Mathematical Excalibur

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Olympiad Corner

The 2009 Czech-Polish-Slovak Math Competition was held on June 21-24. The following were the problems.

Problem 1. Let $\mathbb{R}^+$ be the set of all positive real numbers. Find all functions f: $\mathbb{R}^+\to\mathbb{R}^+$ satisfying

$$(1+yf(x))(1-yf(x+y))=1$$

for all $x,y\in\mathbb{R}^+$.

Problem 2. Given positive integers a and k, the sequence a_1, a_2, a_3, … is defined by $a_1=a$ and $a_{n+1}=a_n+k\rho(a_n)$, where $\rho(m)$ stands for the product of the digits of m in its decimal representation (e.g. $\rho(413)=12$, $\rho(308)=0$). Prove that there exist positive integers a and k such that the sequence a_1, a_2, a_3, … contains exactly 2009 different numbers.

Problem 3. Given ΔABC, let k be the excircle at the side BC. Choose any line p parallel to BC intersecting line segments AB and AC at points D and E. Denote by ℓ the incircle of ΔADE. The tangents from D and E to the circle k not passing through A intersect at P. The tangents from B and C to the circle ℓ not passing through A intersect at Q. Prove that the line PQ passes through a point independent of p.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***December 1, 2009***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Probabilistic Method

Law Ka Ho

Roughly speaking, the probabilistic method helps us solve combinatorial problems via considerations related to probability.

We know that among any 6 people, there exist 3 who know each other or 3 who don't know each other (we assume if A knows B, then B knows A). When 6 is replaced by 5, this is no longer true, as can be seen by constructing a counter-example. When the numbers get large, constructing counterexamples becomes difficult. In this case the probabilistic method helps.

Example 1. Show that among 2^{100} people, there do not necessarily exist 200 people who know each other or 200 people who don't know each other.

Solution. Assign each pair of people to be knowing each other or not by flipping a fair coin. Among a set of 200 people, the probability that they know each other or they don't know each other is thus $2\times 2^{-C_2^{200}}=2^{-19899}$. As there are $C_{200}^{2^{100}}$ choices of 200 people, the probability that there exist 200 people who know each other or 200 people who don't know each other is at most

$$C_{200}^{2^{100}}\times 2^{-19899}<\frac{(2^{100})^{200}}{200!}\times 2^{-19899}$$
$$=\frac{2^{101}}{200!}<1$$

Hence the probability for the non-existence of 200 people who know each other or 200 people who don't know each other is greater than 0, which implies the result.

Here we see that the general rationale is to show that in a random construction of an example, the probability that it satisfies what we want is positive, which means that there exists such an example. Clearly, the

Example 2. In each cell of a 100×100 table, one of the integers 1, 2, …, 5000 is written. Moreover, each integer appears in the table exactly twice. Prove that one can choose 100 cells in the table satisfying the three conditions below:

(1) Exactly one cell is chosen in each row.

(2) Exactly one cell is chosen in each column.

(3) The numbers in the cells chosen are pairwise distinct.

Solution. Take a random permutation a_1, …, a_{100} of {1, …, 100} and choose the a_i-th cell in the i-th row. Such choice satisfies (1) and (2). For j = 1, …, 5000, the probability of choosing both cells written j is

$$\begin{cases} 0 & \text{they are in the same row or column} \\ \dfrac{1}{100}\times\dfrac{1}{99} & \text{otherwise} \end{cases}$$

Hence the probability that such choice satisfies (3) is at least

$$1-5000\times\frac{1}{100}\times\frac{1}{99}>0.$$

Of course, one can easily transform the above two probabilistic solutions to merely using counting arguments (by counting the number of 'favorable outcomes' instead of computing the probabilities), which is essentially the same. But a probabilistic solution is usually neater and more natural.

Another common technique in the probabilistic method is to compute the average (or expected value) – the total is the average times the number of items, and there exists an item which is as good as the average. These are illustrated in the next two examples.

Example 3. (*APMO 1998*) Let F be the set of all n-tuples (A_1 , A_2 , …, A_n) where each A_i , i = 1, 2, …, n, is a subset of {1, 2, …, 1998}. Let $|A|$ denote the number of elements of the set A. Find the number

$$\sum_{(A_1, A_2, \ldots, A_n)} |A_1 \cup A_2 \cup \cdots \cup A_n|.$$

Solution. (*Due to Leung Wing Chung, 1998 Hong Kong IMO team member*) Note that the set {1, 2, …, 1998} has 2^{1998} subsets because we may choose to include or not to include each of the 1998 elements in a subset. Hence there are altogether 2^{1998n} terms in the summation.

Now we compute the average value of each term. For i = 1, 2, …, 1998, i is an element of $A_1 \cup A_2 \cup \cdots \cup A_n$ if and only if i is an element of at least one of $A_1, A_2, \ldots, A_n$. The probability for this to happen is $1-2^{-n}$. Hence the average value of each term in the summation is $1998(1-2^{-n})$, and so the answer is $2^{1998n} \cdot 1998(1-2^{-n})$.

Example 4. In a chess tournament there are 40 players. A total of 80 games have been played, and every two players compete at most once. For certain integer n, show that there exist n players, no two of whom have competed. (Of course, the larger the n, the stronger the result.)

Solution 4.1. If we use a traditional counting approach, we can prove the case $n=4$. Assume on the contrary that among any 4 players, at least one match is played. Then the number of games played is at least $C_4^{40} \div C_2^{38} = 260$, a contradiction. Note that this approach cannot prove the $n=5$ case since $C_5^{40} \div C_3^{38} = 78 < 80$.

Solution 4.2. We use a probabilistic approach to prove the $n=5$ case. Randomly choose some players such that each player has probability 0.25 to be chosen. Then discard all players who had lost in a match with another chosen player. In this way no two remaining players have played with each other.

What is the average number of players left? On average $40 \times 0.25 = 10$ players would be chosen. For each match played, the probability that both players are chosen is 0.25^2, so on average there are $80 \times 0.25^2 = 5$ matches played among the chosen players. After discarding the losers, the average number of players left is at least 5 (in fact greater than 5 since the losers could repeat). That means there exists a choice in which we obtain at least 5 players who have not played against each other.

(Note: if we replace 0.25 by p, then the average number of players left would be $40p - 80p^2 = 5 - 80(p-0.25)^2$ and this explains the choice of the number 0.25.)

Solution 4.3. This time we use another probabilistic approach to prove the $n=8$ case. (!!) We assign a random ranking to the 40 players, and we pick those who have only played against players with lower ranking. Note that in this way no two of the chosen players have competed.

Suppose the i-th player has played d_i games. Since 80 games have been played, we have $d_1 + d_2 + \cdots + d_{40} = 80 \times 2$. Also, the i-th player is chosen if and only if he is assigned the highest ranking among himself and the players with whom he has competed, and the probability for this to happen is $1/(d_i+1)$. Hence the average number of players chosen is

$$\frac{1}{d_1+1} + \cdots + \frac{1}{d_{40}+1} \ge \frac{40^2}{(d_1+1)+\cdots+(d_{40}+1)} = \frac{40^2}{160+40} = 8$$

Here we made use of the Cauchy- Schwarz inequality. This means there exists 8 players, no two of whom have competed.

Remark. Solution 4.3 is the best possible result. Indeed, we may divide the 40 players into eight groups of 5 players each. If two players have competed if and only if they are from the same group, then the number of games played will be $8 \times C_2^5 = 80$ and it is clear that it is impossible to find 9 players, no two of whom have competed.

The above example shows that the probabilistic method can sometimes be more powerful than traditional methods. We conclude with the following example, which makes use of an apparently trivial property of probability, namely the probability of an event always lies between 0 and 1.

Example 5. In a public examination there are n subjects, each offered in Chinese and English. Candidates may sit for as many (or as few) subjects as they like, but each candidate may only choose one language version for each subject. For any two different subjects, there exists a candidate sitting for different language versions of the two subjects. If there are at most 10 candidates sitting for each subject, determine the maximum possible value of n.

Solution. The answer is 1024. The following example shows that n = 1024 is possible. Suppose there are 10 candidates (numbered 1 to 10), each sitting for all 1024 subjects (numbered 0 to 1023). For student i, the j-th subject is taken in Chinese if the i-th digit from the right is 0 in the binary representation of j, and the subject is taken in English otherwise. In this way it is easy to check that the given condition is satisfied. (The answer along with the example is not difficult to get if one begins by replacing 10 with smaller numbers and then observe the pattern.)

To show that 1024 is the maximum, we randomly assign each candidate to be ‘Chinese’ or ‘English’. Let E_j be the event ‘all candidates in the j-th subject are sitting for the language version which matches their assigned identity’. As there are at most 10 candidates in each subject, we have the probability

$$P(E_j) \ge 2^{-10} = \frac{1}{1024}.$$

Since ‘for any two different subjects, there exists a candidate sitting for different language versions of the two subjects’, no two E_j may occur simultaneously. It follows that

$$\begin{aligned} &P(\text{at least one } E_j \text{ happens}) \\ &= P(E_1) + P(E_2) + \cdots + P(E_n) \\ &\ge \frac{n}{1024} \end{aligned}$$

But since the probability of an event is at most 1, the above gives $1 \ge \frac{n}{1024}$, so we have $n \le 1024$ as desired!

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***December 1, 2009.***

Problem 331. For every positive integer n, prove that

$$\sum_{k=0}^{n-1}(-1)^k \cos^n(k\pi/n)=\frac{n}{2^{n-1}}.$$

Problem 332. Let $ABCD$ be a cyclic quadrilateral with circumcenter O. Let BD bisect OC perpendicularly. On diagonal AC, choose the point P such that $PC=OC$. Let line BP intersect line AD and the circumcircle of $ABCD$ at E and F respectively. Prove that PF is the geometric mean of EF and BF in length.

Problem 333. Find the largest positive integer n such that there exist n 4-element sets $A_1, A_2, \ldots, A_n$ such that every pair of them has exactly one common element and the union of these n sets has exactly n elements.

Problem 334. (*Due to FEI Zhenpeng, Northeast Yucai School, China*) Let $x,y \in (0,1)$ and x be the number whose n-th digit after the decimal point is the n^n-th digit after the decimal point of y for all $n = 1,2,3,\ldots.$ Show that if y is rational, then x is rational.

Problem 335. (*Due to Ozgur KIRCAK, Yahya Kemal College, Skopje, Macedonia*) Find all $a\in\mathbb{R}$ for which the functional equation $f\colon \mathbb{R}\to\mathbb{R}$

$$f(x-f(y))=a(f(x)-x)-f(y)$$

for all $x, y \in\mathbb{R}$ has a unique solution.

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Solutions

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Problem 326. Prove that $3^{4^5}+4^{5^6}$ is the product of two integers, each at least 10^{2009}.

Solution. **CHAN Ho Lam Franco** (GT (Ellen Yeung) College, Form 3), **D. Kipp JOHNSON** (Valley Catholic School, Teacher, Beaverton, Oregon, USA), **Manh Dung NGUYEN** (Hanoi University of Technology, Vietnam), **NGUYEN Van Thien** (Luong The Vinh High School, Dong Nai, Vietnam), **O Kin Chit Alex** (GT(Ellen Yeung) College) and **Pedro Henrique O. PANTOJA** (UFRN, Brazil).

Let $a = 3^{256}$ and $b = 4^{3906}$. Then

$$\begin{aligned}3^{4^5}+4^{5^6} &= a^4+4b^4\\ &= (a^4+4a^2b^2+4b^4)-4a^2b^2\\ &= (a^2+2b^2+2ab)(a^2+2b^2-2ab).\end{aligned}$$

Note that $a^2+2b^2+2ab > a^2+2b^2-2ab > 2b^2-2ab = 2b(b-a) > b > 2^{7800} > (10^3)^{780} > 10^{2009}$. The result follows.

Problem 327. Eight pieces are placed on a chessboard so that each row and each column contains exactly one piece. Prove that there is an even number of pieces on the black squares of the board.

(*Source: 1989 USSR Math Olympiad*)

Solution. **G.R.A. 20 Problem Solving Group** (Roma, Italy), **HUNG Ka Kin Kenneth** (Diocesan Boys' School), **LKL Problem Solving Group** (Madam Lau Kam Lung Secondary School of MFBM) and **YUNG Fai**.

Without loss of generality, we may assume the square in row 1, column 1 is not black. Then, for all $i, j = 1,2,\ldots,8$, the square in row i, column j is black if and only if $i + j \equiv 1 \pmod 2$. Since the pieces are in different columns, the position of the piece contained in the i-th row is in column $p(i)$, where p is some permutation of $\{1,2,\ldots,8\}$. Therefore, the number of pieces on the black squares in mod 2 is congruent to

$$\sum_{i=1}^{8}(i+p(i))=\sum_{i=1}^{8}i+\sum_{i=1}^{8}p(i)=72,$$

which is even.

Other commended solvers: **Abby LEE** (SKH Lam Woo Memorial Secondary School) and **NGUYEN Van Thien** (Luong The Vinh High School, Dong Nai, Vietnam).

Problem 328. *(Due to Tuan Le, Fairmont High School, Anaheim, Ca., USA)* Let $a,b,c > 0$. Prove that

$$\frac{\sqrt{a^3+b^3}}{a^2+b^2}+\frac{\sqrt{b^3+c^3}}{b^2+c^2}+\frac{\sqrt{c^3+a^3}}{c^2+a^2}$$
$$\ge\frac{6(ab+bc+ca)}{(a+b+c)\sqrt{(a+b)(b+c)(c+a)}}.$$

Solution 1. **Manh Dung NGUYEN** (Hanoi University of Technology, Vietnam), **NGUYEN Van Thien** (Luong The Vinh High School, Dong Nai, Vietnam),

Below we will use the cyclic notation

$$\sum_{cyc}f(a,b,c)=f(a,b,c)+f(b,c,a)+f(c,a,b).$$

By the Cauchy-Schwarz inequality, we have $(a^3+b^3)(a+b)\ge(a^2+b^2)^2$. Using this, the left side is

$$\sum_{cyc}\frac{\sqrt{a^3+b^3}}{a^2+b^2}\ge\sum_{cyc}\frac{1}{\sqrt{a+b}}$$
$$=\frac{\sum_{cyc}\sqrt{(a+b)(b+c)}}{\sqrt{(a+b)(b+c)(c+a)}}.$$

So it suffices to show

$$\sum_{cyc}\sqrt{(a+b)(b+c)}\ge\frac{6(ab+bc+ca)}{a+b+c}.\quad(*)$$

First we claim that

$$(a+b)(b+c)(c+a)\ge\frac{8}{9}(a+b+c)(ab+bc+ca)$$

and $(a+b+c)^2\ge 3(ab+bc+ca)$.

These follow from

$$9(a+b)(b+c)(c+a)-8(a+b+c)(ab+bc+ca)$$
$$=a(b-c)^2+b(c-a)^2+c(a-b)^2\ge 0$$

and

$$(a+b+c)^2-3(ab+bc+ca)$$
$$=\frac{(a-b)^2+(b-c)^2+(c-a)^2}{2}\ge 0.$$

By the AM-GM inequality,

$$\sum_{cyc}\sqrt{(a+b)(b+c)}\ge 3\sqrt[3]{(a+b)(b+c)(c+a)}.$$

To get (*), it remains to show

$$(a+b+c)\sqrt[3]{(a+b)(b+c)(c+a)}\ge 2(ab+bc+ca).$$

This follows by cubing both sides and using the two inequalities in the claim to get

$$(a+b+c)^3(a+b)(b+c)(c+a)$$
$$\ge\frac{8}{9}(a+b+c)^4(ab+bc+ca)$$
$$\ge 8(ab+bc+ca)^3.$$

Solution 2. **LEE Ching Cheong** (HKUST, Year 1).

Due to the homogeneity of the original inequality, without loss of generality we may assume $ab+bc+ca = 1$. Then

$(a+b)(b+c) = 1+b^2$. The inequality (*) in solution 1 becomes

$$\sum_{cyc}\sqrt{1+b^2} \ge \frac{6}{a+b+c}.$$

Observe that

$$\sqrt{1+x^2} \ge \frac{1}{2}\left(x-\frac{1}{\sqrt{3}}\right)+\frac{2}{\sqrt{3}} = \frac{x+\sqrt{3}}{2},$$

which can be checked by squaring both sides and simplified to $(\sqrt{3}x-1)^2 \ge 0$ (or alternatively, $f(x) = \sqrt{1+x^2}$ is a convex function on $\mathbb{R}$ and $y = (x+\sqrt{3})/2$ is the equation of the tangent line to the graph of $f(x)$ at $(1/\sqrt{3}, 2/\sqrt{3})$.)

Now $(a+b+c)^2 \ge 3(ab+bc+ca)$ can be expressed as

$$\sum_{cyc} b = a+b+c \ge \sqrt{3}.$$

Using these, inequality (*) follows as

$$\sum_{cyc}\sqrt{1+b^2} \ge \frac{\sum_{cyc} b + 3\sqrt{3}}{2}$$

$$\ge 2\sqrt{3} \ge \frac{6}{a+b+c}.$$

Other commended solvers: **Salem MALIKIĆ** (Student, University of Sarajevo, Bosnia and Herzegovina) and **Paolo PERFETTI** (Math Dept, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy).

Problem 329. Let $C(n,k)$ denote the binomial coefficient with value $n!/(k!(n-k)!)$. Determine all positive integers n such that for all $k = 1, 2, \cdots, n-1$, we have $C(2n,2k)$ is divisible by $C(n,k)$.

Solution. **HUNG Ka Kin Kenneth** (Diocesan Boys' School).

For $n < 6$, we can check that $n = 1, 2, 3$ and 5 are the only solutions. For $n \ge 6$, we will show there are no solutions. Observe that after simplification,

$$\frac{C(2n,2k)}{C(n,k)} = \frac{(2n-1)(2n-3)\cdots(2n-2k+1)}{(2k-1)(2k-3)\cdots 1}.$$

Let n be an even integer with $n \ge 6$. Then $n-1 \ge 5$. So $n-1$ has a <u>prime</u> factor $p \ge 3$. Now $1 < (p+1)/2 \le n/2 < n-1$. Let $k = (p+1)/2$. Then $p = 2k-1$, but p is not a factor of $2n-1, 2n-3, \ldots, 2n-2k+1$ since the closest consecutive multiples of p are $2n-2k-1 = 2(n-1)-p$ and $2n - 2 = 2(n-1)$. Hence, $C(2n, 2k)/C(n, k)$ is not an integer. So such n cannot a solution for the problem.

For an odd integer $n \ge 7$, we divide into three cases.

<u>Case 1</u> : ($n-1 \ne 2^a$ for all a=1,2,3,…) Then $n-1$ has a <u>prime</u> factor $p \ge 3$. We repeat the argument above.

<u>Case 2</u> : ($n-2 \ne 3^b$ for all b=1,2,3,…) Then $n-2$ has a <u>prime</u> factor $p \ge 5$. Now $1 < (p+1)/2 \le n/2 < n-1$. Let $k = (p+1)/2$. Then $p=2k-1$, but p is not a factor of $2n-1, 2n-3, \ldots, 2n-2k+1$ since again $2n-2k-3 = 2(n-2)-p$ and $2n-4 = 2(n-2)$ are multiples of p. Hence, $C(2n,2k)/C(n,k)$ is not an integer.

<u>Case 3</u> : ($n-1 = 2^a$ and $n-2 = 3^b$ for some positive integers a and b) Then $2^a - 3^b = 1$. Consider mod 3, we see a is even, say $a = 2c$. Then

$$3^b = 2^a - 1 = 2^{2c} - 1 = (2^c-1)(2^c+1).$$

Since 2^c+1 and 2^c-1 have a difference of 2 and they are powers of 3 by unique prime factorization, we must have $c = 1$. Then $a = 2$ and $n = 5$, which contradicts $n \ge 7$.

Other commended solvers: **G.R.A. 20 Problem Solving Group** (Roma, Italy) and **O Kin Chit Alex** (GT(Ellen Yeung) College).

Problem 330. In ΔABC, $AB = AC = 1$ and $\angle BAC = 90°$. Let D be the midpoint of side BC. Let E be a point inside segment CD and F be a point inside segment BD. Let M be the point of intersection of the circumcircles of ΔADE and ΔABF, other than A. Let N be the point of intersection of the circumcircle of ΔACE and line AF, other than A. Let P be the point of intersection of the circumcircle of ΔAMN and line AD, other than A. Determine the length of segment AP with proof.
(*Source: 2003 Chinese IMO team test*)

Official Solution.

We will show A, B, P, C are concyclic. (Then, by symmetry, AP is a diameter of the circumcircle of ΔABC. We see $\angle ABP = 90°$, $AB = 1$ and $\angle BAP = 45°$, which imply $AP = \sqrt{2}$.)

Consider inversion with center at A and $r = 1$. Let X^* denote the image of point X. Let the intersection of lines XY and WZ be denoted by $XY \cap WZ$. We have $B^* = B$ and $C^* = C$. The line BC is sent to the circumcircle ω of ΔABC. The points F, D, E are sent to the intersection points F^*, D^*, E^* of lines AF, AD, AE with ω respectively.

The circumcircles of ΔADE and ΔABF are sent to lines D^*E^* and BF^*. So $M^* = D^*E^* \cap BF^*$. Also, the circumcircle of ΔACE and line AF are sent to lines CE^* and AF^*. Hence, $N^* = CE^* \cap AF^*$. Next, the circumcircle of ΔAMN and line AD are sent to lines M^*N^* and AD^*. So, $P^* = M^*N^* \cap AD^*$.

Now D^*, E^*, C, B, F^*, A are six points on ω. By Pascal's theorem, $M^* = D^*E^* \cap BF^*$, $N^* = E^*C \cap F^*A$ and $D = CB \cap AD^*$ are collinear. Since $P^* = M^*N^* \cap AD^*$, we get $D = P^*$. Then $P = D^*$ and A, B, P, C are all on ω.

Olympiad Corner

(continued from page 1)

Problem 4. Given a circle k and its chord AB which is not a diameter, let C be any point inside the longer arc AB of k. We denote by K and L the reflections of A and B with respect to the axes BC and AC. Prove that the distance of the midpoints of the line segments KL and AB is independent of the location of point C.

Problem 5. The n-tuple of positive integers $a_1,\ldots,a_n$ satisfies the following conditions:

(i) $1 \le a_1 < a_2 < \cdots < a_n \le 50$;

(ii) for any n-tuple of positive integers $b_1,\ldots, b_n$, there exist a positive integer m and an n-tuple of positive integers $c_1,\ldots, c_n$ such that

$$mb_i = c_i^{a_i} \text{ for } i = 1,\ldots,n.$$

Prove that $n \le 16$ and find the number of different n-tuples $a_1,\ldots, a_n$ satisfying the given conditions for $n = 16$.

Problem 6. Given an integer $n \ge 16$, consider the set

$$G = \{(x,y): x,y \in \{1,2,\ldots,n\}\}$$

consisting of n^2 points in the plane. Let A be any subset of G containing at least $4n\sqrt{n}$ points. Prove that there are at least n^2 convex quadrangles with all their vertices in A such that their diagonals intersect in one common point.