Mathematical Excalibur

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Olympiad Corner

Below are the problems of the 2006 Belarussian Math Olympiad, Final Round, Category C.

Problem 1. Is it possible to partition the set of all integers into three nonempty pairwise disjoint subsets so that for any two numbers a and b from different subsets,

a) there is a number c in the third subset such that $a + b = 2c$?
b) there are two numbers c_1 and c_2 in the third subset such that $a + b = c_1 + c_2$?

Problem 2. Points X, Y, Z are marked on the sides AB, BC, CD of the rhombus $ABCD$, respectively, so that $XY \| AZ$. Prove that XZ, AY and BD are concurrent.

Problem 3. Let a, b, c be real positive numbers such that $abc = 1$. Prove that

$$2(a^2+b^2+c^2)+a+b+c \geq 6+ab+bc+ca.$$

Problem 4. Given triangle ABC with $\angle A = 60^\circ$, $AB = 2005$, $AC = 2006$. Bob and Bill in turn (Bob is the first) cut the triangle along any straight line so that two new triangles with area more than or equal to 1 appear.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***August 20, 2007***.

For individual subscription for the next five issues for the 05-06 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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From *How to Solve It* to Problem Solving in Geometry (II)

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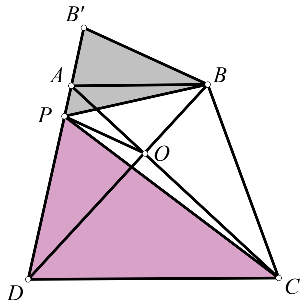
We will continue with more examples.

Example 9. In the trapezium $ABCD$, $AB \| CD$ and the diagonals intersect at O. P, Q are points on AD and BC respectively such that $\angle APB = \angle CPD$ and $\angle AQB = \angle CQD$. Show that $OP = OQ$.

Idea:
We shall try to find OP in terms of "more basic" lengths, e.g. AB, CD, OA, OC, …. To achieve that, we can construct a triangle that is similar to ΔDPC.



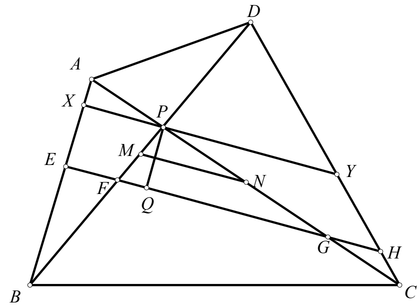
Solution Outline:
(1) Extend DA to B' such that $BB' = BA$. Then $\angle PB'B = \angle B'AB = \angle PDC$. So $\Delta DPC \sim \Delta B'PB$.

(2) It follows that
$$\frac{DP}{PB'} = \frac{CD}{BB'} = \frac{CD}{BA} = \frac{DO}{BO}$$
and so $PO \| BB'$.

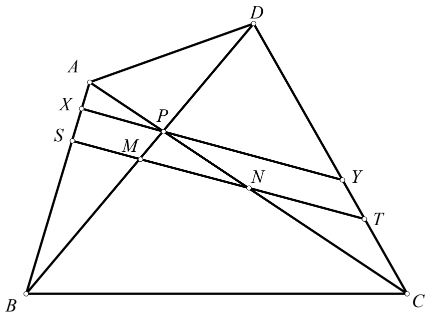
(3) Since $\Delta DPO \sim \Delta DB'B$, we have
$$OP = BB' \times \frac{DO}{DB} = AB \times \frac{DO}{DB}.$$

(4) Similarly, we have $OQ = AB \times \dfrac{CO}{CA}$ and the result follows.

Example 10. In quadrilateral $ABCD$, the diagonals intersect at P. M and N are midpoint of BD and AC respectively. Q is the reflected image of P about MN. The line through P and parallel to MN cuts AB and CD at X and Y respectively. The line through Q parallel to MN cuts AB, BD, AC and CD at E, F, G and H respectively. Prove that $EF = GH$.



Idea:
The diagram is not simple. We shall try to express the lengths involved in terms of "more basic" lengths, e.g. PA, PB, PC and PD.



Solution Outline:
(1) First observe that $PM = MF$ and $PN = NG$, hence $BF = PD$ and $CG = PA$.

(2) $\dfrac{EF}{XP} = \dfrac{BF}{BP} = \dfrac{PD}{BP}$, $EF = \dfrac{PD \times XP}{BP}$.

Similarly, we have $GH = \dfrac{PA \times YP}{CP}$.

(3) Let the line MN cuts AB and CD at S and T respectively. Then

$$\frac{SM}{XP} = \frac{BM}{BP} = \frac{BD}{2BP}, \frac{SN}{XP} = \frac{AN}{AP} = \frac{AC}{2AP}.$$

Subtracting the equalities get
$$\frac{MN}{XP} = \frac{1}{2}\left(\frac{AC}{AP} - \frac{BD}{BP}\right).$$

Similarly, we have
$$\frac{MN}{YP} = \frac{1}{2}\left(\frac{BD}{PD} - \frac{AC}{PC}\right).$$

(4) $EF = GH \Leftrightarrow \dfrac{PD\times XP}{BP} = \dfrac{PA\times YP}{CP}$

$\Leftrightarrow \dfrac{PD\times MN}{BP\times YP} = \dfrac{PA\times MN}{CP\times XP}$. By (3),

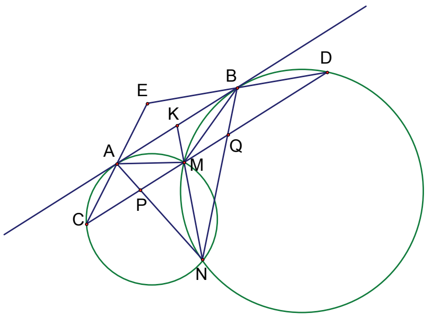
$$\frac{PD}{BP}\left(\frac{BD}{PD}-\frac{AC}{PC}\right) = \frac{PA}{CP}\left(\frac{AC}{AP}-\frac{BD}{BP}\right)$$

$$\Leftrightarrow \frac{BD}{BP}-\frac{PD\times AC}{BP\times PC} = \frac{AC}{CP}-\frac{PA\times BD}{CP\times BP}$$

$$\Leftrightarrow \frac{BD}{BP}+\frac{PA\times BD}{CP\times BP} = \frac{AC}{CP}+\frac{PD\times AC}{BP\times PC}.$$

By addition, both sides of the last equation equal $\dfrac{AC\times BD}{BP\times CP}$.

Example 11. [IMO 2000] Two circles Γ_1 and Γ_2 intersect at M and N. Let L be the common tangent to Γ_1 and Γ_2 so that M is closer to L than N is. Let L touch Γ_1 at A and Γ_2 at B. Let the line through M parallel to L meet the circle Γ_1 again at C and the circle Γ_2 again at D. Lines CA and DB meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.



Idea:
First, note that if $EP = EQ$, then E lies on the perpendicular bisector of PQ.

Observe that $AB \parallel CD$ implies A and B are the midpoints of arc CAM and arc DBM respectively, from which we see ΔACM and ΔBDM are isosceles.

Second, we have $\angle EAB = \angle ECM = \angle AMC = \angle BAM$ and similarly, $\angle EBA = \angle ABM$. That means E is the reflected image of M about AB. In particular, $EM \perp AB$ and hence $EM \perp PQ$.

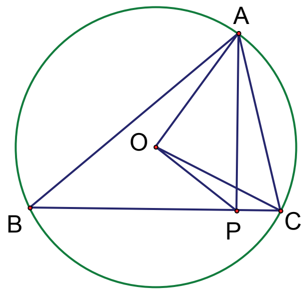
Therefore, the result follows if we can show that M is the midpoint of PQ.

Solution outline:
(1) Extend NM to meet AB at K.

(2) $AK^2 = KN\times KM = BK^2 \Rightarrow K$ is the midpoint of $AB \Rightarrow M$ is the midpoint of PQ.

(3) Following the steps discussed above, we get $EM \perp PQ$ and hence $EP = EQ$.

Example 12. [IMO 2001] Let ABC be an acute-angled triangle with circumcentre O. Let P on BC be the foot of the altitude from A. Suppose that $\angle BCA \ge \angle ABC + 30^\circ$. Prove that $\angle CAB + \angle COP < 90^\circ$.



Idea:
(1) Examine the conclusion $\angle CAB + \angle COP < 90^\circ$, which is equivalent to $2\angle CAB + 2\angle COP < 180^\circ$. That is,

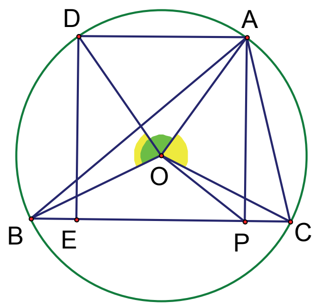
$$\angle COB + 2\angle COP < 180^\circ.$$

On the other hand, we have $\angle COB + 2\angle OCP = 180^\circ$. Therefore, we shall show $\angle COP < \angle OCP$ or $PC < OP$.

(2) Examine the condition $\angle BCA \ge \angle ABC + 30^\circ$, which is equivalent to $2\angle BCA - 2\angle ABC \ge 60^\circ$. That is,

$$\angle BOA - \angle AOC \ge 60^\circ.$$

What is the meaning of $\angle BOA - \angle AOC$?



Solution outline:
(1) Let D and E be the reflected image of A and P about the perpendicular bisector of BC respectively. Let R be the circumradius.

(2) $\angle BCA \ge \angle ABC + 30^\circ$
$\Rightarrow \angle BOA - \angle AOC \ge 60^\circ$
$\Rightarrow \angle DOA \ge 60^\circ$
$\Rightarrow EP = DA \ge R$.

(3) $OP + R = OP + OC = OE + OC$
$> EC = EP + PC \ge R + PC$
$\Rightarrow OP > PC \quad \Rightarrow \angle COP < \angle OCP$.

(4) $2\angle CAB + 2\angle COP$
$= \angle COB + 2\angle COP$
$< \angle COB + 2\angle OCP < 180^\circ$
and the result follows.

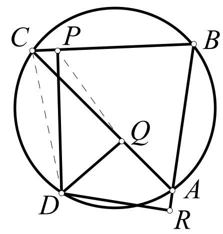
Example 13. [Simson's Theorem] The feet of the perpendiculars drawn from any point on the circumcircle of a triangle to the sides of the triangle are collinear.

Solution:
In the figure below, D is a point on the circumcircle of ΔABC, P, Q, and R are feet of perpendiculars from D to BC, AC, and BA respectively.

Note that $DQAR$, $DCPQ$, and $DPBR$ are cyclic quadrilaterals. So

$$\angle DQR = \angle DAR = \angle BCD$$
$$= 180^\circ - \angle PQD,$$

i.e. $\angle DQR + \angle PQD = 180^\circ$. Thus, P, Q, and R are collinear.



Example 14. [IMO 2003] Let $ABCD$ be a cyclic quadrilateral. Let P, Q and R be the feet of the perpendiculars from D to the lines BC, CA and AB respectively. Show that $PQ = QR$ if and only if the bisector of $\angle ABC$ and $\angle ADC$ meet on AC.

Solution :
From Simson's theorem, P, Q, and R are collinear. Now

$\angle DPC = \angle DQC = 90^\circ$
$\Rightarrow D, P, C$ and Q are concyclic
$\Rightarrow \angle DCA = \angle DPQ = \angle DPR.$

Similarly, since D, Q, R and A are concyclic, we get $\angle DAC = \angle DRP$. It follows that $\Delta DCA \sim \Delta DPR$.

Similarly, $\Delta DAB \sim \Delta DQP$ and $\Delta DBC \sim \Delta DRQ$. So,

$$\frac{DA}{DC} = \frac{DR}{DP} = \frac{DB\cdot\frac{QR}{BC}}{DB\cdot\frac{PQ}{BA}} = \frac{QR}{PQ}\cdot\frac{BA}{BC}.$$

Therefore, $PQ = QR \Leftrightarrow \dfrac{DA}{DC} = \dfrac{BA}{BC}$.

Example 15. [IMO 2001] In a triangle ABC, let AP bisect $\angle BAC$, with P on BC, and let BQ bisect $\angle ABC$, with Q on CA. It is known that $\angle BAC = 60^\circ$ and that $AB + BP = AQ + QB$. What are the possible angles of triangle ABC?

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***August 20, 2007.***

Problem 276. Let n be a positive integer. Given a $(2n-1)\times(2n-1)$ square board with exactly one of the following arrows ↑, ↓, →, ← at each of its cells. A beetle sits in one of the cells. Per year the beetle creeps from one cell to another in accordance with the arrow's direction. When the beetle leaves the cell, the arrow at that cell makes a counterclockwise 90-degree turn. Prove that the beetle leaves the board in at most $2^{3n-1}(n-1)!-3$ years.
(*Source: 2001 Belarussian Math Olympiad*)

Problem 277. (*Due to Koopa Koo, Univ. of Washington, Seattle, WA, USA*) Prove that the equation

$$x^2+y^2+z^2+2xyz=1$$

has infinitely many integer solutions (then try to get all solutions – Editiors).

Problem 278. Line segment SA is perpendicular to the plane of the square $ABCD$. Let E be the foot of the perpendicular from A to line segment SB. Let P, Q, R be the midpoints of SD, BD, CD respectively. Let M, N be on line segments PQ, PR respectively. Prove that AE is perpendicular to MN.

Problem 279. Let R be the set of all real numbers. Determine (with proof) all functions $f: R\to R$ such that for all real x and y,

$$f\big(f(x)+y\big)=2x+f\big(f(f(y))-x\big).$$

Problem 280. Let n and k be fixed positive integers. A basket of peanuts is distributed into n piles. We gather the piles and rearrange them into $n+k$ new piles. Prove that at least $k+1$ peanuts are transferred to smaller piles than the respective original piles that contained them. Also, give an example to show the constant $k+1$ cannot be improved.

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Solutions

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Problem 271. There are 6 coins that look the same. Five of them have the same weight, each of these is called a <u>good</u> coin. The remaining one has a different weight from the 5 good coins and it is called a <u>bad</u> coin. Devise a scheme to weigh groups of the coins using a scale (not a balance) three times only to determine the bad coin and its weight.
(*Source: 1998 Zhejiang Math Contest*)

Solution. **Jeff CHEN** (Virginia, USA), **St. Paul's College Math Team**, **YIM Wing Yin** (HKU, Year 1) and **Fai YUNG**.

Number the coins 1 to 6. For the first weighting, let us weigh coins 1, 2, 3 and let the weight be $3a$. For the second weighting, let us weigh coins 1, 2, 4, 5 and let the weight be $4b$.

If $a = b$, then coin 6 is bad and we can use the third weighting to find the weight of this coin.

If $a \neq b$, then the bad coin is among coins 1 to 5. For the third weighting, let us weigh coins 2, 4 and let the weight be $2c$.

If coin 1 is bad, then c and $4b-3a$ are both the weight of a good coin. So $3a-4b+c=0$. Similarly, if coin 2 or 3 or 4 or 5 is bad, we get respective equations $3a-2b-c=0$, $b-c=0$, $a-2b+c=0$ and $a-c=0$.

We can check that if any two of these equations are satisfied simultaneously, then we will arrive at $a=b$, a contradiction. Therefore, exactly one of these five equations will hold.

If the first equation $3a-4b+c=0$ holds, then coin 1 is bad and its weight can be found by the first and third weightings to be $3a-2c$. Similarly, for $k = 2$ to 5, if the k-th equation holds, then coin k is bad and its weight can be found to be $3c-2b$, $3a-2c$, $4b-3a$ and $4b-3a$ respectively.

Problem 272. $\Delta\, ABC$ is equilateral. Find the locus of all point Q inside the triangle such that

$$\angle QAB+\angle QBC+\angle QCA=90^\circ.$$

(*Source: 2000 Chinese IMO Team Training Test*)

Solution. **Alex Kin-Chit O** (STFA Cheng Yu Tung Secondary School) and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 6).

We take the origin at the center O of $\Delta\, ABC$. Let $\omega\neq 1$ be a cube root of unity and A,B,C,Q correspond to the complex numbers 1, ω, $\omega^2=\overline{\omega}$, z respectively. Then

$$\angle QAB+\angle QBC+\angle QCA=90^\circ$$

if and only if

$$\frac{\omega-1}{z-1}\cdot\frac{\overline{\omega}-\omega}{z-\omega}\cdot\frac{1-\overline{\omega}}{z-\overline{\omega}}=\frac{(\omega-\overline{\omega})\,|\,\omega-1|^2}{z^3-1}$$

is purely imaginary, which is equivalent to z^3 is real. These are the complex numbers whose arguments are multiples of $\pi/3$. Therefore, the required locus is the set of points on the three altitudes.

Commended solvers: **Jeff CHEN** (Virginia, USA), **St. Paul's College Math Team, Simon YAU** and **YIM Wing Yin** (HKU, Year 1).

Problem 273. Let R and r be the circumradius and the inradius of triangle ABC. Prove that

$$\frac{\cos A}{\sin^2 A}+\frac{\cos B}{\sin^2 B}+\frac{\cos C}{\sin^2 C}\geq\frac{R}{r}.$$

(*Source: 2000 Beijing Math Contest*)

Solution. **Jeff CHEN** (Virginia, USA), **Kelvin LEE** (Winchester College, England), **NG Eric Ngai Fung** (STFA Leung Kau Kui College), **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 6) and **YIM Wing Yin** (HKU, Year 1).

Without loss of generality, let a, b, c be the sides and $a\geq b\geq c$. By the extended sine law, $R = a/(2\sin A) = b/(2\sin B) = c/(2\sin C)$. Now the area of the triangle is $(bc\sin A)/2=abc/(4R)$ and is also rs, where $s = (a+b+c)/2$ is the semi- perimeter. So $abc=4Rrs$.

Next, observe that for any positive x and y, we have $(x^2-y^2)(1/x-1/y)\leq 0$, which after expansion yields

$$\frac{x^2}{y}+\frac{y^2}{x}\geq x+y. \qquad (*)$$

By the cosine law and the extended sine law, we get

$$\frac{\cos A}{\sin^2 A}=\frac{(b^2+c^2-a^2)/2bc}{(a/2R)^2}$$

$$=\frac{2R^2}{abc}\left(\frac{b^2+c^2-a^2}{a}\right)=\frac{R}{2rs}\left(\frac{b^2}{a}+\frac{c^2}{a}-a\right).$$

Adding this to the similar terms for B and C, we get

$$\frac{\cos A}{\sin^2 A}+\frac{\cos B}{\sin^2 B}+\frac{\cos C}{\sin^2 C}$$

$$=\frac{R}{2rs}\left(\frac{b^2}{a}+\frac{a^2}{b}+\frac{c^2}{b}+\frac{b^2}{c}+\frac{a^2}{c}+\frac{c^2}{a}-a-b-c\right)$$

$$\ge \frac{R}{2rs}(a+b+c)=\frac{R}{r} \quad \text{by (*).}$$

Commended solvers: **CHEUNG Wang Chi** (Singapore).

Problem 274. Let $n < 11$ be a positive integer. Let p_1, p_2, p_3, p be prime numbers such that $p_1 + p_3^n$ is prime. If $p_1 + p_2 = 3p$, $p_2 + p_3 = p_1^n(p_1 + p_3)$ and $p_2 > 9$, then determine $p_1p_2p_3^n$.
(*Source: 1997 Hubei Math Contest*)

Solution. **CHEUNG Wang Chi** (Singapore), **NG Eric Ngai Fung** (STFA Leung Kau Kui College), **YIM Wing Yin** (HKU, Year 1) and **Fai YUNG**.

Assume $p_1 \ge 3$. Then $p_1+p_2 > 12$ and $3p$ is even, which would imply p is even and at least 5, contradicting p is prime. So $p_1=2$ and $p_2=3p-2$.

Modulo 3, the given equation $p_2 + p_3 = p_1^n(p_1+p_3)$ leads to

$$\begin{aligned} 0 &\equiv 3p \\ &= p_2+2 = 2^n(2+p_3)+2 \\ &= 2^{n+1} + 2 + (2^n-1)p_3 \\ &\equiv (-1)^{n+1}+2+((-1)^n-1)p_3 \pmod 3. \end{aligned}$$

The case n is even results in the contradiction $0 \equiv 1 \pmod 3$. So n is odd and we get $0 \equiv p_3 \pmod 3$. So $p_3 = 3$.

Finally, the cases $n = 1, 3, 5, 7, 9$ lead to $p_1 + p_3^n = 5, 29, 245, 2189, 19685$ respectively. Since 245, 19685 are divisible by 5 and 2189 is divisible by 11, n can only be 1 or 3 for $p_1+p_3^n$ to be prime. Now $p_2 = p_1^n(p_1+p_3) - p_3 = 2^n5-3 > 9$ implies $n = 3$. Then the answer is

$$p_1p_2p_3^n = 2\cdot 37\cdot 3^3 = 1998.$$

Problem 275. There is a group of children coming from 11 countries (at least one child from each of the 11 countries). Their ages are from 7 to 13. Prove that there are 5 children in the group, for each of them, the number of children in the group with the same age is greater than the number of children in the group from the same country.

Solution. **Jeff CHEN** (Virginia, USA).

For i =7 to 13 and j = 1 to 11, let a_{ij} be the number of children of age i from country j in the group. Then

$$b_i = \sum_{j=1}^{11} a_{ij} \ge 0 \text{ and } c_j = \sum_{i=7}^{13} a_{ij} \ge 1$$

are the number of children of age i in the group and the number of children from country j respectively. Note that

$$c_j = \sum_{i=7}^{13} a_{ij} = \sum_{b_i\ne 0} a_{ij} \text{ , where } \sum_{b_i\ne 0} \text{ is}$$

used to denote summing i from 7 to 13 skipping those i for which b_i=0. Now

$$\sum_{b_i\ne 0}\sum_{j=1}^{11} a_{ij}\left(\frac{1}{c_j}-\frac{1}{b_i}\right)$$

$$=\sum_{j=1}^{11}\frac{\sum_{b_i\ne 0}a_{ij}}{c_j}-\sum_{b_i\ne 0}\frac{\sum_{j=1}^{11}a_{ij}}{b_i}$$

$$\ge \sum_{j=1}^{11}1-\sum_{i=7}^{13}1=4.$$

Since $a_{ij}(1/c_j - 1/b_i) < a_{ij}/c_j \le 1$, there are at least five terms $a_{ij}(1/c_j - 1/b_i) > 0$. So there are at least five ordered pairs (i,j) such that $a_{ij} > 0$ (so we can take a child of age i from country j) and we have $b_i > c_j$.

Olympiad Corner

(continued from page 1)

Problem 4. (Cont.) After that an obtused-angled triangle (or any of two right-angled triangles) is deleted and the procedure is repeated with the remained triangle. The player loses if he cannot do the next cutting. Determine, which player wins if both play in the best way.

Problem 5. AA_1, BB_1 and CC_1 are the altitudes of an acute triangle ABC. Prove that the feet of the perpendiculars from C_1 onto the segments AC, BC, BB_1 and AA_1 lie on the same straight line.

Problem 6. Given real numbers a, b, k (k>0). The circle with the center (a,b) has at least three common points with the parabola $y = kx^2$; one of them is the origin (0,0) and two of the others lie on the line $y=kx+b$. Prove that $b \ge 2$.

Problem 7. Let x, y, z be real numbers greater than 1 such that

$$xy^2 - y^2 + 4xy + 4x - 4y = 4004,$$

and $$xz^2 - z^2 + 6xz + 9x - 6z = 1009.$$

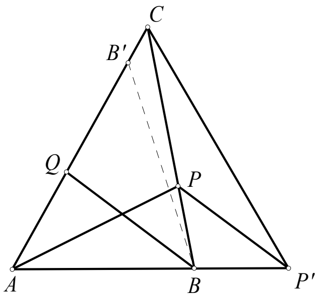
Determine all possible values of the expression $xyz+3xy+2xz-yz+6x-3y-2z$.

Problem 8. A $2n\times 2n$ square is divided into $4n^2$ unit squares. What is the greatest possible number of diagonals of these unit squares one can draw so that no two of them have a common point (including the endpoints of the diagonals)?

From *How to Solve It* to Problem Solving in Geometry (II)

(continued from page 2)

Idea:
By examining the conditions given, we may see that the point C is not too important.



We will focus on how to represent the condition $AB + BP = AQ + QB$ in the diagram. For that, we construct points P' and B' on AB and AQ extended respectively so that $PB = P'B$ and $QB' = QB$. Then

$AB + BP = AQ + QB$
$\Rightarrow AB + BP' = AQ + QB' \Rightarrow AP' = AB'$
$\Rightarrow AP'B'$ is equilateral (as $\angle B'AP' = 60^\circ$).

Solution outline:
(1) Let $\angle ABQ = \angle QBP = \theta$. Since $PB = P'B$, we have $\angle PP'B = \theta$.

(2) Since AP bisects $\angle QAB$ and $\Delta AB'P'$ is equilateral, it follows that B' is the reflected image of P' about AP. So, $PP' = PB'$ and $\angle QB'P = \angle AP'P = \theta$.

(3) Since $QB = QB'$ and $\angle QBP = \theta = \angle QB'P$, by Example 2, P lies on either BB' or the perpendicular bisector of BB'. If P does not lie on BB', we will have $PB = PB' = PP'$. This will imply $\Delta BPP'$ is equilateral, $\theta = 60^\circ$ and $\angle QAB + \angle ABP = 60^\circ + 2\theta = 180^\circ$, which is absurd. So, P must lie on BB'. Therefore, $B' = C$.

(4) Since $QB=QB'=QC$, $\angle QCB = \angle QBC = \theta$. So $\angle QAB + 2\theta + \theta = 180^\circ \Rightarrow 60^\circ + 3\theta = 180^\circ \Rightarrow \theta = 40^\circ$. Therefore, $\angle ABC = 80^\circ$, $\angle ACB = 40^\circ$.