Mathematical Excalibur

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Olympiad Corner

Below are the problems used in the selection of the Indian team for IMO-2010.

Problem 1. Is there a positive integer n, which is a multiple of 103, such that $2^{2n+1} \equiv 2 \pmod{n}$?

Problem 2. Let a, b, c be integers such that b is even. Suppose the equation $x^3+ax^2+bx+c=0$ has roots α, β, γ such that $\alpha^2 = \beta+\gamma$. Prove that α is an integer and $\beta \neq \gamma$.

Problem 3. Let ABC be a triangle in which $BC < AC$. Let M be the midpoint of AB; AP be the altitude from A on to BC; and BQ be the altitude from B on to AC. Suppose QP produced meet AB (extended) in T. If H is the orthocenter of ABC, prove that TH is perpendicular to CM.

Problem 4. Let $ABCD$ be a cyclic quadrilateral and let E be the point of intersection of its diagonals AC and BD. Suppose AD and BC meet in F. Let the midpoints of AB and CD be G and H respectively. If Γ is the circumcircle of triangle EGH, prove that FE is tangent to Γ.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***October 20, 2010***.

For individual subscription for the next five issues for the 10-11 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Lagrange Interpolation Formula

Kin Y. Li

Let n be a positive integer. If we are given two collections of n+1 real (or complex) numbers $w_0, w_1, \ldots, w_n$ and $c_0, c_1, \ldots, c_n$ with the w_k's distinct, then there *exists* a *unique* polynomial $P(x)$ of degree at most n satisfying $P(w_k) = c_k$ for $k = 0,1,\ldots,n$. The uniqueness is clear since if $Q(x)$ is also such a polynomial, then $P(x)-Q(x)$ would be a polynomial of degree at most n and have roots at the n+1 numbers $w_0, w_1, \ldots, w_n$, which leads to $P(x)-Q(x)$ be the zero polynomial.

Now, to exhibit such a polynomial, we define $f_0(x)=(x-w_1)(x-w_2)\cdots(x-w_n)$ and similarly for i from 1 to n, define

$$f_i(x)=(x-w_0)\cdots(x-w_{i-1})(x-w_{i+1})\cdots(x-w_n).$$

Observe that $f_i(w_k) = 0$ if and only if $i \neq k$. Using this, we see

$$P(x) = \sum_{i=0}^{n} c_i \frac{f_i(x)}{f_i(w_i)}$$

satisfies $P(w_k) = c_k$ for $k = 0,1,\ldots,n$. This is the famous *Lagrange interpolation formula*.

Below we will present some examples of using this formula to solve math problems.

Example 1. (*Romanian Proposal to* 1981 *IMO*) Let P be a polynomial of degree n satisfying for $k = 0,1,\ldots,n$,

$$P(k) = \binom{n+1}{k}^{-1}.$$

Determine $P(n+1)$.

Solution. For $k = 0,1,\ldots,n$, let $w_k=k$ and

$$c_k = \binom{n+1}{k}^{-1} = \frac{k!(n+1-k)!}{(n+1)!}.$$

Define $f_0, f_1, \ldots, f_n$ as above. We get

$$f_k(k) = (-1)^{n-k}k!(n-k)!$$

and

$$f_k(n+1) = \frac{(n+1)!}{(n+1-k)}.$$

By the Lagrange interpolation formula,

$$P(n+1) = \sum_{k=0}^{n} c_k \frac{f_k(n+1)}{f_k(k)} = \sum_{k=0}^{n} (-1)^{n-k},$$

which is 0 if n is odd and 1 if n is even.

Example 2. (*Vietnamese Proposal to* 1977 *IMO*) Suppose $x_0, x_1, \ldots, x_n$ are integers and $x_0 > x_1 > \cdots > x_n$. Prove that one of the numbers $|P(x_0)|, |P(x_1)|, \ldots, |P(x_n)|$ is at least $n!/2^n$, where $P(x) = x^n + a_1x^{n-1} + \cdots + a_n$ is a polynomial with real coefficients.

Solution. Define $f_0, f_1, \ldots, f_n$ using $x_0, x_1, \ldots, x_n$. By the Lagrange interpolation formula, we have

$$P(x) = \sum_{i=0}^{n} P(x_i) \frac{f_i(x)}{f_i(x_i)},$$

since both sides are polynomials of degrees at most n and are equal at $x_0, x_1, \ldots, x_n$. Comparing coefficients of x^n, we get

$$1 = \sum_{i=0}^{n} \frac{P(x_i)}{f_i(x_i)}.$$

Since $x_0, x_1, \ldots, x_n$ are strictly decreasing integers, we have

$$|f_i(x_i)| = \prod_{j=0}^{i-1} |x_j - x_i| \prod_{j=i+1}^{n} |x_j - x_i|$$
$$\geq i!(n-i)! = \frac{1}{n!}\binom{n}{i}.$$

Let the maximum of $|P(x_0)|, |P(x_1)|, \ldots, |P(x_n)|$ be $|P(x_k)|$. By the triangle inequality, we have

$$1 \leq \sum_{i=0}^{n} \frac{|P(x_i)|}{|f_i(x_i)|} \leq \frac{|P(x_k)|}{n!} \sum_{i=0}^{n} \binom{n}{i} = \frac{2^n |P(x_k)|}{n!}.$$

Then $|P(x_k)| \geq n!/2^n$.

Example 3. Let P be a point on the plane of ΔABC. Prove that

$$\frac{PA}{BC} + \frac{PB}{CA} + \frac{PC}{AB} \geq \sqrt{3}.$$

Solution. We may take the plane of ΔABC to be the complex plane and let P, A, B, C be corresponded to the complex numbers w, w_1, w_2, w_3 respectively. Then $PA=|w-w_1|$, $BC=|w_2-w_3|$, etc.

Now the only polynomial $P(x)$ of degree at most 2 that equals 1 at w_1, w_2, w_3 is the constant polynomial $P(x) \equiv 1$. So, expressing $P(x)$ by the Lagrange interpolation formula, we have

$$\frac{(x-w_1)(x-w_2)}{(w_3-w_1)(w_3-w_2)}+\frac{(x-w_2)(x-w_3)}{(w_1-w_2)(w_1-w_3)}$$

$$+\frac{(x-w_3)(x-w_1)}{(w_2-w_3)(w_2-w_1)} \equiv 1.$$

Next, setting $x = w$ and applying the triangle inequality, we get

$$\frac{PA}{BC}\frac{PB}{CA}+\frac{PB}{CA}\frac{PC}{AB}+\frac{PC}{AB}\frac{PA}{BC} \geq 1. \quad (*)$$

The inequality $(r+s+t)^2 \geq 3(rs+st+tw)$, after subtracting the two sides, reduces to $[(r-s)^2+(s-t)^2+(t-r)^2]/2 \geq 0$, which is true. Setting $r= PA/BC$, $s=PB/CA$ and $t=PC/AB$, we get

$$\left(\frac{PA}{BC}+\frac{PB}{CA}+\frac{PC}{AB}\right)^2 \geq 3\left(\frac{PA}{BC}\frac{PB}{CA}+\frac{PB}{CA}\frac{PC}{AB}+\frac{PC}{AB}\frac{PA}{BC}\right).$$

Taking square roots of both sides and applying (*), we get the desired inequality.

Example 4. (2002 *USAMO*) Prove that any *monic* polynomial (a polynomial with leading coefficient 1) of degree n with real coefficients is the average of two monic polynomials of degree n with n real roots.

Solution. Suppose $F(x)$ is a monic real polynomial. Choose real $y_1, y_2, \ldots, y_n$ such that for odd i, $y_i < \min\{0,2F(i)\}$ and for even i, $y_i > \max\{0,2F(i)\}$.

By the Lagrange interpolation formula, there is a polynomial of degree less than n such that $P(i) = y_i$ for $i=1,2,\ldots,n$. Let

$$G(x) = P(x)+(x-1)(x-2)\cdots(x-n)$$

and

$$H(x) = 2F(x)-G(x).$$

Then $G(x)$ and $H(x)$ are monic real polynomials of degree n and their average is $F(x)$.

As $y_1, y_3, y_5, \ldots < 0$ and $y_2, y_4, y_6, \cdots > 0$, $G(i)=y_i$ and $G(i+1)=y_{i+1}$ have opposite signs (hence $G(x)$ has a root in $[i,i+1]$) for $i=1,2,\ldots,n-1$. So $G(x)$ has at least $n-1$ real roots. The other root must also be real since non-real roots come in conjugate pair. Therefore, all roots of $G(x)$ are real.

Similarly, for odd i, $G(i) = y_i < 2F(i)$ implies $H(i)=2F(i)-G(i) > 0$ and for even i, $G(i) = y_i > 2F(i)$ implies $H(i) = 2F(i)-G(i) < 0$. These imply $H(x)$ has n real roots by reasoning similar to $G(x)$.

Example 5. Let $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ be real numbers such that $b_i-a_j\neq 0$ for $i,j=1,2,3,4$. Suppose there is a unique set of numbers X_1, X_2, X_3, X_4 such that

$$\frac{X_1}{b_1-a_1}+\frac{X_2}{b_1-a_2}+\frac{X_3}{b_1-a_3}+\frac{X_4}{b_1-a_4}=1,$$

$$\frac{X_1}{b_2-a_1}+\frac{X_2}{b_2-a_2}+\frac{X_3}{b_2-a_3}+\frac{X_4}{b_2-a_4}=1,$$

$$\frac{X_1}{b_3-a_1}+\frac{X_2}{b_3-a_2}+\frac{X_3}{b_3-a_3}+\frac{X_4}{b_3-a_4}=1,$$

$$\frac{X_1}{b_4-a_1}+\frac{X_2}{b_4-a_2}+\frac{X_3}{b_4-a_3}+\frac{X_4}{b_4-a_4}=1.$$

Determine $X_1+X_2+X_3+X_4$ in terms of the a_i's and b_i's.

Solution. Let

$$P(x)=\prod_{i=1}^{4}(x-a_i)-\prod_{i=1}^{4}(x-b_i).$$

Then the coefficient of x^3 in $P(x)$ is

$$\sum_{i=1}^{4} b_i - \sum_{i=1}^{4} a_i.$$

Define f_1, f_2, f_3, f_4 using a_1, a_2, a_3, a_4 as above to get the Lagrange interpolation formula

$$P(x)=\sum_{i=i}^{4} P(a_i)\frac{f_i(x)}{f_i(a_i)}.$$

Since the coefficient of x^3 in $f_i(x)$ is 1, the coefficient of x^3 in $P(x)$ is also

$$\sum_{i=1}^{4}\frac{P(a_i)}{f_i(a_i)}.$$

Next, observe that $P(b_j)/f_i(b_j) = b_j - a_i$, which are the denominators of the four given equations! For $j = 1,2,3,4$, setting $x = b_j$ in the interpolation formula and dividing both sides by $P(b_j)$, we get

$$1=\sum_{i=i}^{4}\frac{P(a_i)}{P(b_j)}\frac{f_i(b_j)}{f_i(a_i)}=\sum_{i=1}^{4}\frac{P(a_i)/f_i(a_i)}{b_j-a_i}.$$

Comparing with the given equations, by uniqueness, we get $X_i=P(a_i)/f_i(a_i)$ for $i = 1,2,3,4$. So

$$\sum_{i=1}^{4} X_i=\sum_{i=1}^{4}\frac{P(a_i)}{f_i(a_i)}=\sum_{i=1}^{4} b_i-\sum_{i=}^{4} a_i.$$

Comment: This example is inspired by problem 15 of the 1984 American Invitational Mathematics Examination.

Example 6. (*Italian Proposal to* 1997 *IMO*) Let p be a prime number and let $P(x)$ be a polynomial of degree d with integer coefficients such that:

(i) $P(0) = 0$, $P(1) = 1$;
(ii) for every positive integer n, the remainder of the division of $P(n)$ by p is either 0 or 1.

Prove that $d \geq p - 1$.

Solution. By (i) and (ii), we see

$P(0)+P(1)+\cdots+P(p-1)\equiv k \pmod p$ (#)

for some $k\in\{1,2,\ldots, p-1\}$.

Assume $d \leq p - 2$. Then $P(x)$ will be uniquely determined by the values $P(0)$, $P(1), \ldots, P(p-2)$. Define $f_0, f_1, \ldots, f_{p-2}$ using $0, 1, \ldots, p-2$ as above to get the Lagrange interpolation formula

$$P(x)=\sum_{k=0}^{p-2} P(k)\frac{f_k(x)}{f_k(k)}.$$

As in example (1), we have

$$f_k(k) = (-1)^{p-2-k}k!(p-2-k)!,$$

$$f_k(p-1)=\frac{(p-1)!}{p-1-k}$$

and so

$$P(p-1)=\sum_{k=0}^{p-2} P(k)(-1)^{p-k}\binom{p-1}{k}.$$

Next, we claim that

$$\binom{p-1}{k}\equiv(-1)^k \pmod p \quad \textit{for} \quad 0\leq k\leq p-2.$$

This is true for $k = 0$. Now for $0 < i < p$,

$$\binom{p}{i}=\frac{p!}{i!(p-i)!}\equiv 0 \pmod p$$

because p divides $p!$, but not $i!(p-i)!$. If the claim is true for k, then

$$\binom{p-1}{k+1}=\binom{p}{k+1}-\binom{p-1}{k}\equiv(-1)^{k+1} \pmod p$$

and the induction step follows. Finally the claim yields

$$P(p-1)\equiv(-1)^p\sum_{k=0}^{p-2} P(k) \pmod p.$$

So $P(0)+P(1)+\cdots+P(p-1)\equiv 0 \pmod p$, a contradiction to (#) above.

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for sending solutions is ***October 20, 2010.***

Problem 351. Let S be a unit sphere with center O. Can there be three arcs on S such that each is a 300° arc on some circle with O as center and no two of the arcs intersect?

Problem 352. (*Proposed by Pedro Henrique O. PANTOJA*, *University of Lisbon, Portugal*) Let a, b, c be real numbers that are at least 1. Prove that

$$\frac{a^2bc}{\sqrt{bc}+1}+\frac{b^2ca}{\sqrt{ca}+1}+\frac{c^2ab}{\sqrt{ab}+1}\ge\frac{3}{2}.$$

Problem 353. Determine all pairs (x, y) of integers such that $x^5-y^2=4$.

Problem 354. For 20 boxers, find the least number n such that there exists a schedule of n matches between pairs of them so that for every three boxers, two of them will face each other in one of the matches.

Problem 355. In a plane, there are two *similar* convex quadrilaterals $ABCD$ and $AB_1C_1D_1$ such that C, D are inside $AB_1C_1D_1$ and B is outside $AB_1C_1D_1$ Prove that if lines BB_1, CC_1 and DD_1 concur, then $ABCD$ is cyclic. Is the converse also true?

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Solutions

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Problem 346. Let k be a positive integer. Divide $3k$ pebbles into five piles (with possibly unequal number of pebbles). Operate on the five piles by selecting three of them and removing one pebble from each of the three piles. If it is possible to remove all pebbles after k operations, then we say it is a *harmonious ending*.

Determine a necessary and sufficient condition for a harmonious ending to exist in terms of the number k and the distribution of pebbles in the five piles.

(*Source*: 2008 *Zhejiang Province High School Math Competition*)

Solution. **CHOW Tseung Man** (True Light Girl's College), **CHUNG Ping Ngai** (MIT Year 1), **HUNG Ka Kin Kenneth** (CalTech Year 1).

The necessary and sufficient condition is every pile has at most k pebbles in the beginning.

The necessity is clear. If there is a pile with more than k pebbles in the beginning, then in each of the k operations, we can only remove at most 1 pebble from that pile, hence we cannot empty the pile after k operations.

For the sufficiency, we will prove by induction. In the case k=1, three pebbles are distributed with each pebble to a different pile. So we can finish in one operation. Suppose the cases less than k are true. For case k, since $3k$ pebbles are distributed. So at most 3 piles have k pebbles. In the first operation, we remove one pebble from each of the three piles with the maximum numbers of pebbles. This will take us to a case less than k. We are done by the inductive assumption.

Problem 347. $P(x)$ is a polynomial of degree n such that for all $w\in\{1, 2, 2^2, \ldots, 2^n\}$, we have $P(w) = 1/w$.

Determine $P(0)$ with proof.

***Solution* 1.** **Carlo PAGANO** (Università di Roma "Tor Vergata", Roma, Italy). **William CHAN Wai-lam** (Carmel Alison Lam Foundation Secondary School) and **Thien Nguyen** (Nguyen Van Thien Luong High School, Dong Nai Province, Vietnam).

Let $Q(x) = xP(x)-1 = a(x-1)(x-2)\cdots(x-2^n)$. For $x\neq 1, 2, 2^2, \ldots, 2^n$,

$$\frac{Q'(x)}{Q(x)}=\frac{1}{x-1}+\frac{1}{x-2}+\cdots+\frac{1}{x-2^n}.$$

Since $Q(0)= -1$ and $Q'(x)=P(x)+xP'(x)$,

$$P(0)=Q'(0)=-\frac{Q'(0)}{Q(0)}=\sum_{k=0}^{n}\frac{1}{2^k}=2-\frac{1}{2^n}.$$

***Solution* 2.** **CHUNG Ping Ngai** (MIT Year 1), **HUNG Ka Kin Kenneth** (CalTech Year 1), **Abby LEE** (SKH Lam Woo Memorial Secondary School, Form 5) and **WONG Kam Wing** (HKUST, Physics, Year 2).

Let $Q(x) = xP(x)-1 = a(x-1)(x-2)\cdots(x-2^n)$. Now $Q(0) = -1 = a(-1)^{n+1}2^s$, where $s = 1+2+\cdots+n$. So $a = (-1)^n 2^{-s}$. Then $P(0)$ is the coefficient of x in $Q(x)$, which is

$$a(-1)^n(2^s+2^{s-1}+\cdots+2^{s-n})=\sum_{k=0}^{n}\frac{1}{2^k}=2-\frac{1}{2^n}.$$

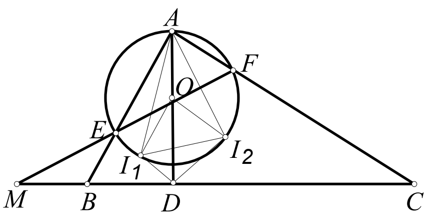
Other commended solvers: **Samuel Liló ABDALLA** (ITA-UNESP, São Paulo, Brazil),

Problem 348. In ΔABC, we have $\angle BAC = 90°$ and $AB < AC$. Let D be the foot of the perpendicular from A to side BC. Let I_1 and I_2 be the incenters of ΔABD and ΔACD respectively. The circumcircle of ΔAI_1I_2 (with center O) intersects sides AB and AC at E and F respectively. Let M be the intersection of lines EF and BC.

Prove that I_1 or I_2 is the incenter of the ΔODM, while the other one is an excenter of ΔODM.

(*Source:* 2008 *Jiangxi Province Math Competition*)

Solution. **CHOW Tseung Man** (True Light Girl's College).



We *claim* EF intersects AD at O. Since $\angle EAF$=90°, EF is a diameter through O. Next we will show O is on AD.

Since AI_1, AI_2 bisect $\angle BAD$, $\angle CAD$ respectively, we get $\angle I_1AI_2$=45°. Then $\angle I_1OI_2$=90°. Since OI_1=OI_2, $\angle OI_1I_2$=45°. Also, DI_1, DI_2 bisect $\angle BDA$, $\angle CDA$ respectively implies $\angle I_1DI_2$=90°. Then D, I_1, O, I_2 are concyclic. So

$$\angle ODI_2=\angle OI_1I_2=45^\circ=\angle ADI_2.$$

Then O is on AD and the claim is true.

Since $\angle EOI_1$ = $2\angle EAI_1$ = $2\angle DAI_1$ = $\angle DOI_1$ and I_1 is on the angle bisector of $\angle ODM$, we see I_1 is the incenter of ΔODM. Similarly, replacing E by F and I_1 by I_2 in the last sentence, we see I_2 is an excenter of ΔODM.

Other commended solvers: **CHUNG Ping Ngai** (MIT Year 1), **HUNG Ka Kin Kenneth** (CalTech Year 1) and **Abby LEE** (SKH Lam Woo Memorial Secondary School, Form 5).

Problem 349. Let $a_1, a_2, \ldots, a_n$ be rational numbers such that for every positive integer m,

$$a_1^m+a_2^m+\cdots+a_n^m$$

is an integer. Prove that $a_1, a_2, \ldots, a_n$ are integers.

Solution. **CHUNG Ping Ngai** (MIT Year 1) and **HUNG Ka Kin Kenneth** (CalTech Year 1).

We may first remove all the integers among $a_1, a_2, \ldots, a_n$ since their m-th powers are integers, so the rest of $a_1, a_2, \ldots, a_n$ will still have the same property. Hence, without loss of generality, we may assume all $a_1, a_2, \ldots, a_n$ are rational numbers and not integers. First write every a_i in simplest term. Let Q be their least common denominator and for all $1 \le i \le n$, let $a_i = k_i/Q$. Take a prime factor p of Q. Then p is not a prime factor of one of the k_i's. So one of the remainders r_i when k_i is divided by p is nonzero! Since $k_i \equiv r_i (\text{mod } p)$, so for every positive integer m,

$$\sum_{i=1}^{n} r_i^m \equiv \sum_{i=1}^{n} k_i^m = \left(\sum_{i=1}^{n} a_i^m\right) Q^m \equiv 0 (\text{mod} p^m).$$

This implies $p^m \le \sum_{i=1}^{n} r_i^m$. Since $r_i < p$,

$$1 \le \lim_{m\to\infty} \frac{1}{p^m} \sum_{i=1}^{n} r_i^m = \lim_{m\to\infty} \sum_{i=1}^{n} \left(\frac{r_i}{p}\right)^m = 0,$$

which is a contradiction.

Comments: In the above solution, it does not need all positive integers m, just an infinite sequence of positive integers m with the given property will be sufficient.

Problem 350. Prove that there exists a positive constant c such that for all positive integer n and all real numbers $a_1, a_2, \ldots, a_n$, if

$$P(x) = (x - a_1)(x - a_2) \cdots (x - a_n),$$

then

$$\max_{x\in[0,2]} |P(x)| \le c^n \max_{x\in[0,1]} |P(x)|.$$

(*Ed.*-Both solutions below show the conclusion holds for any polynomial!)

Solution 1. **LEE Kai Seng.**

Let S be the maximum of $|P(x)|$ for all $x \in [0,1]$. For $i=0,1,2,\ldots,n$, let $b_i = i/n$ and

$$f_i(x) = (x-b_0)\cdots(x-b_{i-1})(x-b_{i+1})\cdots(x-b_n).$$

By the Lagrange interpolation formula, for all real x,

$$P(x) = \sum_{i=0}^{n} P(b_i) \frac{f_i(x)}{f_i(b_i)}.$$

For every $w \in [0,2]$, $|w - b_k| \le |2 - b_k|$ for all $k = 0,1,2,\ldots,n$. So

$$|f_i(w)| \le |f_i(2)| = \prod_{i=0}^{n} \left(2 - \frac{i}{n}\right)$$

$$= \frac{2n(2n-1)(2n-2)\cdots(n+1)}{n^n}$$

$$= \frac{(2n)!}{n!n^n}.$$

Also, $|P(b_i)| \le S$ and

$$|f_i(b_i)| = \frac{i!(n-i)!}{n^n}.$$

By the triangle inequality,

$$|P(w)| \le \sum_{i=0}^{n} |P(b_i)| \frac{|f_i(w)|}{|f_i(b_i)|} \le S \sum_{i=0}^{n} \binom{2n}{i}\binom{2n-i}{n}.$$

Finally,

$$\sum_{i=0}^{n} \binom{2n}{i}\binom{2n-i}{n} \le \sum_{i=0}^{n} \binom{2n}{i}\binom{2n}{n} = 2^{2n}\binom{2n}{n} \le 2^{4n}.$$

Then

$$\max_{w\in[0,2]} |P(w)| \le 2^{4n} S = 16^n \max_{x\in[0,1]} |P(x)|.$$

Solution 2. **G.R.A.20 Problem Solving Group** (Roma, Italy).

For a bounded closed interval I and polynomial $f(x)$, let $\|f\|_I$ denote the maximum of $|f(x)|$ for all x in I. The <u>Chebyschev polynomial of order n</u> is defined by $T_0(x) = 1$, $T_1(x) = x$ and

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \text{ for } n \ge 2.$$

(*Ed.*-By <u>induction</u>, we can obtain

$$T_n(x) = 2^n x^n + c_{n-1}x^{n-1} + \cdots + c_0$$

and $T_n(\cos\theta) = \cos n\theta$. So $T_n(\cos(\pi k/n)) = (-1)^k$, which implies all n roots of $T_n(x)$ are in $(-1,1)$ as it changes sign n times.)

It is known that for any polynomial $Q(x)$ with degree at most $n>0$ and all $t \notin [-1,1]$,

$$|Q(t)| \le \|Q\|_{[-1,1]} |T_n(t)|. \qquad (!)$$

To see this, we may assume $\|Q\|_{[-1,1]} = 1$ by dividing $Q(x)$ by such maximum. Assume $x_0 \notin [-1,1]$ and $|Q(x_0)| > |T_n(x_0)|$. Let

$$a = T(x_0)/Q(x_0) \text{ and } R(x) = aQ(x) - T_n(x).$$

For $k = 0, 1, 2, \cdots, n$, since $T_n(\cos(\pi k/n)) = (-1)^k$ and $|a|<1$, we see $R(\cos(\pi k/n))$ is positive or negative depending on whether k is odd or even. (In particular, $R(x) \not\equiv 0$.) By continuity, $R(x)$ has $n+1$ distinct roots on $[-1,1] \cup \{x_0\}$, which contradicts the degree of $R(x)$ is at most n.

Next, for the problem, we claim that for every $t \in [1,2]$, we have $|P(t)| \le 6^n \|P\|_{[0,1]}$.

(*Ed.*-Observe that the change of variable $t = (s+1)/2$ is a bijection between $s \in [-1,1]$ and $t \in [0,1]$. It is also a bijection between $s \in [1,3]$ and $t \in [1,2]$.) By letting $Q(s) = P((s+1)/2)$, the claim is equivalent to proving that for every $s \in [1,3]$, we have $|Q(s)| \le 6^n \|Q\|_{[-1,1]}$. By (!) above, it suffices to show that $|T_n(s)| \le 6^n$ for every $s \in [1,3]$.

Clearly, $|T_0(s)| = 1 = 6^0$. For $n=1$ and $s \in [1,3]$, $|T_1(s)| = s \le 3 < 6$. Next, since the largest root of T_n is less than 1, we see all $T_n(s) > 0$ for all $s \in [1,3]$. Suppose cases $n-2$ and $n-1$ are true. Then for all $s \in [1,3]$, we have $2sT_{n-1}(s)$, $T_{n-2}(s) > 0$ and so

$$\begin{aligned} |T_n(s)| &= |2sT_{n-1}(s) - T_{n-2}(s)| \\ &\le \max(2sT_{n-1}(s), T_{n-2}(s)) \\ &\le \max(6 \cdot 6^{n-1}, 6^{n-2}) = 6^n. \end{aligned}$$

This finishes everything.

Olympiad Corner

(continued from page 1)

Problem 5. Let $A = (a_{jk})$ be a 10×10 array of positive real numbers such that the sum of the numbers in each row as well as in each column is 1. Show that there exist $j<k$ and $l<m$ such that

$$a_{jl}a_{km} + a_{jm}a_{kl} \ge \frac{1}{50}.$$

Problem 6. Let ABC be a triangle. Let AD, BE, CF be cevians such that $\angle BAD = \angle CBE = \angle ACF$. Suppose these cevians concur at a point Ω. (Such a point exists for each triangle and it is called a Brocard point.) Prove that

$$\frac{A\Omega^2}{BC^2} + \frac{B\Omega^2}{CA^2} + \frac{C\Omega^2}{AB^2} \ge 1.$$

(*Ed.*-A <u>cevian</u> is a line segment which joins a vertex of a triangle to a point on the opposite side or its extension.)

Problem 7. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x+y) + xy = f(x)f(y)$$

for all reals x,y.

Problem 8. Prove that there are infinitely many positive integers m for which there exist consecutive odd positive integers p_m, q_m $(=p_m+2)$ such that the pairs (p_m, q_m) are all distinct and

$$p_m^2 + p_mq_m + q_m^2, \quad p_m^2 + mp_mq_m + q_m^2$$

are both perfect squares.