Mathematical Excalibur

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Olympiad Corner

Below are the problems of the 2011 Asia Pacific Math Olympiad, which was held in March 2011.

Problem 1. Let a, b, c be positive integers. Prove that it is impossible to have all of the three numbers a^2+b+c, b^2+c+a, c^2+a+b to be perfect squares.

Problem 2. Five points A_1, A_2, A_3, A_4, A_5 lie on a plane in such a way that no three among them lie on a same straight line. Determine the maximum possible value that the minimum value for the angles $\angle A_iA_jA_k$ can take where i, j, k are distinct integers between 1 and 5.

Problem 3. Let ABC be an acute triangle with $\angle BAC=30°$. The internal and external angle bisectors of $\angle ABC$ meet the line AC at B_1 and B_2, respectively, and the internal and external angle bisectors of $\angle ACB$ meet the line AB at C_1 and C_2, respectively. Suppose that the circles with diameters B_1B_2 and C_1C_2 meet inside the triangle ABC at point P. Prove that $\angle BPC = 90°$.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***June 25, 2011***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

Dr. Kin-Yin LI, Math Dept., Hong Kong Univ. of Science and Technology, Clear Water Bay, Kowloon, Hong Kong

Fax: (852) 2358 1643
Email: makyli@ust.hk

Harmonic Series (II)

Leung Tat-Wing

As usual, for integers a, b, n (with $n > 0$), we write $a\equiv b \pmod n$ to mean $a-b$ is divisible by n. If $b\neq 0$ and n are relatively prime (i.e. they have no common prime divisor), then $0, b, 2b, \ldots, (n-1)b$ are distinct (mod n) because for $0 \le s < r < n$, $rb \equiv sb \pmod n$ implies $(r-s)b = kn$. Since b, n have no common prime divisor, this means b divides k. Then $0 < (k/b)n = r-s < n$, contradicting $b \le k$. Hence, there is a unique r among $1, \ldots, n-1$ such that $rb\equiv 1 \pmod n$. We will denote this r as b^{-1} or $1/b$ (mod n). Further, we can extend (mod n) to fractions by defining $a/b \equiv ab^{-1} \pmod n$. We can easily check that the usual properties of fractions holds in mod n arithmetic.

Next, we will introduce Wolstenholme's theorem, which is an important relation concerning harmonic series.

Theorem (Wolstenholme): For a prime $p \ge 5$,

$$H(p-1)=1+\frac{1}{2}+\ldots+\frac{1}{p-1}\equiv 0 \pmod{p^2}.$$

(More precisely, for a prime $p \ge 5$, if

$$H(p-1)=1+\frac{1}{2}+\ldots+\frac{1}{p-1}=\frac{a}{b},$$

then $p^2 \mid a$.)

Example We have

$$H(10)=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{10}=\frac{7381}{2520}$$

and $11^2 \mid 7381$.

First proof We have

$$H(p-1)=1+\frac{1}{2}+\cdots+\frac{1}{p-1}$$
$$=\sum_{n=1}^{(p-1)/2}\left(\frac{1}{n}+\frac{1}{p-n}\right)=p\sum_{n=1}^{(p-1)/2}\frac{1}{n(p-n)}.$$

So we need to prove

$$\sum_{n=1}^{(p-1)/2}\frac{1}{n(p-n)}\equiv 0 \pmod p.$$

Now $\displaystyle\sum_{n=1}^{(p-1)/2}\frac{1}{n(p-n)}\equiv -\sum_{n=1}^{(p-1)/2}\frac{1}{n^2} \pmod p.$

Since every $1/n^2$ is congruent to exactly one of the numbers $1^2, 2^2, \ldots, [(p-1)/2]^2$ (mod p) and $1/n^2$ are all distinct for $n=1,2,\ldots,(p-1)/2$, we have when $p \ge 5$,

$$\sum_{n=1}^{(p-1)/2}\frac{1}{n^2}\equiv\sum_{k=1}^{(p-1)/2}k^2=\frac{(p^2-1)p}{24}\equiv 0 \pmod p.$$

Wolstenholme's theorem follows.

Second proof (using polynomials mod p) We use a theorem of Lagrange, which says if $f(x) = c_0 + c_1x + \cdots + c_nx^n$ is a polynomial of degree n, with integer coefficients, and if $f(x) \equiv 0 \pmod p$ has more than n solutions, where p is prime, then every coefficient of $f(x)$ is divisible by p. The proof is not hard. It can be done basically by induction and the division algorithm mod p. The statement is false if p is not prime. For instance, $x^2 - 1 \equiv 0 \pmod 8$ has 4 solutions. Here is the other proof.

From Fermat's Little theorem, $x^{p-1} \equiv 1 \pmod p$ has $1, 2, \ldots, p-1$ as solutions. Thus $x^{p-1}-1 \equiv (x-1)(x-2)\cdots(x-p+1) \pmod p$. Let

$$(x-1)(x-2)\cdots(x-p+1)$$
$$=x^{p-1}-s_1x^{p-2}+\cdots-s_{p-2}x+s_{p-1}. \quad (*)$$

By Wilson's theorem, $s_{p-1} = (p-1)! \equiv -1 \pmod p$. Thus

$$0 \equiv s_1x^{p-2}+\cdots-s_{p-2}x \pmod p.$$

The formula is true for every integer x. By Lagrange's theorem, p divides each of $s_1, s_2, \ldots, s_{p-2}$. Putting $x = p$ in (*), we get $(p-1)!=p^{p-1}-s_1p^{p-2}+\cdots-s_{p-2}p+s_{p-1}$. Canceling out $(p-1)!$ and dividing both sides by p, we get

$$0 = p^{p-2}-s_1p^{p-3}+\cdots+s_{p-3}p-s_{p-2}.$$

As $p \ge 5$, each of the terms is congruent to 0 (mod p^2). Hence, we have $s_{p-2} \equiv 0 \pmod{p^2}$. Finally,

$$s_{p-2}=(p-1)!(1+\frac{1}{2}+\ldots+\frac{1}{p-1})=(p-1)!\frac{a}{b}.$$

This proves Wolstenholme's theorem.

Using Wolstenholme's theorem and setting $x = kp$ in (*), we get

$$(kp-1)(kp-2)\cdots(kp-p+1)$$
$$= (kp)^{p-1} - s_1(kp)^{p-2} + \cdots + s_{p-3}(kp)^2 - s_{p-2}kp + s_{p-1}$$
$$\equiv s_{p-3}(kp)^2 - s_{p-2}kp + s_{p-1}$$
$$\equiv (p-1)! \pmod{p^3}.$$

Upon dividing by $(p-1)!$, we have

$$\binom{kp-1}{p-1} \equiv 1 \pmod{p^3}, \quad k = 1, 2, \ldots.$$

This result may in fact be taken as the statement of Wolstenholme's theorem.

Here are a few further remarks. Wolstenholme's theorem on the congruence of harmonic series is related to the Bernoulli numbers B_n. For instance, we have

$$\binom{kp-1}{p-1} \equiv 1 - \frac{1}{3}(k^2-k)p^3 B_{p-3} \pmod{p^4},$$

which is usually called *Glaisher's congruence*. These numbers are related to Fermat's Last Theorem. It is known that for any prime $p \geq 5$,

$$\binom{kp-1}{p-1} \equiv 1 \pmod{p^3}.$$

Are there primes satisfying

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^4}?$$

These primes are called *Wolstenholme primes*. (So far, we only know 16843 and 2124679 are such primes). In another direction, one can ask if there exist composite numbers n such that

$$\binom{kn-1}{n-1} \equiv 1 \pmod{n^3}?$$

All these are very classical questions.

Example 10 (APMO 2006): Let $p \geq 5$ be a prime and let r be the number of ways of placing p checkers on a $p\times p$ checkerboard so that not all checkers are in the same row (but they may all be in the same column). Show that r is divisible by p^5.

Solution Observe that

$$r = \binom{p^2}{p} - p = p\left(\frac{(p^2-1)\cdots(p^2-(p-1))}{(p-1)!} - 1\right).$$

Hence it suffices to show that

$$(p^2-1)(p^2-2)\cdots(p^2-(p-1)) - (p-1)! \equiv 0 \pmod{p^4} \qquad (1)$$

Now let

$$f(x) = (x-1)(x-2)\cdots(x-(p-1)) = x^{p-1} + s_1x^{p-2} + \cdots + s_{p-2}x + s_{p-1}. \qquad (2)$$

Thus the first congruence relation is the same as $f(p^2) - (p-1)! \equiv 0 \pmod{p^4}$. Therefore it suffices to show that $s_{p-2}p^2 \equiv 0 \pmod{p^4}$ or $s_{p-2} \equiv 0 \pmod{p^2}$, which is exactly Wolstenholme's theorem.

Example 11 (Putnam 1996): Let p be a prime number greater than 3 and $k = [2p/3]$. Show that

$$\binom{p}{1} + \binom{p}{2} + \ldots + \binom{p}{k} \equiv 0 \pmod{p^2}$$

For example,

$$\binom{7}{1} + \binom{7}{2} + \binom{7}{3} + \binom{7}{4} = 98 \equiv 0 \pmod{7^2}.$$

Solution Recall

$$\binom{p}{i} = \frac{p(p-1)\ldots(p-i+1)}{1\cdot 2\cdot\ldots i}.$$

This is a multiple of p if $1 \leq i \leq p-1$. Modulo p, the right side after divided by p is congruent to

$$\frac{(-1)\cdots(-(i-1))}{1\cdot 2\cdots i} = (-1)^{i-1}\frac{1}{i}.$$

Hence, to prove the congruence, it suffices to show

$$1 - \frac{1}{2} + \frac{1}{3} - \cdots + (-1)^{k-1}\frac{1}{k} \equiv 0 \pmod{p}.$$

Now observe that

$$-\frac{1}{2i} \equiv \frac{1}{2i} + \frac{1}{p-i} \pmod{p}.$$

This allows us to replace the sum by

$$1 + \frac{1}{2} + \ldots + \frac{1}{p-1} \equiv 0 \pmod{p},$$

which is Wolstenholme's theorem.

We can also give a more detailed proof as follow. Let

$$H(n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

and

$$P(n) = 1 - \frac{1}{2} + \ldots + (-1)^{n-1}\frac{1}{n}.$$

Then the problem is reduced to showing that for any $p > 3$, p divides the numerator of $P([2p/3])$. First we note that p divides the numerator of $H(p-1)$ because

$$2H(p-1)$$
$$= \left(1 + \frac{1}{p-1}\right) + \left(\frac{1}{2} + \frac{1}{p-2}\right) + \cdots + \left(\frac{1}{p-1} + 1\right)$$
$$= \frac{p}{p-1} + \frac{p}{2(p-2)} + \cdots + \frac{p}{p-1} \equiv 0 \pmod{p}.$$

Next we have two cases.

Case 1 ($p = 3n+1$) Then $[2p/3] = 2n$. So we must show p divides the numerator of $P(2n)$. Now

$$H(3n) - P(2n)$$
$$= 2\left(1 + \frac{1}{2} + \ldots + \frac{1}{2n}\right) + \left(\frac{1}{2n+1} + \frac{1}{2n+2} + \ldots + \frac{1}{3n}\right)$$
$$= \left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right) + \left(\frac{1}{2n+1} + \frac{1}{2n+2} + \ldots + \frac{1}{3n}\right)$$
$$= \left(1 + \frac{1}{p-1}\right) + \left(2 + \frac{1}{p-2}\right) + \ldots + \left(\frac{1}{n} + \frac{1}{p-n}\right)$$
$$= \frac{p}{p-1} + \frac{p}{2(p-2)} + \ldots + \frac{p}{n(p-n)}.$$

So p divides the numerators of both $H(3n)$ and $H(3n) - P(2n)$, hence also the numerator of $P(2n)$.

Case 2 ($p = 3n+2$) Then $[2p/3] = 2n+1$. So we must show p divides the numerator of $P(2n+1)$. Now

$$H(3n+1) - P(2n+1)$$
$$= 2\left(1 + \frac{1}{2} + \ldots + \frac{1}{2n}\right) + \left(\frac{1}{2n+2} + \frac{1}{2n+2} + \ldots + \frac{1}{3n+1}\right)$$
$$= \left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right) + \left(\frac{1}{2n+2} + \frac{1}{2n+2} + \ldots + \frac{1}{3n+1}\right)$$
$$= \left(1 + \frac{1}{p-1}\right) + \left(2 + \frac{1}{p-2}\right) + \ldots + \left(\frac{1}{n} + \frac{1}{p-n}\right)$$
$$= \frac{p}{p-1} + \frac{p}{2(p-2)} + \ldots + \frac{p}{n(p-n)}.$$

So, p divides the numerator of $H(3n+1) - P(2n+1)$, and hence $P(2n+1)$.

Example 12: Let $p \geq 5$ be a prime, show that if

$$1 + \frac{1}{2} + \ldots + \frac{1}{p} = \frac{a}{b},$$

then $p^4 \mid ap - b$.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***June 25, 2011.***

Problem 371. Let a_1, a_2, a_3, … be a sequence of nonnegative rational numbers such that $a_m+a_n= a_{mn}$ for all positive integers *m, n*. Prove that there exist two terms that are equal.

Problem 372. (*Proposed by Terence ZHU)* For all *a,b,c* > 0 and *abc* = 1, prove that

$$\frac{1}{a(a+1)+ab(ab+1)}+\frac{1}{b(b+1)+bc(bc+1)}$$

$$+\frac{1}{c(c+1)+ca(ca+1)}\ge\frac{3}{4}.$$

Problem 373. Let *x* and *y* be the sums of positive integers $x_1,x_2,\ldots,x_{99}$ and $y_1,y_2,\ldots,y_{99}$ respectively. Prove that there exists a 50 element subset *S* of {1,2,…,99} such that the sum of all x_n with *n* in *S* is at least *x*/2 and the sum of all y_n with *n* in *S* is at least *y*/2.

Problem 374. *O* is the circumcenter of acute ΔABC and *T* is the circumcenter of ΔAOC. Let *M* be the midpoint of side *AC*. On sides *AB* and *BC*, there are points *D* and *E* respectively such that $\angle BDM=\angle BEM=\angle ABC$. Prove that $BT\perp DE$.

Problem 375. Find (with proof) all odd integers *n* > 1 such that if *a*, *b* are divisors of *n* and are relatively prime, then *a*+*b*−1 is also a divisor of *n*.

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Solutions

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Problem 366. Let *n* be a positive integer in base 10. For *i*=1,2,…,9, let *a*(*i*) be the number of digits of *n* that equal *i*. Prove that

$$2^{a(1)}3^{a(2)}\cdots 9^{a(8)}10^{a(9)}\le n+1$$

and determine all equality cases.

Solution. **LAU Chun Ting** (St. Paul's Co-educational College, Form 2).

Let $f(n)=2^{a(1)}3^{a(2)}\cdots 9^{a(8)}10^{a(9)}$. If *n* is a number with one digit, then $f(n) = n+1$. Suppose all numbers *A* with *k* digits satisfy the given inequality $f(A) \le A+1$. For any (*k*+1) digit number, it is of the form 10*A*+*B*, where *A* is a *k* digit number and $0 \le B \le 9$. We have

$$\begin{aligned} f(10A+B) &= (B+1)f(A) \le (B+1)(A+1)\\ &= (B+1)A+B+1 \le 10A+B+1.\end{aligned}$$

Equality holds if and only if $f(A) = A+1$ and *B* = 9. By induction, the inequality holds for all positive integers *n* and equality holds if and only if all but the leftmost digits of *n* are 9's.

Other commended solvers: **CHAN Long Tin** (Diocesan Boys' School), **LEE Tak Wing** (Carmel Alison Lam Foundation Secondary School), **Gordon MAN Siu Hang** (CCC Ming Yin College) and **YUNG Fai**.

Problem 367. For *n* = 1,2,3,…, let x_n and y_n be positive real numbers such that

$$x_{n+2}=x_n+x_{n+1}^2$$

and

$$y_{n+2}=y_n^2+y_{n+1}.$$

If x_1, x_2, y_1, y_2 are all greater than 1, then prove that there exists a positive integer *N* such that for all *n* > *N*, we have $x_n > y_n$.

Solution. **LAU Chun Ting** (St. Paul's Co-educational College, Form 2) and **Gordon MAN Siu Hang** (CCC Ming Yin College).

Since x_1, x_2, y_1, y_2 are all greater than 1, by induction, we can get $x_{n+1}> x_n^2 > 1$ and $y_{n+1}> 1+y_n > n$ for $n \ge 2$. Then $x_{n+2} = x_n+x_{n+1}^2 > x_{n+1}^2 > x_n^4$ and $y_{n+2} = y_n^2+y_{n+1} = y_n^2+y_{n-1}^2+y_n < 3y_n^2 < y_n^3$ for all $n \ge 4$.

Hence, $\log x_{n+2} > 4 \log x_n$ and $\log y_{n+2} < 3 \log y_n$. So for $n \ge 4$,

$$\frac{\log x_{n+2}}{\log y_{n+2}}>\frac{4}{3}\left(\frac{\log x_n}{\log y_n}\right).\qquad (*)$$

As 4/3 > 1, by taking logarithm, we can solve for a positive integer *k* satisfying the inequality

$$\left(\frac{4}{3}\right)^k\min\left\{\frac{\log x_4}{\log y_4},\frac{\log x_5}{\log y_5}\right\}>1.$$

Let *N* = 2*k*+3. If *n* > *N*, then either *n* = 2*m*+4 or *n* = 2*m*+5 for some integer $m \ge k$.

Applying (*) *m* times, we have

$$\frac{\log x_n}{\log y_n}>\left(\frac{4}{3}\right)^m\min\left\{\frac{\log x_4}{\log y_4},\frac{\log x_5}{\log y_5}\right\}>1.$$

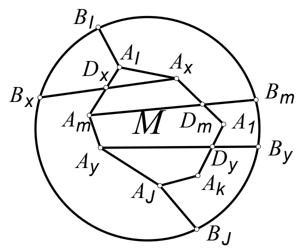
This implies $x_n > y_n$.

Other commended solvers: **LEE Tak Wing** (Carmel Alison Lam Foundation Secondary School) and **NGUYEN Van Thien** (Luong The Vinh High School, Dong Nai, Vietnam).

Problem 368. Let *C* be a circle, A_1, A_2, …, A_n be distinct points inside *C* and B_1, B_2, …, B_n be distinct points on *C* such that no two of the segments A_1B_1, A_2B_2,…, A_nB_n intersect. A grasshopper can jump from A_r to A_s if the line segment A_rA_s does not intersect any line segment A_tB_t ($t\ne r,s$). Prove that after a certain number of jumps, the grasshopper can jump from any A_u to any A_v.

Solution. **William PENG**.

The cases *n* = 1 or 2 are clear. Suppose $n \ge 3$. By reordering the pairs A_i, B_i, we may suppose the convex hull of A_1, A_2, …, A_n is the polygonal region *M* with vertices A_1, A_2, …, A_k ($k\le n$). For $1\le m\le k$, if every A_mB_m intersects *M* only at A_m, then the *n*-th case follows by removing two pairs of A_m, B_m separately and applying case *n* −1.



Otherwise, there exists a segment A_mB_m intersecting *M* at more than 1 point. Let it intersect the perimeter of *M* again at D_m. Since A_iB_i's do not intersect, so A_jD_j's (being subsets of A_iB_i's) do not intersect. In particular, D_m is not a vertex of *M*.

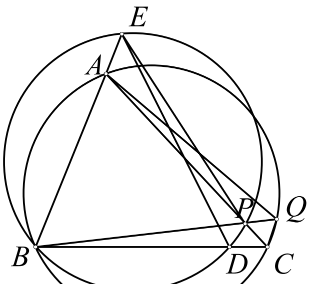
Now A_mD_m divides the perimeter of *M* into two parts. Moving from A_m to D_m <u>*clockwise*</u> on the perimeter of *M*, there are points A_x, D_x such that there is no D_w between them. As D_x is not a vertex, there is a vertex A_I between A_x and D_x. Then A_IB_I only intersect *M* at A_I. Also, moving from A_m to D_m <u>*anti-clockwise*</u> on the perimeter of *M*, there is A_J such that A_JB_J only intersects *M* at A_J. Then A_IB_I and A_JB_J do not intersect any diagonal of *M* with endpoints different from A_I and A_J.

Removing A_I, B_I and applying case $n-1$, the grasshopper can jump between any two of the points $A_1,\ldots,A_{I-1}, A_{I+1},\ldots,A_n$. Also, removing A_J, B_J and applying case $n-1$, the grasshopper can jump between any two of the points $A_1,\ldots,A_{J-1}, A_{J+1},\ldots,A_n$. Using these two cases, we see the grasshopper can jump from any A_u to any A_v via A_t ($t\neq I,J$).

Other commended solvers: **T. h. G**.

Problem 369. *ABC* is a triangle with $BC > CA > AB$. D is a point on side BC and E is a point on ray BA beyond A so that $BD=BE=CA$. Let P be a point on side AC such that E, B, D, P are concyclic. Let Q be the intersection point of ray BP and the circumcircle of $\triangle ABC$ different from B. Prove that $AQ+CQ=BP$.

Solution. **CHAN Long Tin** (Diocesan Boys' School), **Giorgos KALANTZIS** (Demenica's Public High School, Patras, Greece) and **LAU Chun Ting** (St. Paul's Co-educational College, Form 2).



Since A,B,C,Q are concyclic and E,P,D,B are concyclic, we have

$$\angle AQC = 180^\circ - \angle ABC = \angle EPD$$

and

$$\angle PED = \angle PBD = \angle QAC.$$

Hence, $\triangle AQC$ and $\triangle EPD$ are similar. So we have $AQ/AC=PE/DE$ and $CQ/AC = PD/DE$. Cross-multiplying and adding these two equations, we get

$$(AQ+CQ)\times DE = (PE+PD)\times AC. \quad (*)$$

For cyclic quadrilateral $EPDB$, by the Ptolemy theorem, we have

$$\begin{aligned} BP\times DE &= PE\times BD+PD\times BE \\ &= (PE+PD)\times AC \qquad (**) \end{aligned}$$

Comparing (*) and (**), we have $AQ+CQ=BP$.

Other commended solvers: **LEE Tak Wing** (Carmel Alison Lam Foundation Secondary School).

Problem 370. On the coordinate plane, at every lattice point (x,y) (these are points where x, y are integers), there is a light. At time $t = 0$, exactly one light is turned on. For $n = 1, 2, 3, \ldots$, at time $t = n$, every light at a lattice point is turned on if it is at a distance 2005 from a light that was on at time $t = n - 1$. Prove that every light at a lattice point will eventually be turned on at some time.

Solution. **LAU Chun Ting** (St. Paul's Co-educational College, Form 2), **LEE Tak Wing** (Carmel Alison Lam Foundation Secondary School), **Gordon MAN Siu Hang** (CCC Ming Yin College) and **Emanuele NATALE** (Università di Roma "Tor Vergata", Roma, Italy).

We may assume the light that was turned on at $t = 0$ was at the origin.

Let $z = 2005 = 5\times401 = (2^2+1^2)(20^2+1^2) = |(2+i)(20+i)|^2 = |41+22i|^2 = 41^2+22^2$. Let $x = 41^2-22^2=1037$ and $y = 2\times41\times22 = 1716$. Then $x^2+y^2 = z^2$.

By the Euclidean algorithm, we get gcd(1037, 1716) = 1. By eliminating the remainders in the calculations, we get $84\times1716-139\times1037=1$.

Let V_1, V_2, V_3, V_4, V_5 be the vectors from the origin to (2005,0), (1037, 1716), (1037, −1716), (1716, 1037), (1716, −1037) respectively. We have $V_2+V_3 = (2\times1037,0)$ and $V_4+V_5 = (2\times1716,0)$. Then we can get $(1,0) = 1003[84(V_4+V_5)-139(V_2+V_3)]-V_1$.

So, from the origin, following these vector movements, we can get to the point (1,0). Similarly, we can get to the point (0,1). As $(a,b) = a(1,0) + b(0,1)$, we can get to any lattice point.

Olympiad Corner

(continued from page 1)

Problem 4. Let n be a fixed positive odd integer. Take $m+2$ distinct points P_0, P_1, …, P_{m+1} (where m is a non-negative integer) on the coordinate plane in such a way that the following 3 conditions are satisfied:

(1) $P_0=(0,1)$, $P_{m+1}=(n+1,n)$, and for each integer i, $1 \le i \le m$, both x- and y-coordinates of P_i are integers lying in between 1 and n (1 and n inclusive).

(2) For each integer i, $0 \le i \le m$, P_iP_{i+1} is parallel to the x-axis if i is even, and is parallel to the y-axis if i is odd.

(3) For each pair i,j with $0 \le i < j \le m$, line segments P_iP_{i+1} and P_jP_{j+1} share at most 1 point.

Determine the maximum possible value that m can take.

Problem 5. Find all functions $f:\mathbb{R}\to\mathbb{R}$, where $\mathbb{R}$ is the set of all real numbers, satisfying the following 2 conditions:

(1) There exists a real number M such that for every real number x, $f(x) < M$ is satisfied.

(2) For every pair of real numbers x and y, $f(xf(y)) + yf(x) = xf(y) + f(xy)$ is satisfied.

Harmonic Series (II)

(continued from page 2)

__Solution__ By Wolstenholme's theorem,

$$p^2 \left| (p-1)!\left(1+\frac{1}{2}+\cdots+\frac{1}{p-1}\right)\right..$$

So,

$$1+\frac{1}{2}+\ldots+\frac{1}{p-1} = p^2\frac{x}{y},$$

where x, y are integers with y not divisible by p. So we have

$$\frac{a}{b}-\frac{1}{p} = p^2\frac{x}{y},$$

which implies $ap-b = p^3bx/y$. Finally,

$$\frac{a}{b} = \frac{2\cdot3\cdots p+1\cdot3\cdot4\cdots p+\cdots+1\cdot2\cdots(p-1)}{p!}$$

and the numerator of the right side is of the form $mp+(p-1)!$. Hence, it is not divisible by p. So $p \mid b$ and $p^4 \mid p^3bx/y = ap-b$.

__Example 13__: Let p be an odd prime, then prove that

$$1-\frac{1}{2^2}+\frac{1}{3^2}-\ldots+(-1)^{p-2}\frac{1}{(p-1)^2} \equiv 0 \pmod{p}.$$

__Solution__ The proof is not hard. Indeed,

$$\sum_{k=1}^{p-1}(-1)^{k-1}\frac{1}{k^2}$$

$$= -\sum_{k=1}^{(p-1)/2}\left((-1)^k\frac{1}{k^2}+(-1)^{p-k}\frac{1}{(p-k)^2}\right)$$

$$\equiv -\sum_{k=1}^{(p-1)/2}\left((-1)^k\frac{1}{k^2}+(-1)^{1-k}\frac{1}{(-k)^2}\right) = 0 \pmod{p}.$$