Mathematical Excalibur

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Olympiad Corner

The following are the four problems of the 2008 Balkan Mathematical Olympiad.

Problem 1. An acute-angled scalene triangle ABC is given, with $AC > BC$. Let O be its circumcenter, H its orthocenter and F the foot of the altitude from C. Let P be the point (other than A) on the line AB such that $AF=PF$ and M be the midpoint of AC. We denote the intersection of PH and BC by X, the intersection of OM and FX by Y and the intersection of OF and AC by Z. Prove that the points F, M, Y and Z are concyclic.

Problem 2. Does there exist a sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ of positive real numbers satisfying both of the following conditions:

(i) $\sum_{i=1}^{n} a_i \le n^2$, for every positive integer n;

(ii) $\sum_{i=1}^{n} \frac{1}{a_i} \le 2008$, for every positive integer n?

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***August 20, 2008***.

For individual subscription for the next five issues for the 05-06 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Geometric Transformations I

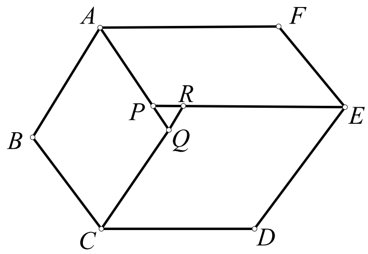
Kin Y. Li

Too often we *stare* at a figure in solving a geometry problem. In this article, we will *move* parts of the figure to better positions to facilitate the way to a solution.

Below we shall denote the vector from X to Y by the boldface italics $\boldsymbol{XY}$. On a plane, a *translation* by a vector v moves every point X to a point Y such that $\boldsymbol{XY} = v$. We denote this translation by $T(v)$.

Example 1. The opposite sides of a hexagon $ABCDEF$ are parallel. If $BC-EF = ED-AB = AF-CD > 0$, show that all angles of $ABCDEF$ are equal.

Solution. One idea is to move the side lengths closer to do the subtractions. Let $T(\boldsymbol{FA})$ move E to P, $T(\boldsymbol{BC})$ move A to Q and $T(\boldsymbol{DE})$ move C to R.



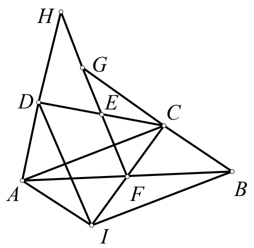
Hence, $EFAP$, $ABCQ$, $CDER$ are parallelograms. Since the opposite sides of the hexagon are parallel, P is on AQ, Q is on CR and R is on EP. Then, we get $BC - EF = AQ - AP = PQ$. Similarly, $ED - AB = QR$ and $AF - CD = RP$. Hence, ΔPQR is equilateral.

Now, $\angle ABC = \angle AQC = 120°$. Also, $\angle BCD = \angle BCQ + \angle DCQ = 60° + 60° = 120°$. Similarly, $\angle CDE = \angle DEF = \angle EFA = \angle FAB = 120°$.

Example 2. $ABCD$ is a convex quadrilateral with $AD = BC$. Let E, F be midpoints of CD, AB respectively. Suppose rays AD, FE intersect at H and rays BC, FE intersect at G. Show that

$$\angle AHF = \angle BGF.$$

Solution. One idea is to move BC closer to AD. Let $T(\boldsymbol{CB})$ move A to I.



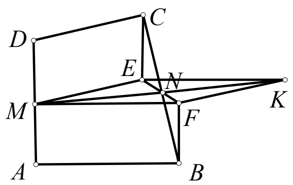
Then $BCAI$ is a parallelogram. Since F is the midpoint of AB, so F is also the midpoint of CI. Applying the midpoint theorem to ΔCDI, we get $EF||DI$. Using this and $CB||AI$, we get $\angle BGF = \angle AID$. From $AI = BC = AD$, we get $\angle AID = \angle ADI$. Since $EF || DI$, $\angle AHF = \angle ADI = \angle AID = \angle BGF$.

Example 3. Let M and N be the midpoints of sides AD and BC of quadrilateral $ABCD$ respectively. If

$$2MN = AB+CD,$$

then prove that $AB||CD$.

Solution. One idea is to move AB, CD closer to MN. Let $T(\boldsymbol{DC})$ move M to E and $T(\boldsymbol{AB})$ move M to F.



Then we can see $CDME$ and $BAMF$ are parallelograms. Since $\boldsymbol{EC} = ½\boldsymbol{AD} = \boldsymbol{BF}$, $BFCE$ is a parallelogram. Since N is the midpoint of BC, so N is also the midpoint of EF.

Next, let $T(\boldsymbol{ME})$ move F to K. Then $EMFK$ is a parallelogram and

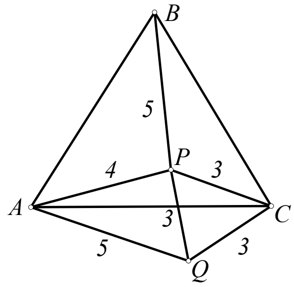
$$MK = 2MN = AB+CD = MF+EM = MF+FK.$$

So F, M, K, N are collinear and $AB||MN$. Similarly, $CD||MN$. Therefore, $AB||CD$.

On a plane, a <u>rotation</u> about a center O by angle α moves every point X to a point Y such that $OX = OY$ and $\angle XOY = \alpha$ (anticlockwise if $\alpha > 0$, clockwise if $\alpha < 0$). We denote this rotation by $R(O,\alpha)$.

<u>Example 4.</u> Inside an equilateral triangle ABC, there is a point P such that PC=3, PA=4 and PB=5. Find the perimeter of $\triangle ABC$.

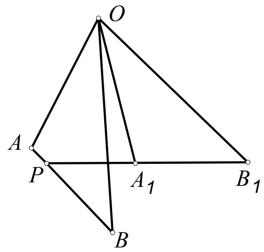
<u>Solution.</u> One idea is to move PC, PA, PB to form a triangle. Let $R(C,60°)$ move $\triangle CBP$ to $\triangle CAQ$.



Now $CP=CQ$ and $\angle PCQ = 60°$ imply $\triangle PCQ$ is equilateral. As $AQ = BP = 5$, $AP = 4$ and $PQ = PC = 3$, so $\angle APQ = 90°$. Then $\angle APC = \angle APQ + \angle QPC = 90°+60° = 150°$. So the perimeter of $\triangle ABC$ is

$$3AC = 3\sqrt{3^2+4^2-12\cos 150^\circ}$$
$$= 3\sqrt{25+12\sqrt{3}}.$$

For our next example, we will point out a property of rotation, namely



if $R(O,\alpha)$ moves a line AB to the line A_1B_1 and P is the intersection of the two lines, then these lines intersect at an angle α.

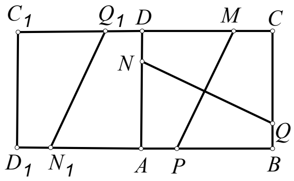
This is because $\angle OAB = \angle OA_1B_1$ implies O,A,P,A_1 are concyclic so that $\angle BPB_1 = \angle AOA_1 = \alpha$.

<u>Example 5.</u> $ABCD$ is a unit square. Points P,Q,M,N are on sides AB, BC, CD, DA respectively such that

$$AP + AN + CQ + CM = 2.$$

Prove that $PM \perp QN$.

<u>Solution.</u> One idea is to move AP, AN together and CQ, CM together. Let $R(A,90°)$ map $B \to D$, $C \to C_1$, $D \to D_1$, $Q \to Q_1$, $N \to N_1$ as shown below.



Then $AN=AN_1$ and $CQ=C_1Q_1$. So

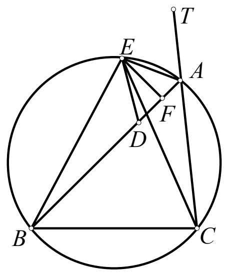
$$PN_1 = AP+AN_1 = AP+AN = 2-(CM+CQ)$$
$$= CC_1-(CM+C_1Q_1) = MQ_1.$$

Hence, PMQ_1N_1 is a parallelogram and $MP \| Q_1N_1$. By the property before the example, lines QN and Q_1N_1 intersect at 90°. Therefore, $PM \perp QN$.

<u>Example 6.</u> (*1989 Chinese National Senoir High Math Competition*) In $\triangle ABC$, $AB > AC$. An external bisector of $\angle BAC$ intersects the circumcircle of $\triangle ABC$ at E. Let F be the foot of perpendicular from E to line AB. Prove that

$$2AF = AB-AC.$$

<u>Solution.</u> One idea is to move AC to coincide with a part of AB. To do that, consider $R(E,\angle CEB)$.



Observe that $\angle EBC = \angle EAT = \angle EAB = \angle ECB$ implies $EC=EB$. So $R(E,\angle CEB)$ move C to B. Let $R(E,\angle CEB)$ move A to D. Since $\angle CAB = \angle CEB$, by the property above and $AB > AC$, D is on segment AB.

So $R(E,\angle CEB)$ moves $\triangle AEC$ to $\triangle DEB$. Then $\angle DAE = \angle EAT = \angle EDA$ implies $\triangle AED$ is isosceles. Since $EF \perp AD$,

$$2AF=AD=AB-BD=AB-AC.$$

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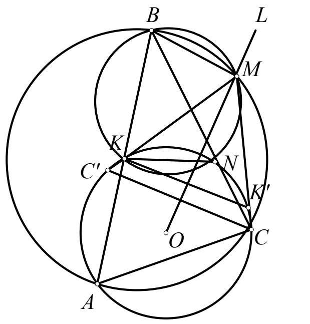
On a plane, a <u>reflection</u> across a line moves every point X to a point Y such that the line is the perpendicular bisector of segment XY. We say Y is the <u>mirror image</u> of X with respect to the line.

<u>Example 7.</u> (*1985 IMO*) A circle with center O passes through vertices A and C of $\triangle ABC$ and cuts sides AB, BC at K, N respectively. The circumcircles of $\triangle ABC$ and $\triangle KBN$ intersect at B and M. Prove that $\angle OMB = 90°$.

<u>Solution.</u> Let L be the line through O perpendicular to line BM. We are done if we can show M is on L.



Let the reflection across L maps $C \to C'$ and $K \to K'$. Then $CC' \perp L$ and $KK' \perp L$, which imply lines CC', KK', BM are parallel. We have

$$\angle KC'C = \angle KAC = \angle BNK = \angle BMK,$$

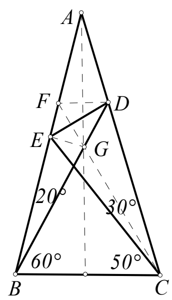
which implies C',K,M collinear. Now

$$\angle C'CK' = \angle CC'K = \angle CAK$$
$$= \angle CAB = 180° - \angle BMC$$
$$= \angle C'CM,$$

which implies C,K',M collinear. Then lines $C'K$ and CK' intersect at M. Since lines $C'K$ and CK' are symmetric with respect to L, so M is on L.

<u>Example 8.</u> Points D and E are on sides AB and AC of $\triangle ABC$ respectively with $\angle ABD = 20°$, $\angle DBC = 60°$, $\angle ACE = 30°$ and $\angle ECB = 50°$. Find $\angle EDB$.

<u>Solution.</u> Note $\angle ABC = \angle ACB$. Consider the reflection across the perpendicular bisector of side BC. Let the mirror image of D be F. Let BD intersect CF at G. Since $BG = CG$, lines BD, CF intersect at 60° so that $\triangle BGC$ and $\triangle DGF$ are equilateral. Then $DF=DG$.



We claim $EF = EG$ (which implies $\triangle EFD \cong \triangle EGD$. So $\angle EDB = ½\angle FDG = 30°$). For the claim, we have $\angle EFG = \angle CDG = 40°$ and $\angle FGB = 120°$.

Next $\angle BEC = 50°$. So $BE=BC$. As $\triangle BGC$ is equilateral, so $BE = BC = BG$. This gives $\angle EGB = 80°$. Then

$$\angle EGF = \angle FGB - \angle EGB$$
$$= 40° = \angle EFG,$$

which implies the claim.

(Continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***August 20, 2008.***

Problem 301. Prove that it is possible to decompose two congruent regular hexagons into a total of six pieces such that they can be rearranged to form an equilateral triangle with no pieces overlapping.

Problem 302. Let $\mathbb{Z}$ denotes the set of all integers. Determine (with proof) all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that for all x, y in $\mathbb{Z}$, we have $f(x+f(y)) = f(x) - y$.

Problem 303. In base 10, let N be a positive integer with all digits nonzero. Prove that there do not exist two permutations of the digits of N, forming numbers that are different (integral) powers of two.

Problem 304. Let M be a set of 100 distinct lattice points (i.e. coordinates are integers) chosen from the x-y coordinate plane. Prove that there are at most 2025 rectangles whose vertices are in M and whose sides are parallel to the x-axis or the y-axis.

Problem 305. A circle Γ_2 is internally tangent to the circumcircle Γ_1 of ΔPAB at P and side AB at C. Let E, F be the intersection of Γ_2 with sides PA, PB respectively. Let EF intersect PC at D. Lines PD, AD intersect Γ_1 again at G, H respectively. Prove that F, G, H are collinear.

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Solutions

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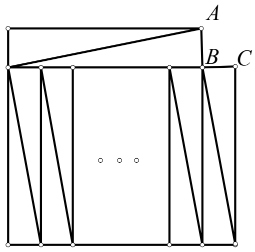
Problem 296. Let $n > 1$ be an integer. From a $n\times n$ square, one 1×1 corner square is removed. Determine (with proof) the least positive integer k such that the remaining areas can be partitioned into k triangles with equal areas.
(*Source 1992 Shanghai Math Contest*)

Solution. **Jeff CHEN** (Virginia, USA), **O Kin Chit Alex** (GT Ellen Yeung College), **PUN Ying Anna** (HKU Math Year 2), **Simon YAU Chi-Keung** (City University of Hong Kong) and **Fai YUNG**.



The figure above shows the least k is at most $2n+2$. Conversely, suppose the required partition is possible for some k. Then one of the triangles must have a side lying in part of segment AB or in part of segment BC. Then the length of that side is at most 1. Next, the altitude perpendicular to that side is at most $n - 1$. Hence, that triangle has an area at most $(n-1)/2$. That is $(n^2-1)/k \le (n-1)/2$. So $k \ge 2n + 2$. Therefore, the least k is $2n+2$.

Problem 297. Prove that for every pair of positive integers p and q, there exist an integer-coefficient polynomial $f(x)$ and an open interval with length $1/q$ on the real axis such that for every x in the interval, $|f(x) - p/q| < 1/q^2$.
(*Source:1983 Finnish Math Olympiad*)

Solution. **Jeff CHEN** (Virginia, USA) and **PUN Ying Anna** (HKU Math Year 2).

If $q = 1$, then take $f(x) = p$ works for any interval of length $1/q$. If $q > 1$, then define the interval $I = \left(\frac{1}{2q}, \frac{3}{2q}\right)$.

Choosing a positive integer m greater than $(\log q)/(\log 2q/3)$, we get $[3/(2q)]^m < 1/q$. Let $a = 1-[1/(2q)]^m$. Then for all x in I, we have $0 < 1 - qx^m < a < 1$.

Choosing a positive integer n greater than $-(\log pq)/(\log a)$, we get $a^n < 1/(pq)$. Let

$$f(x) = \frac{p}{q}[1-(1-qx^m)^n].$$

Now

$$f(x) = \frac{p}{q}[1-(1-qx^m)]\sum_{k=0}^{n-1}(1-qx^m)^k$$

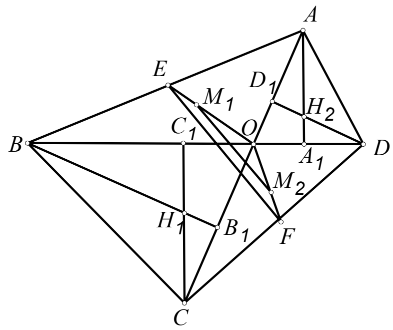
$$= px^m\sum_{k=0}^{n-1}(1-qx^m)^k$$

has integer coefficients. For x in I, we have

$$\left|f(x)-\frac{p}{q}\right| = \frac{p}{q}\left|(1-qx^m)^n\right| < \frac{p}{q}a^n < \frac{1}{q^2}.$$

Problem 298. The diagonals of a convex quadrilateral $ABCD$ intersect at O. Let M_1 and M_2 be the centroids of ΔAOB and ΔCOD respectively. Let H_1 and H_2 be the orthocenters of ΔBOC and ΔDOA respectively. Prove that $M_1M_2\perp H_1H_2$.

Solution. **Jeff CHEN** (Virginia, USA).



Let A_1, C_1 be the feet of the perpendiculars from A, C to line BD respectively. Let B_1, D_1 be the feet of the perpendiculars from B, D to line AC respectively. Let E, F be the midpoints of sides AB, CD respectively. Since

$$OM_1/OE = 2/3 = OM_2/OF,$$

we get $EF \parallel M_1M_2$. Thus, it suffices to show $H_1H_2\perp EF$.

Now the angles AA_1B and BB_1A are right angles. So A, A_1, B, B_1 lie on a circle Γ_1 with E as center. Similarly, C, C_1, D, D_1 lie on a circle Γ_2 with F as center.

Next, since the angles AA_1D and DD_1A are right angles, points A,D,A_1,D_1 are concyclic. By the intersecting chord theorem, $AH_2\cdot H_2A_1 = DH_2\cdot H_2D_1$.

This implies H_2 has equal power with respect to Γ_1 and Γ_2. Similarly, H_1 has equal power with respect to Γ_1 and Γ_2. Hence, line H_1H_2 is the radical axis of Γ_1 and Γ_2. Since the radical axis is perpendicular to the line joining the centers of the circles, we get $H_1H_2\perp EF$.

Comments: For those who are not familiar with the concepts of power and radical axis of circles, please see <u>*Math. Excalibur*</u>, *vol.* 4, *no.* 3, *pp.* 2,4.

Commended solvers: **PUN Ying Anna** (HKU Math Year 2) and **Simon YAU Chi-Keung** (City University of Hong Kong).

Problem 299. Determine (with proof) the least positive integer n such that in every way of partitioning $S = \{1,2,\ldots,n\}$ into two subsets, one of the subsets will contain two distinct numbers a and b such that ab is divisible by $a+b$.

Solution. **Jeff CHEN** (Virginia, USA),

PUN Ying Anna (HKU Math Year 2).

Call a pair (a,b) of distinct positive integers a <u>*good*</u> pair if and only if ab is divisible by $a+b$. Here is a list of good pairs with $1 < a < b < 50$:

(3,6), (4,12), (5,20), (6,12), (6,30), (7,42), (8,24), (9,18), (10,15), (10,40), (12,24), (12,36), (14,35), (15,30), (16,48), (18,36), (20,30), (21,28), (21,42), (24,40), (24,48), (30,45), (36,45).

Now we try to put the positive integers from 1 to 39 into one of two sets S_1, S_2 so that no good pair is in the same set. If a positive integer is not in any good pair, then it does not matter which set it is in, say we put it in S_1. Then we get

S_1={1, 2, 3, 5, 8, 10, 12, 13, 14, 18, 19, 21, 22, 23, 30, 31, 32, 33, 34, 36} and S_2={4, 6, 7, 9, 11,15, 17, 20, 24, 25, 26, 27, 28, 29, 35, 37, 38, 39}.

So 1 to 39 do not have the property.

Next, for $n = 40$, we observe that any two consecutive terms of the sequence 6, 30, 15, 10, 40, 24, 12, 6 forms a good pair. So no matter how we divide the numbers 6, 30, 15, 10, 40, 24, 12 into two sets, there will be a good pair in one of them. So, $n = 40$ is the least case.

Problem 300. Prove that in base 10, every odd positive integer has a multiple all of whose digits are odd.

Solution. **Jeff CHEN** (Virginia, USA) and **G.R.A. 20 Problem Solving Group** (Roma, Italy), **PUN Ying Anna** (HKU Math Year 2).

We first show by induction that <u>for every positive integer k, there is a k-digit number n_k whose digits are all odd and n_k is a multiple of 5^k</u>. We can take n_1=5. Suppose this is true for k. We will consider the case $k+1$. If n_k is a multiple of 5^{k+1}, then take n_{k+1} to be $n_k + 5\times10^k$. Otherwise, n_k is of the form $5^k(5i+j)$, where i is a nonnegative integer and j = 1, 2, 3 or 4. Since $\gcd(5,2^k) = 1$, one of the numbers 10^k+n_k, $3\times10^k+n_k$, $7\times10^k+n_k$, $9\times10^k+n_k$ is a multiple of 5^{k+1}. Hence we may take it to be n_{k+1}, which completes the induction.

Now for the problem, let m be an odd number. Let $N(a,b)$ denote the number whose digits are those of a written b times in a row. For example, $N(27,3)$ = 272727.

Observe that m is of the form 5^kM, where k is a nonnegative integer and $\gcd(M,5) = 1$. Let $n_0 = 1$ and for $k > 0$, let n_k be as in the underlined statement above. Consider the numbers $N(n_k,1)$, $N(n_k,2)$, …, $N(n_k, M+1)$. By the pigeonhole principle, two of these numbers, say $N(n_k, i)$ and $N(n_k, j)$ with $1 \le i < j \le M+1$, have the same remainder when dividing by M. Then $N(n_k, j) - N(n_k, i) = N(n_k, j-i) \times 10^{ik}$ is a multiple of M and 5^k.

Finally, since $\gcd(M, 10) = 1$, $N(n_k, j-i)$ is also a multiple of M and 5^k. Therefore, it is a multiple of m and it has only odd digits.

Olympiad Corner

(continued from page 1)

Problem 3. Let n be a positive integer. The rectangle $ABCD$ with side lengths AB=90n+1 and BC=90n+5 is partitioned into unit squares with sides parallel to the sides of $ABCD$. Let S be the set of all points which are vertices of these unit squares. Prove that the number of lines which pass through at least two points from S is divisible by 4.

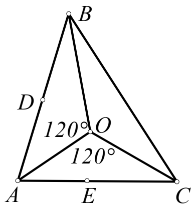
Problem 4. Let c be a positive integer. The sequence $a_1, a_2, \ldots, a_n, \ldots$ is defined by $a_1=c$ and $a_{n+1}=a_n^2+a_n+c$ for every positive integer n. Find all values of c for which there exist some integers $k \ge 1$ and $m \ge 2$ such that $a_k^2+c^3$ is the m^{th} power of some positive integer.

Geometric Transformations I

(continued from page 2)

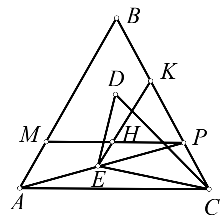
On a plane, a <u>*spiral similarity*</u> with center O, angle α and ratio k moves every point X to a point Y such that $\angle XOY = \alpha$ and $OY/OX = k$, i.e. it is a rotation with a homothety. We denote it by $S(O,\alpha, k)$.

<u>***Example 9.***</u> (*1996 St. Petersburg Math Olympiad*) In ΔABC, $\angle BAC$=60°. A point O is inside the triangle such that $\angle AOB = \angle BOC = \angle COA$. Points D and E are the midpoints of sides AB and AC, respectively. Prove that A, D, O, E are concyclic.



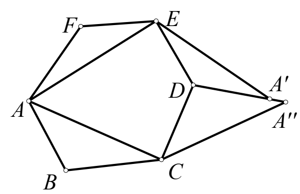
<u>***Solution.***</u> Since $\angle AOB = \angle COA$=120° and $\angle OBA$=60°−$\angle OAB = \angle OAC$, we see $\Delta AOB \sim \Delta COA$. Then the spiral similarity $S(O,120°,OC/OA)$ maps $\Delta AOB \rightarrow \Delta COA$ and also $D \rightarrow E$. Then $\angle DOE$ = 120° = 180°−$\angle BAC$, which implies A, D, O, E concyclic.

<u>***Example 10.***</u> (*1980 All Soviet Math Olympiad*) ΔABC is equilateral. M is on side AB and P is on side CB such that $MP||AC$. D is the centroid of ΔMBP and E is the midpoint of PA. Find the angles of ΔDEC.



<u>***Solution.***</u> Let H and K be the midpoints of PM and PB respectively. Observe that $S(D,-60°,1/2)$ maps $P \rightarrow H$, $B \rightarrow K$ and so $PB \rightarrow HK$. Now H, K, E are collinear as they are midpoints of PM, PB, PA. Note $BC/BP = BA/BM = KE/KH$, which implies $S(D,-60°,1/2)$ maps $C \rightarrow E$. Then $\angle EDC$ = 60° and DE=½DC. So we have $\angle DEC$ = 90° and $\angle DCE$ = 30°.

<u>***Example 11.***</u> (*1998 IMO Proposal by Poland*) Let $ABCDEF$ be a convex hexagon such that $\angle B + \angle D + \angle F$ = 360° and $(AB/BC)(CD/DE)(EF/FA)$=1. Prove $(BC/CA)(AE/EF)(FD/DB)$=1.



<u>***Solution.***</u> Since $\angle B + \angle D + \angle F$ =360°, $S(E, \angle FED, ED/EF)$ maps $\Delta FEA \rightarrow \Delta DEA'$ and $S(C, \angle BCD, CD/CB)$ maps $\Delta BCA \rightarrow \Delta DCA''$. So $\Delta FEA \sim \Delta DEA'$ and $\Delta BCA \sim \Delta DCA''$. These yield $BC/CA = DC/CA''$, $DE/EF = DA'/FA$ and using the given equation, we get

$$\frac{A''D}{DC} = \frac{AB}{BC} = \frac{DE}{CD}\frac{FA}{EF} = \frac{DA'}{CD},$$

which implies $A' = A''$. Next $\angle AEF = \angle A'ED$ implies $\angle DEF = \angle A'EA$. As $DE/FE = A'E/AE$, so $\Delta DEF \sim \Delta A'EA$ and $AE/FE = AA'/FD$. Similarly, we get $\Delta DCB \sim \Delta A'CA$ and $DC/A'C = DB/A'A$. Therefore,

$$\frac{BC}{CA}\frac{AE}{EF}\frac{FD}{DB} = \frac{DC}{CA''}\frac{AA'}{DB} = 1.$$