Mathematical Excalibur

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Olympiad Corner

Below are the problems of the 2011 Canadian Math Olympiad, which was held on March 23, 2011.

Problem 1. Consider 70-digit numbers n, with the property that each of the digits 1, 2, 3, …, 7 appears in the decimal expansion of n ten times (and 8, 9 and 0 do not appear). Show that no number of this form can divide another number of this form.

Problem 2. Let $ABCD$ be a cyclic quadrilateral whose opposite sides are not parallel, X the intersection of AB and CD, and Y the intersection of AD and BC. Let the angle bisector of $\angle AXD$ intersect AD, BC at E, F respectively and let the angle bisector of $\angle AYB$ intersect AB, CD at G, H respectively. Prove that $EGFH$ is a parallelogram.

Problem 3. Amy has divided a square up into finitely many white and red rectangles, each with sides parallel to the sides of the square. Within each white rectangle, she writes down its width divided by its height. Within each red rectangle, she writes down its height divided by its width. Finally, she calculates x, the sum of these numbers.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***May 29, 2011***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Harmonic Series (I)

Leung Tat-Wing

A series of the form

$$\frac{1}{m}+\frac{1}{m+d}+\frac{1}{m+2d}+\cdots,$$

where m, d are numbers such that the denominators are never zero, is called a *harmonic series*. For example, the series

$$H(n)=H(1,n)=1+\frac{1}{2}+\ldots+\frac{1}{n}$$

is a harmonic series, or more generally

$$H(m,n)=\frac{1}{m}+\frac{1}{m+1}+\ldots+\frac{1}{n}$$

is also a harmonic series. Below we always assume $1 \le m < n$. There are many interesting properties concerning this kind of series.

Example 1: $H(1,n)$ is unbounded, i.e. for any positive number A, we can find n big enough, so that $H(1,n) \ge A$.

Solution For any positive integer r, note

$$\frac{1}{r+1}+\frac{1}{r+2}+\cdots+\frac{1}{2r}\ge\frac{1}{2},$$

which can be proved by induction. Hence we can take enough pieces of these fractions to make $H(1,n)$ as large as possible.

Example 2: $H(m,n)$ is never an integer.

Solution (i) For the special case $m = 1$, let s be such that $2^s \le n < 2^{s+1}$. We then multiply $H(1,n)$ by $2^{s-1}Q$, where Q is the product of all odd integers in $[1, n]$. All terms in $H(1,n)$ will become an integer except the term 2^s will become an integer divided by 2 (a half integer). This implies $H(1,n)$ is not an integer.

(ii) Alternatively, for the case $m = 1$, let p be the greatest prime number not exceeding n. By Bertrand's postulate there is a prime q with $p < q < 2p$. Therefore we have $n < 2p$. If $H(1,n)$ is an integer, then

$$n!H(n)=\sum_{i=1}^{n}\frac{n!}{i}$$

is an integer divisible by p. However the term $n!/p$ (an addend) is not divisible by p but all other addends are.

(iii) We deal with the case $m > 1$. Suppose $2^{\alpha} \mid k$ but $2^{\alpha+1}$ does not divide k (write this as $2^{\alpha} \parallel k$), then we call α the "parity order" of k. Now observe 2^{α}, $3\cdot2^{\alpha}$, $5\cdot2^{\alpha}$, ⋯ all have the same parity order. Between these numbers, there are $2\cdot2^{\alpha}$, $4\cdot2^{\alpha}$, $6\cdot2^{\alpha}$, ⋯, all have greater parity orders. Hence, between any two numbers of the same parity order, there is one with greater parity order. This implies among m, $m+1$, …, n, there is a unique integer with the greatest parity order, say q of parity order μ. Now multiply

$$\frac{1}{m}+\frac{1}{m+1}+\ldots+\frac{1}{n}$$

by $2^{\mu}L$, where L is the product of all odd integers in $[m, n]$. Then $2^{\mu}L\cdot H(m,n)$ is an odd number. Hence

$$H(m,n)=\frac{2r+1}{2^{\mu}L}=\frac{q}{p},$$

where p is even and q is odd and so is not an integer.

Example 3 (APMO 1997): Given that

$$S=1+\frac{1}{1+\frac{1}{3}}+\frac{1}{1+\frac{1}{3}+\frac{1}{6}}+\ldots+\frac{1}{1+\frac{1}{3}+\frac{1}{6}+\ldots+\frac{1}{1993006}},$$

where the denominators contains partial sum of the sequence of reciprocals of triangular numbers. Prove that $S > 1001$.

Solution Let T_n be the nth triangular number. Then $T_n=n(n+1)/2$ and hence

$$\frac{1}{T_1}+\frac{1}{T_2}+\ldots+\frac{1}{T_n}=\frac{2}{1\cdot2}+\frac{2}{2\cdot3}+\ldots+\frac{2}{n(n+1)}$$
$$=2(1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\ldots+\frac{1}{n}-\frac{1}{n+1})=2(1-\frac{1}{n+1})=\frac{2n}{n+1}.$$

Since 1993006=1996·1997/2, we get

$$S=\frac{1}{2}\left(\frac{2}{1}+\frac{3}{2}+\cdots+\frac{1997}{1996}\right)$$
$$>\frac{1}{2}\left(1996+1+\frac{1}{2}+\left(\frac{1}{3}+\cdots+\frac{1}{1024}\right)\right).$$

Hence, $S > (1996+6)/2=1001$ using example 1 that $H(r+1,2r) \ge 1/2$ for $r = 2$, 4, 8, 16, 32, 64, 128, 256, 512.

Congruence relations of harmonic series are of some interest. First, let us look at an example.

Example 4 (IMO 1979): Let p, q be natural numbers such that

$$\frac{p}{q}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots-\frac{1}{1318}+\frac{1}{1319}.$$

Prove that p is divisible by 1979.

Solution We will prove the famous Catalan identity (due to N. Botez (1872) and later used by Catalan):

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots-\frac{1}{2n}=\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}.$$

It is proved as follows:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots-\frac{1}{2n}$$

$$=\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2n}\right)-2\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n}\right)$$

$$=\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2n}\right)-\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)$$

$$=\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}.$$

Thus

$$\frac{p}{q}=\frac{1}{660}+\frac{1}{661}+\ldots+\frac{1}{1318}+\frac{1}{1319}$$

$$=\frac{1}{2}\left(\frac{1}{660}+\frac{1}{1319}+\frac{1}{661}+\frac{1}{1318}+\ldots+\frac{1}{1319}+\frac{1}{660}\right)$$

$$=\frac{1}{2}\left(\frac{1979}{660\cdot1319}+\frac{1979}{661\cdot1318}+\ldots+\frac{1979}{1319\cdot660}\right)$$

$$=1979\cdot\frac{A}{B},$$

where B is the product of some positive integers less than 1319. However, 1979 is prime, hence $1979|p$.

For another proof using congruence relations, observe that if $(k,1979)=1$, then by Fermat's little theorem, $k^{1978}\equiv 1 \pmod{1979}$. Hence, we can consider $1/k\equiv k^{1977} \pmod{1979}$. Then

$$\sum_{k=1}^{1319}(-1)^{k-1}\frac{1}{k}\equiv\sum_{k=1}^{1319}(-1)^{k-1}k^{1977}$$

$$=\sum_{k=1}^{1319}k^{1977}-2\sum_{k=1}^{659}(2k)^{1977}$$

$$=\sum_{k=1}^{1319}k^{1977}-2\cdot2^{1977}\sum_{k=1}^{659}k^{1977}$$

$$\equiv\sum_{k=1}^{1319}k^{1977}-\sum_{k=1}^{659}k^{1977}$$

$$=\sum_{k=660}^{1319}k^{1977}=\sum_{k=660}^{989}(k^{1977}+(1979-k)^{1977})$$

$$\equiv\sum_{k=660}^{989}(k^{1977}+(-k)^{1977})=0 \pmod{1979}.$$

Note that $1/k \pmod p$ (as well as many fraction mod p) makes sense if $k\not\equiv 0 \pmod p$. Also, as a generalization, we have

Example 5: If $H(m,n)=q/p$ and $m+n$ is an odd prime number, then $m+n \mid q$.

Solution Note that $H(m,n)$ has an even number of terms and it equals

$$\sum_{j=0}^{(n-m-1)/2}\left(\frac{1}{m+j}+\frac{1}{n-j}\right)$$

$$=\sum_{j=0}^{(n-m-1)/2}\frac{m+n}{(m+j)(n-j)}=(m+n)\frac{s}{r}.$$

where $\gcd(s,r)=1$. Since $m+n$ is prime, $\gcd(r,m+n)=1$. Then $q/p=(m+n)s/r$ and $m+n \mid q$.

The Catalan identity is also used in the following example.

Example 6 (Rom Math Magazine, July 1998): Let

$$A=\frac{1}{1\cdot2}+\frac{1}{3\cdot4}+\ldots+\frac{1}{2011\cdot2012}$$

and

$$B=\frac{1}{1007\cdot2012}+\frac{1}{1008\cdot2011}+\ldots+\frac{1}{2012\cdot1007}.$$

Evaluate A/B.

Solution

$$A=\sum_{k=1}^{1006}\frac{1}{(2k-1)2k}=\sum_{k=1}^{1006}\left(\frac{1}{2k-1}-\frac{1}{2k}\right)$$

$$=\frac{1}{1007}+\frac{1}{1008}+\cdots+\frac{1}{2012}$$

$$=\frac{1}{2}\left(\frac{1}{1007}+\frac{1}{2012}+\frac{1}{1008}+\frac{1}{2011}+\cdots+\frac{1}{2012}+\frac{1}{1007}\right)$$

$$=\frac{1}{2}\left(\frac{3019}{1007\cdot2012}+\frac{3019}{1008\cdot2011}+\cdots+\frac{3019}{2012\cdot1007}\right)$$

$$=\frac{3019B}{2}.$$

Hence $\dfrac{A}{B}=\dfrac{3019}{2}.$

Example 7: Given any proper fraction m/n, where m, n are positive integers satisfying $0<m<n$, then prove it is the sum of fractions of the form

$$\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_k},$$

where $x_1, x_2, \ldots, x_k$ are distinct positive integers.

Solution We use the "greedy method". Let x_1 be the positive integer such that

$$\frac{1}{x_1}\le\frac{m}{n}<\frac{1}{x_1-1},$$

i.e. x_1 is the least integer greater than or equal to n/m. If $1/x_1=m/n$, then the problem is done. Otherwise

$$\frac{m}{n}-\frac{1}{x_1}=\frac{mx_1-n}{nx_1}=\frac{m_1}{nx_1},$$

where $m_1=mx_1-n<m$ (due to $m/n<1/(x_1-1)$) and obviously $nx_1>n$. Let x_2 be another positive integer such that

$$\frac{1}{x_2}\le\frac{m_1}{nx_1}<\frac{1}{x_2-1}.$$

The procedure can be repeated until $m>m_1>m_2>\cdots>m_k>0$ and

$$\frac{m}{n}=\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_k},$$

where $1\le k\le m$. (Note: writing

$$\frac{1}{n}=\frac{1}{n+1}+\frac{1}{n(n+1)},$$

we observe actually there are infinitely many ways of writing any proper fractions as sum of fractions of this kind. These fractions are called *unit fractions* or *Egyptian fractions*.)

Example 8: Remove those terms in

$$1+\frac{1}{2}+\cdots+\frac{1}{n}+\cdots$$

such that its denominator in decimal expansion contains the digit "9", then prove that the sequence is bounded.

Solution The integers without the digit 9 in the interval $[10^{m-1}, 10^m-1]$ are m-digit numbers. The first digit from the left cannot be the digits "0" and "9", (8 choices), the other digits cannot contain "9", hence nine choices 0, 1, 2, 3, 4, 5, 6, 7 and 8. Altogether there are $8\cdot9^{m-1}$ such integers. The sum of their reciprocals is less than

$$\frac{8\cdot9^{m-1}}{10^{m-1}}=8\left(\frac{9}{10}\right)^{m-1}.$$

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***May 29, 2011.***

Problem 366. Let n be a positive integer in base 10. For i =1,2,…,9, let $a(i)$ be the number of digits of n that equal i. Prove that

$$2^{a(1)}3^{a(2)}\cdots 9^{a(8)}10^{a(9)} \le n+1$$

and determine all equality cases.

Problem 367. For n = 1,2,3,…, let x_n and y_n be positive real numbers such that

$$x_{n+2} = x_n + x_{n+1}^2$$

and

$$y_{n+2} = y_n^2 + y_{n+1}.$$

If x_1, x_2, y_1, y_2 are all greater than 1, then prove that there exists a positive integer N such that for all $n > N$, we have $x_n > y_n$.

Problem 368. Let C be a circle, A_1, A_2, …, A_n be distinct points inside C and B_1, B_2, …, B_n be distinct points on C such that no two of the segments A_1B_1, A_2B_2,…, A_nB_n intersect. A grasshopper can jump from A_r to A_s if the line segment A_rA_s does not intersect any line segment A_tB_t ($t \neq r,s$). Prove that after a certain number of jumps, the grasshopper can jump from any A_u to any A_v.

Problem 369. ABC is a triangle with $BC > CA > AB$. D is a point on side BC and E is a point on ray BA beyond A so that $BD=BE=CA$. Let P be a point on side AC such that E, B, D, P are concyclic. Let Q be the intersection point of ray BP and the circumcircle of $\triangle ABC$ different from B. Prove that $AQ+CQ=BP$.

Problem 370. On the coordinate plane, at every lattice point (x,y) (these are points where x, y are integers), there is a light. At time $t = 0$, exactly one light is turned on. For n = 1, 2, 3, …, at time $t = n$, every light at a lattice point is turned on if it is at a distance 2005 from a light that was turned on at time $t = n - 1$. Prove that every light at a lattice point will eventually be turned on at some time.

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Solutions

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Problem 361. Among all real numbers a and b satisfying the property that the equation $x^4+ax^3+bx^2+ax+1=0$ has a real root, determine the minimum possible value of a^2+b^2 with proof.

Solution. **U. BATZORIG** (National University of Mongolia) and **Evangelos MOUROUKOS** (Agrinio, Greece).

Consider all a,b such that the equation has x as a real root. The equation implies $x \neq 0$. Using the Cauchy-Schwarz inequality (or looking at the equation as the line $(x^3 + x)a + x^2b + (x^4 + 1) = 0$ in the (a,b)-plane and computing its distance from the origin), as

$$(a^2+2b^2+a^2)\left(x^6+\frac{x^4}{2}+x^2\right)$$

$$\ge (ax^3+bx^2+ax)^2 = (x^4+1)^2,$$

we get $a^2+b^2 \ge \dfrac{(x^4+1)^2}{2x^6+x^4+2x^2}$ with equality if and only if $x = \pm 1$ (at which both sides are 4/5). For x = 1, (a,b) = (−4/5, −2/5). For x = −1, (a,b) = (−2/5,4/5). Finally,

$$\frac{(x^4+1)^2}{2x^6+x^4+2x^2} \ge \frac{4}{5}$$

by calculus or rewriting it as

$$5(x^4+1)^2 - 4(2x^6+x^4+2x^2)$$
$$= (x^2-1)^2(5x^4+2x^2+5) \ge 0.$$

So the minimum of $a^2 + b^2$ is 4/5.

Other commended solvers: **CHAN Long Tin** (Diocesan Boys' School), **Hong Kong Joint School Math Society, LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **LKL Excalibur** (Madam Lau Kam Lung Secondary School of MFBM), **Raymond LO** (King's College), **Paolo PERFETTI** (Math Dept, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy), **Anna PUN Ying** (HKU Math), **The 7B Math Group** (Carmel Alison Lam Foundation Secondary School) and **Alice WONG Sze Nga** (Diocesan Girls' School).

Problem 362. Determine all positive rational numbers x,y,z such that

$$x+y+z,\ \ xyz,\ \ \frac{1}{x}+\frac{1}{y}+\frac{1}{z}$$

are integers.

Solution. **CHAN Long Tin** (Diocesan Boys' School), **Hong Kong Joint School Math Society, Raymond LO** (King's College), **Anna PUN Ying** (HKU Math) and **The 7B Math Group** (Carmel Alison Lam Foundation Secondary School).

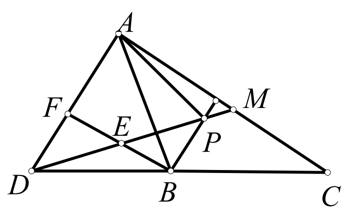
Let $A = x + y + z$, $B = xyz$ and $C = 1/x + 1/y + 1/z$, then A, B, C are integers. Since $xy + yz + zx = BC$, so x,y,z are the roots of the equation $t^3 - At^2 + BCt - B = 0$. Since the coefficients are integers and the coefficient of t^3 is 1, by Gauss lemma or the rational root theorem, the roots x, y, z are integers.

Since they are positive, without loss of generality, we may assume $z \ge y \ge x \ge 1$. Now $1 \le 1/x + 1/y + 1/z \le 3/x$ lead to x=1, 2 or 3. For x = 1, $1/y + 1/z$ = 1 or 2, which yields (y,z) = (1,1) or (2,2). For x = 2, $1/y + 1/z = 1/2$, which yields (y,z) = (3,6) or (4,4). For x = 3, $1/y + 1/z = 2/3$, which yields (y,z) = (3,3). So the solutions are (x,y,z) = (1,1,1), (1,2,2), (2,3,6), (2,4,4), (3,3,3) and permutations of coordinates.

Other commended solvers: **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College) and **Alice WONG Sze Nga** (Diocesan Girls' School).

Problem 363. Extend side CB of triangle ABC beyond B to a point D such that $DB=AB$. Let M be the midpoint of side AC. Let the bisector of $\angle ABC$ intersect line DM at P. Prove that $\angle BAP = \angle ACB$.

Solution. **Raymond LO** (King's College).



Construct line $BF \parallel$ line CA with F on line AD. Let DM intersect BF at E.

Since $BD=AB$, we get $\angle BDF = \angle BAF = ½\angle ABC = \angle ABP = \angle CBP$. Then line $FD \parallel$ line PB. Hence, $\triangle DFE$ is similar to $\triangle PBE$.

Since $BF \parallel CA$ and M is the midpoint of

AC, so E is the midpoint of FB, i.e. $FE=BE$. Then $\triangle DFE$ is congruent to $\triangle PBE$. Hence, $FD=PB$.

This along with $DB = BA$ and $\angle BDF = \angle ABP$ imply $\triangle BDF$ is congruent to $\triangle ABP$. Therefore, $\angle BAP = \angle DBF = \angle ACB$.

Other commended solvers: **U. BATZORIG** (National University of Mongolia), **CHAN Long Tin** (Diocesan Boys' School), **Hong Kong Joint School Math Society, Abby LEE Shing Chi** (SKH Lam Woo Memorial Secondary School), **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **LKL Excalibur** (Madam Lau Kam Lung Secondary School of MFBM), **Anna PUN Ying** (HKU Math), **The 7B Math Group** (Carmel Alison Lam Foundation Secondary School), **Ercole SUPPA** (Liceo Scientifico Statale E.Einstein, Teramo, Italy) and **Alice WONG Sze Nga** (Diocesan Girls' School).

Problem 364. Eleven robbers own a treasure box. What is the least number of locks they can put on the box so that there is a way to distribute the keys of the locks to the eleven robbers with no five of them can open all the locks, but every six of them can open all the locks? The robbers agree to make enough duplicate keys of the locks for this plan to work.

Solution. **CHAN Long Tin** (Diocesan Boys' School), **Hong Kong Joint School Math Society, LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **LKL Excalibur** (Madam Lau Kam Lung Secondary School of MFBM), **Raymond LO** (King's College), **Emanuele NATALE** (Università di Roma "Tor Vergata", Roma, Italy), **Anna PUN Ying** (HKU Math), **The 7B Math Group** (Carmel Alison Lam Foundation Secondary School) and **Alice WONG Sze Nga** (Diocesan Girls' School).

Let n be the least number of locks required. If for every group of 5 robbers, we put a new lock on the box and give a key to each of 6 other robbers only, then the plan works. Thus

$$n \le \binom{11}{5} = 462.$$

Conversely, in the case when there are n locks, for every group G of 5 robbers, there exists a lock $L(G)$, which they do not have the key, but the other 6 robbers all have keys to $L(G)$. Assume there exist $G \ne G'$ such that $L(G)=L(G')$. Then there is a robber in G and not in G'. Since G is one of the 6 robbers not in G', he has a key to $L(G')$, which is $L(G)$, contradiction. So $G \ne G'$ implies $L(G) \ne L(G')$. Then the number of locks is at least as many groups of 5 robbers. So

$$n \ge \binom{11}{5} = 462.$$ Therefore, $n = 462$.

Problem 365. For nonnegative real numbers a,b,c satisfying $ab+bc+ca = 1$, prove that

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}-\frac{1}{a+b+c} \ge 2.$$

Solution. **CHAN Long Tin** (Diocesan Boys' School) and **Alice WONG Sze Nga** (Diocesan Girls' School).

Since $a, b, c \ge 0$ and $ab+bc+ca = 1$, none of the denominators can be zero. Multiplying both sides by $a+b+c$, we need to show

$$\frac{c}{a+b}+\frac{a}{b+c}+\frac{b}{c+a}+2 \ge 2(a+b+c).$$

This follows from using the Cauchy-Schwarz inequality and expanding $(c+a+b-2)^2 \ge 0$ as shown below

$$2\left(\frac{c}{a+b}+\frac{a}{b+c}+\frac{b}{c+a}\right)$$

$$=((a+b)c+(b+c)a+(c+a)b)\left(\frac{c}{a+b}+\frac{a}{b+c}+\frac{b}{c+a}\right)$$

$$\ge (c+a+b)^2$$

$$\ge 4(a+b+c)-4.$$

Other commended solvers: **Andrea FANCHINI** (Cantu, Italy), **D. Kipp JOHNSON** (Valley Catholic School, Teacher, Beaverton, Oregon, USA), **LI Pak Hin** (PLK Vicwood K. T. Chong Sixth Form College), **Paolo PERFETTI** (Math Dept, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy), **Anna PUN Ying** (HKU Math) and **The 7B Math Group** (Carmel Alison Lam Foundation Secondary School).

Olympiad Corner

(continued from page 1)

Problem 3. (Cont.) If the total area of the white rectangles equals the total area of the red rectangles, what is the smallest possible value of x?

Problem 4. Show that there exists a positive integer N such that for all integers $a > N$, there exists a contiguous substring of the decimal expansion of a which is divisible by 2011. (For instance, if $a = 153204$, then 15, 532, and 0 are all contiguous substrings of a. Note that 0 is divisible by 2011.)

Problem 5. Let d be a positive integer. Show that for every integer S there exists an integer $n > 0$ and a sequence $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$, where for any k, $\varepsilon_k = 1$ or $\varepsilon_k = -1$, such that

$$S = \varepsilon_1(1+d)^2 + \varepsilon_2(1+2d)^2 + \varepsilon_3(1+3d)^2 + \cdots + \varepsilon_n(1+nd)^2.$$

Harmonic Series (I)

(continued from page 2)

The sum of reciprocals of all such numbers is therefore less than

$$\sum_{m=0}^{\infty} 8\left(\frac{9}{10}\right)^m = \frac{8}{1-\frac{9}{10}} = 80.$$

<u>**Example 9**</u>: Let $m > 1$ be a positive integer. Show that $1/m$ is the sum of consecutive terms in the sequence

$$\sum_{j=1}^{\infty} \frac{1}{j(j+1)}.$$

<u>**Solution**</u> Since

$$\frac{1}{j(j+1)} = \frac{1}{j} - \frac{1}{j+1},$$

the problem is reduced to finding integers a and b such that

$$\frac{1}{m} = \frac{1}{a} - \frac{1}{b} \quad (*).$$

One obvious solution is $a = m-1$ and $b = m(m-1)$. To find other solutions of (*), we note that $1/a > 1/m$, so $m > a$.

Let $a = m-c$, then $b = (m^2/c)-m$. For each c satisfying $c \mid m^2$ and $1 \le c \le m$, there exists one and only one pair of a and b satisfying (*), and because $a < b$, the representation is unique. Let $d(n)$ count the number of factors of n. Now consider all factors of m^2 except m, there are $d(m^2)-1$ of them. If c is one of them, then exactly one of c or m^2/c will be less than m. Hence the number of solutions of (*) is $[d(m^2)-1]/2$.