Mathematical Excalibur

Volume 11, Number 5 *January 2007 – February 2007*

Olympiad Corner

Below are the 2006 British Math Olympiad (Round 2) problems.

Problem 1. Find the minimum possible value of $x^2 + y^2$ given that x and y are real numbers satisfying $xy(x^2 - y^2) = x^2 + y^2$ and $x \neq 0$.

Problem 2. Let x and y be positive integers with no prime factors larger than 5. Find all such x and y which satisfy $x^2 - y^2 = 2^k$ for some non-negative integer k.

Problem 3. Let ABC be a triangle with $AC > AB$. The point X lies on the side BA extended through A, and the point Y lies on the side CA in such a way that $BX = CA$ and $CY = BA$. The line XY meets the perpendicular bisector of side BC at P. Show that $\angle BPC + \angle BAC = 180°$.

Problem 4. An exam consisting of six questions is sat by 2006 children. Each question is marked right or wrong. Any three children have right answers to at least five of the six questions between them. Let N be the total number of right answers achieved by all the children (i.e. the total number of questions solved by child 1 + the total solved by child 2 + ··· + the total solved by child 2006). Find the least possible value of N.

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***March 25, 2007***.

For individual subscription for the next five issues for the 05-06 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Difference Operator

Kin Y. Li

Let h be a nonzero real number and $f(x)$ be a function. When $f(x+h)$ and $f(x)$ are real numbers, we call

$$\Delta_h f(x) = f(x+h) - f(x)$$

the *first difference of f at x with step h*. For functions f, g and real number c, we have

$$\Delta_h(f+g)(x) = \Delta_h f(x) + \Delta_h g(x) \text{ and}$$
$$\Delta_h(cf)(x) = c\Delta_h f(x).$$

Also, $\Delta_h^0 f(x)$ or $I f(x)$ stands for $f(x)$.

For any integer $n \geq 1$, we define the *n-th difference* by $\Delta_h^n f(x) = \Delta_h(\Delta_h^{n-1} f)(x)$.

For example,

$$\Delta_h^2 f(x) = f(x+2h) - 2f(x+h) + f(x),$$
$$\Delta_h^3 f(x) = f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x).$$

By induction, we can check that

$$\Delta_h^n f(x) = \sum_{k=0}^{n} (-1)^{n-k} C_n^k f(x+kh), \qquad (\alpha)$$

where $C_n^0 = 1$ and for $k > 0$,

$$C_n^k = \binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}.$$

(*Note*: for these formulas, we may even let n be a real number!!!)

If h=1, we simply write Δ and omit the subscript h. For example, in case of a sequence $\{x_n\}$, we have $\Delta x_n = x_{n+1} - x_n$.

Facts.(1) For function $f(x)$, n=0,1,2,…,

$$f(x+n) = \sum_{k=0}^{n} C_n^k \Delta^k f(x);$$

in particular, if $\Delta^m f(n)$ is a nonzero constant for every positive integer n, then $f(n) = \sum_{k=0}^{m} C_n^k \Delta^k f(0)$;

(2) if $P(x) = ax^n + \cdots$ is a polynomial of degree n, then for all x,
$\Delta_h^n P(x) = an!h^n$ and $\Delta_h^m P(x) = 0$ for $m>n$.

Let k be a positive integer. As a function of x, C_x^k has the properties:

(a) $C_x^{k-1} + C_x^k = C_{x+1}^k$ (so $\Delta C_x^k = C_x^{k-1}$);

(b) for $0 \leq r \leq k$, $\Delta^r C_x^k = C_x^{k-r}$;
for $r > k$, $\Delta^r C_x^k = 0$;

(c) $C_1^k + C_2^k + \cdots + C_n^k = C_{n+1}^{k+1}$ (just add $C_1^{k+1} = 0$ to the left and apply (a) repeatedly).

Similar to fact (1), if $f(x)$ is a degree m polynomial, then

$$f(x) = \sum_{k=0}^{m} C_x^k \Delta^k f(0). \qquad (\beta)$$

(This is because both sides are degree m polynomials and from property (b), the k-th differences at 0 are the same for $k = 0$ to m, which implies the values of both sides at 0, 1, 2, …, m are the same.)

Example 1. Sum $S_n = 1^4 + 2^4 + \cdots + n^4$ in terms of n.

Solution. Let $f(x) = x^4$. By (β) and (c),

$$S_n = \sum_{j=1}^{n} f(j) = \sum_{j=1}^{n}\sum_{k=0}^{4} C_j^k \Delta^k f(0)$$
$$= \sum_{k=0}^{4}\Big(\sum_{j=1}^{n} C_j^k\Big)\Delta^k f(0) = \sum_{k=0}^{4} C_{n+1}^{k+1} \Delta^k f(0).$$

x :	0	1	2	3	4
$f(x)$:	0	1	16	81	256
$\Delta f(x)$:	1	15	65	175	
$\Delta^2 f(x)$:	14	50	110		
$\Delta^3 f(x)$:	36	60			
$\Delta^4 f(x)$:	24				

Therefore,

$$S_n = \binom{n+1}{2} + 14\binom{n+1}{3} + 36\binom{n+1}{4} + 24\binom{n+1}{5}$$
$$= n(n+1)(2n+1)(3n^2+3n-1)/30.$$

Example 2. (*2000 Chinese IMO Team Selection Test*) Given positive integers k, m, n satisfying $1 \leq k \leq m \leq n$. Find

$$\sum_{i=0}^{n} (-1)^i \frac{1}{n+k+i} \frac{(m+n+i)!}{i!(n-i)!(m+i)!}.$$

Solution. Define

$$g(x) = \frac{(x+m+1)(x+m+2)\cdots(x+m+n)}{x+n+k}.$$

From $1 \leq k \leq m \leq n$, we see $m+1 \leq n+k \leq m+n$. So $g(x)$ is a polynomial of degree $n-1$. By fact (2) and formula (α),

$$0 = (-1)^n \Delta^n g(0) = \sum_{i=0}^{n} (-1)^i C_n^i g(i)$$
$$= \sum_{i=0}^{n} (-1)^i \frac{n!}{i!(n-i)!} \frac{(m+n+i)!}{(m+i)!} \frac{1}{n+k+i}.$$

The required sum is $(-1)^n\ \Delta^n g(0)/n! = 0$.

Example 3. (*1949 Putnam Exam*) The sequence $x_0, x_1, x_2, \ldots$ is defined by the conditions $x_0 = a$, $x_1 = b$ and for $n \geq 1$,

$$x_{n+1} = \frac{x_{n-1} + (2n-1)x_n}{2n},$$

where a and b are given numbers. Express $\lim_{n\to\infty} x_n$ in terms of a and b.

Solution. The recurrence relation can be written as

$$\Delta x_n = -\frac{\Delta x_{n-1}}{2n}.$$

Repeating this $n - 2$ times, we get

$$\Delta x_n = \left(-\frac{1}{2}\right)^n \frac{1}{n!}\Delta x_1 = \left(-\frac{1}{2}\right)^n \frac{1}{n!}(b-a).$$

Then

$$x_n = x_0 + \sum_{i=0}^{n-1} \Delta x_i = a + (b-a)\sum_{i=0}^{n-1}\left(-\frac{1}{2}\right)^i \frac{1}{i!}.$$

Using the fact

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!},$$

we get $\lim_{n\to\infty} x_n = a + (b - a)e^{-1/2}$.

Example 4. (*2004 Chinese Math Olympiad*) Given a positive integer c, let $x_1, x_2, x_3, \ldots$ satisfy $x_1 = c$ and

$$x_n = x_{n-1} + \left[\frac{2x_{n-1} - (n+2)}{n}\right] + 1$$

for $n = 2,3,\ldots$, where $[x]$ is the greatest integer less than or equal to x. Find a general formula of x_n in terms of n.

Solution. First tabulate some values.

	x_1	x_2	x_3	x_4	x_5	x_6
c=1	1	1	1	1	1	1
c=2	2	3	4	5	6	7
c=3	3	5	7	10	13	17
c=4	4	7	11	16	22	29
c=5	5	9	14	20	27	35
c=6	6	11	17	25	34	45
c=7	7	13	21	31	43	57

Next tabulate first differences in each column.

column 1: 1,1,1,1,1,1,…
column 2: 2,2,2,2,2,2,…
column 3: 3,3,4,3,3,4,…
column 4: 4,5,6,4,5,6,…
column 5: 5,7,9,5,7,9,…
column 6: 6,10,12,6,10,12,…

We suspect they are periodic with period 3. Let $x(c,n)$ be the value of x_n for the sequence with $x_1 = c$. For rows 1 and 2, the first differences seem to be constant and for row 4, the second differences seem to be constant. Using fact (1) and induction, we get

$$x(1,n) = 1, \quad x(2,n) = n + 1 \qquad \text{(i)}$$

and $x(4,n) = (n^2+3n+4)/2$ for all n. Now

$$x(4,n) - x(1,n) = \frac{(n+1)(n+2)}{2}.$$

To check the column difference periodicity, we claim that for a fixed c,

$$x(c + 3, n) = x(c, n) + (n + 1)(n + 2)/2.$$

If $n = 1$, then $x(c + 3,1) = c + 3 = x(c,1) + 3$ and so case $n = 1$ is true. Suppose the case $n-1$ is true. By the recurrence relation, $x(c+3,n)$ equals

$$x(c+3,n-1) + \left[\frac{2x(c+3,n-1)-(n+2)}{n}\right] + 1.$$

From the case $n - 1$, we get $x(c + 3, n - 1) = x(c, n - 1) + n(n + 1)/2$. Using this, the displayed expression simplifies to

$$x(c,n-1) + \left[\frac{2x(c,n-1)-(n+2)}{n}\right] + \frac{n^2+3n+4}{2},$$

which is $x(c, n) + (n + 1)(n + 2)/2$ by the recurrence relation. This completes the induction for the claim.

Now the claim implies

$$x(c,n) = x(d,n) + \left(\frac{c-d}{3}\right)\frac{(n+1)(n+2)}{2}, \qquad \text{(ii)}$$

where $d = 1,2$ or 3 subject to $c \equiv d \pmod 3$. Since $x(1,n)$ and $x(2,n)$ are known, all we need to find is $x(3,n)$.

For the case $c = 3$, studying $x_1, x_3, x_5, \ldots$ and $x_2, x_4, x_6, \ldots$ separately, we can see that the second differences of these sequences seem to be constant. Using fact (1) and induction, we get

$x(3,n) = (n^2+4n+7)/4$ if n is odd and
$x(3,n) = (n^2+4n+8)/4$ if n is even. (iii)

Formula (ii) along with formulas (i) and (iii) provided the required answer for the problem.

Example 5. Let $g(x)$ be a polynomial of degree n with real coefficients. If $a \geq 3$, then prove that one of the numbers $|1 - g(0)|, |a - g(1)|, |a^2 - g(2)|, \ldots, |a^{n+1} - g(n+1)|$ is at least 1.

Solution. Let $f(x) = a^x - g(x)$. We have

$$\Delta a^x = a^{x+1} - a^x = (a-1)a^x,$$
$$\Delta^2 a^x = (a-1)\,\Delta a^x = (a-1)^2 a^x,$$
$$\ldots,$$
$$\Delta^{n+1} a^x = (a-1)^{n+1} a^x.$$

In particular, $\Delta^{n+1} a^0 = (a-1)^{n+1}$. Now

$$\Delta^{n+1} f(0) = \Delta^{n+1} a^0 - \Delta^{n+1} g(0) = (a-1)^{n+1}.$$

Since $a \geq 3$, we get $2^{n+1} \geq \Delta^{n+1} f(0)$. Assume $|a^k - g(k)| < 1$ for $k = 0,1,\ldots,n+1$. Then

$$\Delta^{n+1} f(0) = \sum_{k=0}^{n+1} (-1)^{n+1-k} C_{n+1}^k \left(a^k - g(k)\right)$$

$$< \sum_{k=0}^{n+1} C_{n+1}^k = 2^{n+1},$$

which is a contradiction.

Example 6. (*1984 USAMO*) Let $P(x)$ be a polynomial of degree $3n$ such that

$$P(0) = P(3) = \cdots = P(3n) = 2,$$
$$P(1) = P(4) = \cdots = P(3n-2) = 1,$$
$$P(2) = P(5) = \cdots = P(3n-1) = 0.$$

If $P(3n + 1)$, then find n.

Solution. By fact (2) and (α),

$$0 = (-1)^{n+1}\Delta^{3n+1}P(0) = \sum_{k=0}^{3n+1}(-1)^k C_{3n+1}^k P(3n+1-k)$$

$$= 730 + 2\sum_{j=0}^{n}(-1)^{3j+1}C_{3n+1}^{3j+1} + \sum_{j=1}^{n}(-1)^{3j}C_{3n+1}^{3j}.$$

We can write this as $2a+b = -3^6$, where

$$a = \sum_{j=0}^{n}(-1)^{3j+1}C_{3n+1}^{3j+1}, \quad b = \sum_{j=0}^{n}(-1)^{3j}C_{3n+1}^{3j}.$$

To find a and b, we consider the cube root of unity $\omega = e^{2\pi i/3}$, the binomial expansion of $f(x) = (1-x)^{3n+1}$ and let

$$c = \sum_{j=1}^{n}(-1)^{3j-1}C_{3n+1}^{3j-1}.$$

Now $0 = f(1) = b-a-c$, $f(\omega) = b-a\omega-c\omega^2$ and $f(\omega^2) = b-a\omega^2-c\omega$. Solving, we see

$$a = -\left(f(1) + \omega^2 f(\omega) + \omega f(\omega^2)\right)/3$$
$$= 2\left(\sqrt{3}\right)^{3n-1}\cos\frac{3n-1}{6}\pi$$

and $b = \left(f(1) + f(\omega) + f(\omega^2)\right)/3$

$$= 2\left(\sqrt{3}\right)^{3n-1}\cos\frac{3n+1}{6}\pi.$$

Studying the equation $2a + b = -3^6$, we find that it has no solution when n is odd and one solution when n is even, namely when $n = 4$.

Example 7. (*1980 Putnam Exam*) For which real numbers a does the sequence defined by the initial condition $u_0 = a$ and the recursion $u_{n+1} = 2u_n - n^2$ have $u_n > 0$ for all $n \geq 0$?

Solution. Among all sequences satisfying $u_{n+1} = 2u_n - n^2$ for all $n \geq 0$, the difference v_n of any two such sequences will satisfy $v_{n+1} = 2v_n$ for all $n \geq 0$. Then $v_n = 2^n v_0$ for all $n \geq 0$.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***March 25, 2007.***

Problem 266. Let

$$N = 1+10+10^2+\cdots+10^{1997}.$$

Determine the 1000th digit after the decimal point of the square root of N in base 10.

Problem 267. For any integer a, set

$$n_a = 101a - 100\cdot 2^a.$$

Show that for $0 \le a, b, c, d \le 99$, if

$$n_a + n_b \equiv n_c + n_d \pmod{10100},$$

then $\{a,b\} = \{c,d\}$.

Problem 268. In triangle ABC, $\angle ABC = \angle ACB = 40^\circ$. Points P and Q are inside the triangle such that $\angle PAB = \angle QAC = 20^\circ$ and $\angle PCB = \angle QCA = 10^\circ$. Must B, P, Q be collinear? Give a proof.

Problem 269. Let $f(x)$ be a polynomial with integer coefficients. Define a sequence $a_0, a_1, \ldots$ of integers such that $a_0 = 0$, $a_{n+1} = f(a_n)$ for all $n \ge 0$. Prove that if there exists a positive integer m for which $a_m = 0$, then either $a_1 = 0$ or $a_2 = 0$.

Problem 270. The distance between any two of the points A, B, C, D on a plane is at most 1. Find the minimum of the radius of a circle that can cover these four points.

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Solutions

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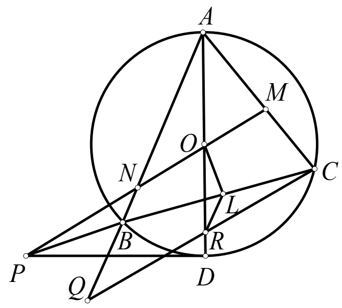
Problem 261. Prove that among any 13 consecutive positive integers, one of them has sum of its digits (in base 10) divisible by 7.

Solution. **Jeff CHEN** (Virginia, USA), **CHEUNG Wang Chi** (Raffles Junior College, Singapore), **G.R.A. 20 Math Problem Group** (Roma, Italy), **Naoki S. D. LING**, **Anna Ying PUN** (HKU, Math, Year 1), **Simon YAU Chi Keung**, **YIM Wing Yin** (HKU, Year 1) and **Fai YUNG**.

Consider the tens digits of the 13 consecutive positive integers. By the pigeonhole principle, there are at least $[13/2] + 1 = 7$ of them with the same tens digit. The sums of digits for these 7 numbers are consecutive. Hence, one of the sums of digits is divisible by 7.

Problem 262. Let O be the center of the circumcircle of ΔABC and let AD be a diameter. Let the tangent at D to the circumcircle intersect line BC at P. Let line PO intersect lines AC, AB at M, N respectively. Prove that $OM=ON$.

Solution 1. **Jeff CHEN** (Virginia, USA).



We may assume B is between P and C (otherwise interchange B and C, then N and M). Through C, draw a line parallel to line MN and intersect line AN at Q. Let line AO intersect line CQ at R. Since $MN \| CQ$, triangles AMN and ACQ are similar. To show $OM = ON$, it suffices to show $RC = RQ$.

Let L be the midpoint of BC. We will show $LR \parallel BQ$ (which implies $RC = RQ$).

Now $\angle OLP = \angle OLB = 90^\circ = \angle ODP$, which implies O, P, D, L are concyclic. Then $\angle ODL = \angle OPL$. From $OP \parallel RC$, we get $\angle RDL = \angle RCL$, which implies L,R,D,C are concyclic. Then

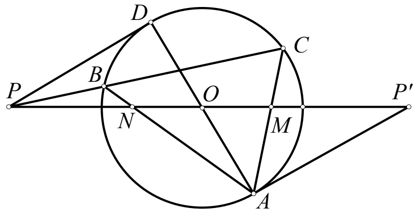
$$\angle RLC = 180^\circ - \angle RDC = 180^\circ - \angle ADC$$
$$= 180^\circ - \angle ABC = \angle QBC.$$

Therefore, $LR \parallel BQ$ as claimed.

Solution 2. **CHEUNG Wang Chi** (Raffles Junior College, Singapore).

Set O as the origin and line MN as the x-axis.

Let P' be the reflection of P with respect to O. Then the coordinates of P and P' are of the form $(p,0)$ and $(-p,0)$.



The equation of the circumcircle as a conic section is of the form $x^2+y^2-r^2=0$. The equation of the pair of lines AP' and BC as a (degenerate) conic section is

$$\big(y - m(x+p)\big)\big(y - n(x-p)\big) = 0,$$

where m is the slope of line AP' and n is the slope of line BC. Since these two conic sections intersect at A, B, C, so the equation of the pair of lines AB and AC as a (degenerate) conic section is of the form

$$x^2 + y^2 - r^2 = \lambda\big(y - m(x+p)\big)\big(y - n(x-p)\big),$$

for some real number λ. When we set $y = 0$, we see the x-coordinates of M and N satisfies $x^2 - r^2 = \lambda mn(x^2 - p^2)$, whose roots are some positive number and its negative. Therefore, $OM = ON$.

Commended solvers: **Courtis G. CHRYSSOSTOMOS** (Larissa, Greece, teacher) and **Anna Ying PUN** (HKU, Math, Year 1).

Problem 263. For positive integers m, n, consider a $(2m+1)\times(2n+1)$ table, where in each cell, there is exactly one ant. At a certain moment, every ant moves to a horizontal or vertical neighboring cell. Prove that after that moment, there exists a cell with no ant.

Solution. **Jeff CHEN** (Virginia, USA), **CHEUNG Wang Chi** (Raffles Junior College, Singapore), **G.R.A. 20 Math Problem Group** (Roma, Italy), **Naoki S. D. LING**, **Anna Ying PUN** (HKU, Math, Year 1), **YIM Wing Yin** (HKU, Year 1) and **Fai YUNG**.

Assign the value $(-1)^{i+j}$ to the cell in the i-th row, j-th column of the table. Then two horizontal or vertical neighboring cells will have values of opposite sign. Since $2m+1$ and $2n+1$ are odd, there is exactly one more cell with negative values than cells with positive values. Before the moment, there is one more ant in cells with negative values than ants in cells with positive values. After the moment, two of the ants from cells with negative values will occupy a common cell with a positive value. Then there exists a cell with no ant.

Problem 264. For a prime number $p > 3$ and arbitrary integers a, b, prove that $ab^p - ba^p$ is divisible by $6p$.

Solution. **Samuel Liló ABDALLA** (São Paulo, Brazil), **Claudio ARCONCHER** (Jundiaí, Brazil), **Jeff CHEN** (Virginia, USA), **CHEUNG Wang Chi** (Raffles Junior College, Singapore), **G.R.A. 20 Math Problem Group** (Roma, Italy), **HO Ka Fai** (Carmel Divine Grace Foundation Secondary School, Form 6), **D. Kipp JOHNSON** (Valley Catholic School, Teacher, Beaverton, Oregon, USA), **Anna Ying PUN** (HKU, Math, Year 1), **Simon YAU Chi Keung**, **YIM Wing Yin** (HKU, Year 1) and **Fai YUNG**.

Observe that

$ab^p - ba^p = ab[(b^{p-1}-1)-(a^{p-1}-1)].$

For $q = 2$, 3 or p, if a or b is divisible by q, then the right side is divisible by q.

Otherwise, a and b are relatively prime to q. Now $p - 1$ is divisible by $q - 1$, which is 1, 2 or $p - 1$. By Fermat's little theorem, both $a^{q-1}, b^{q-1} \equiv 1 \pmod{q}$. So $a^{p-1}, b^{p-1} \equiv 1 \pmod{q}$. Hence, the bracket factor above is divisible by q. Thus $ab^p - ba^p$ is divisible by 2, 3 and p. Therefore, it is divisible by $6p$.

Problem 265. Determine (with proof) the maximum of

$$\sum_{j=1}^{n}(x_j^4 - x_j^5),$$

where $x_1, x_2, \ldots, x_n$ are nonnegative real numbers whose sum is 1.
(*Source: 1999 Chinese IMO Team Selection Test*)

Solution. **Jeff CHEN** (Virginia, USA), **D. Kipp JOHNSON** (Valley Catholic School, Teacher, Beaverton, Oregon, USA), **Anna Ying PUN** (HKU, Math, Year 1) and **YIM Wing Yin** (HKU, Year 1).

Let $f(x) = x^4 - x^5 = x^4(1-x)$. Since $f''(x) = 4x^2(3-5x)$, we see that $f(x)$ is strictly convex on [0, 3/5]. Suppose $n \geq 3$. Without loss of generality, we may assume $x_1 \geq x_2 \geq \cdots \geq x_n$. If $x_1 \leq 3/5$, then since

$$\left(\frac{3}{5},\frac{2}{5},0,\cdots,0\right) \succ (x_1, x_2, \cdots, x_n),$$

by the majorization inequality (see Math Excalibur, *vol.* 5, *no.* 5, *pp.* 2,4),

$$\sum_{i=1}^{n} f(x_i) \leq f\left(\frac{3}{5}\right) + f\left(\frac{2}{5}\right).$$

If $x_1 > 3/5$, then $1 - x_1, x_2, \ldots, x_n$ are in [0, 2/5]. Since

$$\left(1-x_1, 0, \ldots, 0\right) \succ (x_2, \ldots, x_n),$$

by the majorization inequality,

$$\sum_{i=1}^{n} f(x_i) \leq f(x_1) + f(1-x_1).$$

Thus the problem is reduced to the case $n = 2$. So now consider nonnegative a, b with $a + b = 1$. We have

$$\begin{aligned} f(a)+f(b) &= a^4(1-a)+b^4(1-b) \\ &= a^4b+b^4a = ab(a^3+b^3) \\ &= ab[(a+b)^3-3ab(a+b)] \\ &= 3ab(1-3ab)/3 \\ &\leq 1/12 \end{aligned}$$

by the *AM-GM* inequality. Equality case holds when $ab = 1/6$ in addition to $a + b = 1$, for example when

$$(a,b) = \left(\frac{3+\sqrt{3}}{6}, \frac{3-\sqrt{3}}{6}\right).$$

Therefore, the maximum is 1/12.

Difference Operator

(continued from page 2)

Next we look for a particular solution of $u_{n+1} = 2u_n - n^2$ for all $n \geq 0$. Observe that $n^2 = u_n - (u_{n+1} - u_n) = (I - \Delta)u_n$. From the sum of geometric series, we guess

$$u_n = (I-\Delta)^{-1}n^2 = (I+\Delta+\Delta^2+\cdots)n^2$$

$$= n^2+(2n+1)+2 = n^2+2n+3$$

should work. Indeed, this is true since $(n+1)^2 + 2(n+1) + 3 = 2(n^2 + 2n + 3) - n^2$.

Combining, we see that the general solution to $u_{n+1} = 2u_n - n^2$ for all $n \geq 0$ is

$$u_n = n^2 + 2n + 3 + 2^n v_0 \text{ for any real } v_0.$$

Finally, to have $u_0 = a$, we must choose $v_0 = a - 3$. Hence, the sequence we seek is

$$u_n = n^2+2n+3+2^n(a-3) \text{ for all } n \geq 0.$$

Since $$\lim_{n\to\infty}\frac{2^n}{n^2+2n+3} = +\infty,$$

u_n will be negative for large n if $a - 3 < 0$. Conversely, if $a - 3 \geq 0$, then all $u_n > 0$. Therefore, the answer is $a \geq 3$.

Example 8. (*1971 Putnam Exam*) Let c be a real number such that n^c is an integer for every positive integer n. Show that c is a non-negative integer.

Solution. 2^c is an integer implies $c \geq 0$.

Next we will do the case c is between 0 and 1 using the mean value theorem. This motivates and clarifies the general argument for the case $c \geq 1$. Assume $0 \leq c < 1$. Then choose a positive integer $n > c^{1/(1-c)}$. Applying the mean value theorem to $f(x) = x^c$ on $[n, n + 1]$, we know there exists a number w between n and $n + 1$ such that $\Delta n^c = (n+1)^c - n^c = cw^{c-1}$. On the left side, we have an integer, but on the right side, since $w > n > c^{1/(1-c)}$, we have $0 \leq cw^{c-1} < 1$. Hence, $c = 0$.

For $c \geq 1$, let us mention there is an extension of the mean value theorem, which asserts that if f is continuous on $[a,b]$, k-times differentiable on (a,b), $0<h \leq (b-a)/k$ and $x + kh \leq b$, then there exists a number v such that $a < v < b$ and

$$\frac{\Delta_h^k f(x)}{h^k} = f^{(k)}(v).$$

Taking this for the moment, we will finish the argument as follows. Let k be the integer such that $k-1 \leq c < k$. Choose an integer n so large that

$$c(c-1)(c-2)\cdots(c-k+1)n^{c-k} < 1.$$

Applying the extension of the mean value theorem mentioned above to $f(x) = x^c$ on $[n, n + k]$, there is a number v between n and $n + k$ such that

$$\Delta^k n^c = c(c-1)(c-2)\cdots(c-k+1)v^{c-k}.$$

Again, the left side is an integer, but the right side is in the interval [0, 1). Therefore, both sides are 0 and $c = k-1$.

Now the extension of the mean value theorem can be proved by doing math induction on k. The case $k = 1$ is the mean value theorem. Next, suppose the extension is true for the case $k-1$. Let $0<h \leq (b-a)/k$. On $[a,b-h]$, define

$$g(x) = \frac{\Delta_h f(x)}{h} = \frac{f(x+h)-f(x)}{h}.$$

Applying the case $k-1$ to $g(x)$, we know there exists a number v_0 such that $a < v_0 < b-h$ and

$$\frac{\Delta_h^k f(x)}{h^k} = \frac{\Delta_h^{k-1} g(x)}{h^{k-1}} = g^{(k-1)}(v_0).$$

By the mean value theorem, there exists h_0 such that $0 < h_0 < h$ and

$$g^{(k-1)}(v_0) = \frac{f^{(k-1)}(v_0+h) - f^{(k-1)}(v_0)}{h}$$

$$= f^{(k)}(v_0 + h_0).$$

Finally, $v = v_0 + h_0$ is between a and b.