.

Pole and Polar

(continued from page 2)

<u>***Example 7.***</u> (*1998 IMO*) Let I be the incenter of triangle ABC. Let the incircle of ABC touch the sides BC, CA and AB at K, L and M respectively. The line through B parallel to MK meets the lines LM and LK at R and S respectively. Prove that angle RIS is acute.



<u>***Solution.***</u> Consider the pole-polar transformation with respect to the incircle. Due to tangency, the polars of B, K, L, M are lines MK, BC, CA, AB respectively. Observe that B is sent to $B' = IB\cap MK$ under the inversion with respect to the incircle. Since B' is on line MK, which is the polar of B, by La Hire's theorem, B is on the polar of B'. Since $MK||RS$, so the polar of B' is line RS. Since R,B,S are collinear, their polars concur at B'.

Next, since the polars of K, L intersect at C and since L, K, S are collinear, their polars concur at C. Then the polar of S is $B'C$. By the definition of polar, we get $IS\perp B'C$. By a similar reasoning, we also get $IR\perp B'A$. Then $\angle RIS = 180^\circ - \angle AB'C$.

To finish, we will show B' is inside the circle with diameter AC, which implies $\angle AB'C > 90^\circ$ and hence $\angle RIS < 90^\circ$. Let T be the midpoint of AC. Then

$$2\overrightarrow{B'T} = \overrightarrow{B'C}+\overrightarrow{B'A}$$
$$=(\overrightarrow{B'K}+\overrightarrow{KC})+(\overrightarrow{B'M}+\overrightarrow{MA})$$
$$=\overrightarrow{KC}+\overrightarrow{MA}.$$

Since $\overrightarrow{KC}$ and $\overrightarrow{MA}$ are nonparallel,

$$B'T<\frac{KC+MA}{2}=\frac{CL+AL}{2}=\frac{AC}{2}.$$

Therefore, B' is inside the circle with diameter AC.