Mathematical Excalibur

Volume 16, Number 2 *June - October 2011*

Olympiad Corner

Below are the problems of the 28th Balkan Math Olympiad, which was held in May 6, 2011. Time allowed was 4½ hours.

Problem 1. Let $ABCD$ be a cyclic quadrilateral which is not a trapezoid and whose diagonals meet at E. The midpoints of AB and CD are F and G respectively, and ℓ is the line through G parallel to AB. The feet of the perpendiculars from E onto the lines ℓ and CD are H and K, respectively. Prove that the lines EF and HK are perpendicular.

Problem 2. Given real numbers x, y, z such that $x+y+z = 0$, show that

$$\frac{x(x+2)}{2x^2+1}+\frac{y(y+2)}{2y^2+1}+\frac{z(z+2)}{2z^2+1}\geq 0.$$

When does equality hold?

Problem 3. Let S be a finite set of positive integers which has the following property: if x is a member of S, then so are all positive divisors of x. A non-empty subset T of S is *good* if whenever $x, y \in T$ and $x < y$, the ration y/x is a power of a prime number.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***November 10, 2011***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

Dr. Kin-Yin LI, Math Dept., Hong Kong Univ. of Science and Technology, Clear Water Bay, Kowloon, Hong Kong

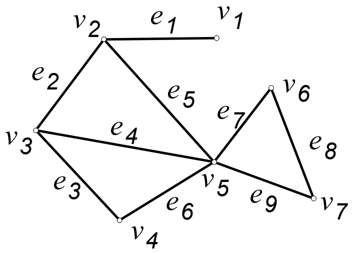
Fax: (852) 2358 1643
Email: makyli@ust.hk

Euler's Planar Graph Formula

Kin Y. Li

A *graph* G is consisted of a *nonempty* set $V(G)$ (its elements are called *vertices*) and a set $E(G)$ (its elements are called *edges*), where an edge is to be thought of as a continuous curve joining a vertex u in $V(G)$ to a vertex v in $V(G)$. A graph G is *finite* if and only if $V(G)$ is a finite set. It is *simple* if and only if each edge in $E(G)$ joins some pair of distinct vertices in $V(G)$ and no other edge joins the same pair. *In this article, all graphs are understood to be finite and simple*.

A graph is *connected* if and only if for every pair of distinct vertices a, b, there is a sequence of edges $e_1, e_2, \ldots, e_n$ such that for i from 1 to n, edge e_i joins v_i and v_{i+1} with $v_1 = a$ and $v_{n+1} = b$. A graph is *planar* if and only if it can be drawn on a plane with no pair of edges intersect at any point other than a vertex of the graph. A planar graph divides the plane into regions (bounded by edges) called *faces*.



In the graph above, there are 7 vertices (labeled v_1 to v_7), 9 edges (labeled e_1 to e_9) and 4 faces (the 3 triangular regions and the outside region bounded by $e_1, e_5, e_7, e_8, e_9, e_6, e_3, e_2, e_1$). The following theorem due to Euler relates the number of vertices, the number of edges and the number of regions for a connected planar graph and is the key tool in solving some interesting problems.

Euler's Theorem on Planar Graphs

Let V, E, F denote the number of vertices, the number of edges, the number of faces respectively for a connected planar (finite simple) graph. Then $V - E + F = 2$, which we will called *Euler's formula*.

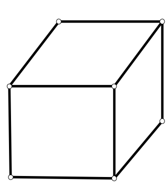
We will *sketch* the usual mathematical induction proof on E. If $E = 0$, then since $V(G)$ is nonempty and G is connected, we have $V = 1$ and $F = 1$. So $V-E+F=2$. Also, if $E =1$, then $V = 2$, $F =1$ and again the formula is true.

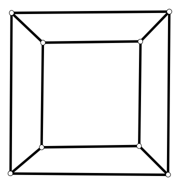
Suppose the cases $E < k$ are true. For the case $E=k$, either there is a cycle (that is a sequence of edges $e_1, e_2, \ldots, e_n$ such that for i from 1 to n, edge e_i joins v_i and v_{i+1} with $v_1 = v_{n+1}$) or no cycle.

In the former case, removing e_n will result in a connected graph with E decreases by 1, V stays the same and F decreases by 1 (since the two regions sharing e_n in their boundaries will become one). The formula still holds.

In the latter case, we call these graphs *trees*. It can be proved that they satisfy $E=V-1$ and $F=1$ (which implies Euler's formula). Basically, removing any edge will split such a graph into two connected graphs with each having no cycle. This observation would allow us to do the induction on E.

Before presenting some examples, we remark that Euler's formula also applies to convex polyhedrons. These are the boundary surfaces of three dimensional convex solids obtained by intersecting finitely many (half-spaces on certain sides of) planes. For example, take the surface of a cube, V=8, E=12, F=6 so that $V - E + F = 2$. For any convex polyhedron, we can obtain a connected planar graph by choosing a face as base, stretching the base sufficiently big and taking a top view projection onto the plane containing the base. The following is a cube and a planar graph for its boundary surface.





Example 1. There are $n > 3$ points on a circle. Each pair of them is connected by a chord such that no three of these chords intersect at the same point inside the circle. Find the number of regions formed inside the circle.

Solution. Removing the n arcs on the circle, we get a *simple* connected planar graph, where the vertices are the n points on the circle and the intersection points inside the circle. For every 4 of the n points, we can draw two chords intersecting at a point inside the circle. So the number of vertices is $V = n + {}_nC_4$.

Since there are $n-1$ edges incident with each of the n points on the circle, 4 edges incident with every intersection point inside the circle and each edge is counted twice, so the number of edges is $E = (n(n-1)+4\ {}_nC_4)/2$.

By Euler's formula, the number of faces for this graph is $F = 2 - V + E$. Excluding the outside face and adding the n regions having the n arcs as boundary, the number of regions inside the circle is $F - 1 + n = n + 1 - V + E = 1 + {}_nC_4 + n(n-1)/2$.

For the next few examples, we define the degree of a vertex v in a graph to be the number of edges meeting at v. Below $d(v)$ will denote the degree of v. *The sum of degrees of all vertices equals twice the number of edges since each edge is counted twice at its two endponts.*

Example 2. A square region is partitioned into n convex polygonal regions. Find the maximal number of edges in the figure.

Solution. Let V, E, F be the number of vertices, edges, faces respectively in the graph. Euler's formula yields

$$n+1 = F = 2-V+E \quad \text{or} \quad V = E + 1 - n.$$

Let A, B, C, D be the vertices of the square, then $t = d(A) + d(B) + d(C) + d(D) \geq 8$ as each term is at least 2.

Let W be the set of vertices inside the square. For any v in W, we have $d(v) \geq 3$ since angles of convex polygons are less than 180°. Let s be the sum of $d(v)$ for all v in W. Since there are $V-4$ vertices in W, we have $s \geq 3(V-4)$.

Now summing degree of all vertices, we get $s + t = 2E$. Then

$$2E-8 \geq 2E - t = s \geq 3(V-4) = 3(E-3-n),$$

which simplifies to $E \leq 3n+1$.

Finally, the case $E = 3n+1$ is possible by partitioning the square region into n rectangles using $n - 1$ line segments parallel to a side of the square. So the maximum possible value of E is $3n+1$.

Example 3. (*2000 Belarussian Math Olympiad*) In a convex polyhedron with m triangular faces (and possibly faces of other shapes), exactly four edges meet at each vertex. Find the minimum possible value of m.

Solution. Let V, E, F be the number of vertices, edges, faces respectively on such a polyhedron. Since each vertex is met by 4 distinct edges, summing all degrees, we have $2E = 4V$.

Next, summing the number of edges in the F faces and observing that each edge is counted twice on the 2 faces sharing it, we get $2E \geq 3m+4(F-m)$.

By Euler's formula, we have

$$2 = V - E + F = (E/2) - E + F = F - E/2,$$

which implies

$$4F - 8 = 2E \geq 3m + 4(F-m).$$

This simplifies to $m \geq 8$. A regular octahedron is an example of the case $m = 8$. So the minimum possible m is 8.

Example 4. (*1985 IMO proposal by Federal Republic of Germany*) Let M be the set of edge-lengths of an octahedron whose faces are congruent quadrilaterals. Prove that M has at most three elements.

Solution. The octahedron has $(4\times8)/2=16$ edges. By Euler's formula, it has $V = 2 + E - F = 2 + 16 - 8 = 10$ vertices.

Next, let n_i be the number of vertices v with $d(v) = i$. Then, counting vertices and edges respectively in terms of n_i's, we have

$$V = n_3 + n_4 + n_5 + \cdots = 10$$

and

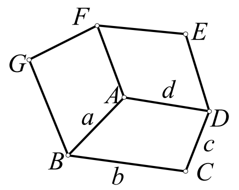
$$2E = 3n_3 + 4n_4 + 5n_5 + \cdots = 2\times16.$$

Eliminating n_3, we get

$$n_4 + 2n_5 + 3n_6 + \cdots = 2.$$

Hence, $n_4 \leq 2$, $n_5 \leq 1$ and $n_i = 0$ for $i \geq 6$. Then $n_3 = 10 - n_4 - n_5 > 0$.

Let A be a vertex with degree 3. Assume M has 4 distinct elements a, b, c, d. Then the 3 faces about A are like the figure below, where we may take $AB = a$, $BC = b$, $CD = c$ and $DA = d$.



Since $ABCD$ and $ABGF$ are congruent, so $AF = b$ or d. Also, since $ABCD$ and $AFED$ are congruent, so $AF = a$ or c. Hence, two of a, b, c, d must be equal, contradiction. Therefore, M has at most 3 elements.

Example 5. Let n be a positive integer. A convex polyhedron has $10n$ faces. Prove that n of the faces have the same number of edges.

Solution. Let V be the number of vertices of this polyhedron. For the $10n$ faces, let these faces be polygons with $a_1, a_2, \ldots, a_{10n}$ sides respectively, where the a_i's are arranged in ascending order. Then the number of edges of the polyhedron is $E = (a_1 + a_2 + \cdots + a_{10n})/2$. By Euler's formula, we have

$$V - \frac{a_1 + a_2 + \cdots + a_n}{2} + 10n = 2. \quad (*)$$

Also, since the degree of every vertex is at least 3, we get

$$a_1+a_2+\cdots+a_{10n} \geq 3V. \quad (**)$$

Using (*) and (**), we can eliminate V and solve for $a_1+a_2+\cdots+a_{10n}$ to get

$$a_1+a_2+\cdots+a_{10n} \leq 60n - 12. \quad (***)$$

Assume no n faces have equal number of edges. Then we have $a_1, a_2, \ldots, a_{n-1} \geq 3$, $a_n, a_{n+1}, \ldots, a_{2n-2} \geq 4$ and so on. This leads to

$$\begin{aligned} & a_1 + a_2 + \cdots + a_{10n} \\ \geq\ & (3 + 4 + \cdots + 12)(n-1) + 13\times10 \\ =\ & 75n + 55. \end{aligned}$$

Comparing with (***), we get $75n + 55 \leq 60n - 12$, which is false for n.

Example 6. (*1975 Kiev Math Olympiad and 1987 East German Math Olympiad*) An arrowhead is drawn on every edge of a convex polyhedron H such that at every vertex, there are at least one arrowhead pointing toward the vertex and another arrowhead pointing away from the vertex. Prove that there exist at least two faces of H, the arrowheads on each of its boundary form a (clockwise or counterclockwise) cycle.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***November 10, 2011.***

Problem 376. A polynomial is monic if the coefficient of its greatest degree term is 1. Prove that there exists a monic polynomial $f(x)$ with integer coefficients such that for every prime p, $f(x) \equiv 0 \pmod{p}$ has solutions in integers, but $f(x) = 0$ has no solution in integers.

Problem 377. Let n be a positive integers. For i=1,2,…,n, let z_i and w_i be complex numbers such that for all 2^n choices of $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ equal to ±1, we have

$$\left|\sum_{i=1}^{n}\varepsilon_i z_i\right| \le \left|\sum_{i=1}^{n}\varepsilon_i w_i\right|.$$

Prove that $\sum_{i=1}^{n}|z_i|^2 \le \sum_{i=1}^{n}|w_i|^2$.

Problem 378. Prove that for every positive integers m and n, there exists a positive integer k such that $2^k - m$ has at least n distinct positive prime divisors.

Problem 379. Let ℓ be a line on the plane of ΔABC such that ℓ does not intersect the triangle and none of the lines AB, BC, CA is perpendicular to ℓ.

Let A', B' C' be the feet of the perpendiculars from A, B, C to ℓ respectively. Let A'', B'', C'' be the feet of the perpendiculars from A', B', C' to lines BC, CA, AB respectively.

Prove that lines $A'A''$, $B'B''$, $C'C''$ are concurrent.

Problem 380. Let $S = \{1,2,\ldots,2000\}$. If A and B are subsets of S, then let $|A|$ and $|B|$ denote the number of elements in A and in B respectively. Suppose the product of $|A|$ and $|B|$ is at least 3999. Then prove that sets $A-A$ and $B-B$ contain at least one common element, where $X-X$ denotes $\{s-t : s, t \in X$ and $s \ne t\}$.

****

Solutions

****

Problem 371. Let a_1, a_2, a_3, … be a sequence of nonnegative rational numbers such that $a_m + a_n = a_{mn}$ for all positive integers m, n. Prove that there exists two terms that are equal.

Solution. **U. BATZORIG** (National University of Mongolia), **CHUNG Kwan** (King's College) and **F7B Pure Math Group** (Carmel Alison Lam Foundation Secondary School).

Let p and q be distinct primes. If a_p and a_q are zeros, then we are done. Otherwise, consider

$$m = p^{Na_q} \quad and \quad n = q^{Na_p},$$

where N is a positive integer that makes both Na_q and Na_p integers. Obviously, we have $m \ne n$ and

$$a_m = (Na_q)a_p = (Na_p)a_q = a_n.$$

Other commended solvers: **Samuel Liló ABDALLA** (ITA-UNESP, São Paulo, Brazil).

Problem 372. (*Proposed by Terence ZHU*) For all $a,b,c > 0$ and abc=1, prove that

$$\frac{1}{a(a+1)+ab(ab+1)} + \frac{1}{b(b+1)+bc(bc+1)} + \frac{1}{c(c+1)+ca(ca+1)} \ge \frac{3}{4}.$$

Solution. **V. ADIYASUREN** (National University of Mongolia) and **B. SANCHIR** (Mathematics Institute of the National University of Mongolia), **F7B Pure Math Group** (Carmel Alison Lam Foundation Secondary School) and **Kipp JOHNSON** (Valley Catholic School, Teacher, Beaverton, Oregon, USA).

Substituting $a = z/y$, $b = x/z$, $c = y/x$ (say by choosing x=ab=1/c, y=1, z=a) into the inequality and simplifying, we get

$$\sum_{cyc} f(x,y,z) \ge \frac{3}{4},$$

where

$$f(x,y,z) = \frac{y^2}{z(z+y)+x(x+y)} \quad \text{and}$$

$$\sum_{cyc} f(x,y,z) = f(x,y,z) + f(y,z,x) + f(z,x,y).$$

Let $g(x,y,z) = y^2(z^2 + zy + x^2 + xy)$. By the Cauchy-Schwarz inequality, we have

$$\sum_{cyc} f(x,y,z) \sum_{cyc} g(x,y,z) \ge \left(\sum_{cyc} y^2\right)^2.$$

So it is enough to prove

$$\left(\sum_{cyc} y^2\right)^2 \Big/ \left(\sum_{cyc} g(x,y,z)\right) \ge \frac{3}{4}. \quad (*)$$

Expanding and factorizing, we get

$$4\left(\sum_{cyc} y^2\right)^2 - 3\left(\sum_{cyc} g(x,y,z)\right)$$

$$= 4\sum_{cyc} y^4 + 2\sum_{cyc} x^2y^2 - 3\sum_{cyc} xy(x^2+y^2)$$

$$= 3\sum_{cyc}(x-y)^2(x^2+y^2) + \sum_{cyc}(x^2-y^2)^2 \ge 0.$$

This implies (*), which implies the desired inequality.

Other commended solvers: **CHUNG Kwan** (King's College), **NGUYEN Van Thien** (Luong The Vinh High School, Dong Nai, Vietnam) and **Paolo PERFETTI** (Math Dept, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy).

Problem 373. Let x and y be the sums of positive integers $x_1, x_2, \ldots, x_{99}$ and $y_1, y_2, \ldots, y_{99}$ respectively. Prove that there exists a 50 element subset S of $\{1,2,\ldots,99\}$ such that the sum of all x_n with n in S is at least $x/2$ and the sum of all y_n with n in S is at least $y/2$.

Solution. **William Peng** and **Jeff Peng**.

Arrange the numbers $x_1, x_2, \ldots, x_{99}$ in descending order, say $x_{n(1)} \ge x_{n(2)} \ge \cdots \ge x_{n(99)}$ so that

$$\{n(1), n(2), \ldots, n(99)\} = \{1,2,\ldots,99\}.$$

Let $A = \{n(2), n(4), \ldots, n(98)\}$ and $B = \{n(3), n(5), \ldots, n(99)\}$. We have

$$x_{n(1)} + \sum_{j\in B} x_j > \sum_{i\in A} x_i \ge \sum_{j\in B} x_j.$$

If $\sum_{i\in A} y_i \ge \sum_{j\in B} y_j$, then let $S = A \cup \{n(1)\}$.

Now S has 50 elements. Also,

$$\sum_{i\in S} x_i > \sum_{i\in A} x_i \ge \sum_{j\in B} x_j$$

and

$$\sum_{i\in S} y_i > \sum_{i\in A} y_i \ge \sum_{j\in B} y_j.$$

So the sum of all x_n with n in S is at least $x/2$ and the sum of all y_n with n in S is at least $y/2$.

If $\sum_{i\in A} y_i < \sum_{j\in B} y_j$, then let $S=B\cup\{n(1)\}$.

Again S has 50 elements. Now

$$\sum_{i\in S} x_i = x_{n(1)} + \sum_{j\in B} x_j > \sum_{i\in A} x_i \ge \sum_{j\in B} x_j$$

and

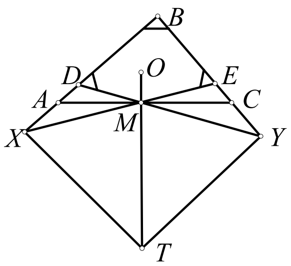
$$\sum_{i\in S} y_i > \sum_{j\in B} y_j > \sum_{i\in A} y_i.$$

So the sum of all x_n with n in S is at least $x/2$ and the sum of all y_n with n in S is at least $y/2$.

Other commended solvers: **U. BATZORIG** (National University of Mongolia) and **F7B Pure Math Group** (Carmel Alison Lam Foundation Secondary School),

Problem 374. O is the circumcenter of acute ΔABC and T is the circumcenter of ΔAOC. Let M be the midpoint of side AC. On sides AB and BC, there are points D and E respectively such that $\angle BDM=\angle BEM=\angle ABC$. Prove that $BT\perp DE$.

***Solution*. William Peng** and **Jeff Peng**.



By the exterior angle theorem, $\angle ABC = \angle BDM > \angle BAM$ and also $\angle ABC = \angle BEM > \angle BCM$. So $\angle ABC$ is the largest angle in ΔABC. Then we have $60° < \angle ABC < 90°$. This implies O is on the same side of line AC as B. Then T will be on the opposite side of line AC as O. Also, O, M, T are on the perpendicular bisector of line AC.

Let X be the intersection of lines AB and ME. Let Y be the intersection of lines CB and MD. Now

$$\begin{aligned}\angle DXE &= 180° - \angle XBE - \angle BEX\\ &= 180° - 2\angle ABC\end{aligned}$$

and similarly $\angle EYD = 180° - 2\angle ABC$. So $\angle DXE = \angle EYD$, which implies D, X, Y, E are concyclic.

Next, since T is the circumcenter of ΔAOC, so

$$\begin{aligned}\angle ATM &= \angle ATO = 2\angle ACO\\ &= 2(90° - \angle BXE)\\ &= 180° - 2\angle ABC\\ &= \angle BXE = \angle AXM.\end{aligned}$$

This implies A, M, T, X are concyclic. So $\angle AXT = 180° - \angle AMT = 90°$. Similarly, $\angle CYT = 90°$. Then $\angle BXT = \angle BYT$, which implies B, X, T, Y are concyclic. So

$$\angle TBY = \angle TXY = 90° - \angle BXY. \quad (*)$$

Since D, X, Y, E are concyclic,

$$\begin{aligned}&\angle BED + \angle TBE\\ =\ &\angle BXY + \angle TBY\\ =\ &90° \qquad \text{by } (*),\end{aligned}$$

which implies $BT\perp DE$.

Other commended solvers: **F7B Pure Math Group** (Carmel Alison Lam Foundation Secondary School),

Problem 375. Find (with proof) all odd integers $n > 1$ such that if a, b are divisors of n and are relatively prime, then $a+b-1$ is also a divisor of n.

Solution. **U. BATZORIG** (National University of Mongolia), **William Peng** and **Jeff Peng**.

For such odd n, let p be its <u>least</u> prime divisor. Then $n = p^m a$, where m is the exponent of p in the prime factorization of n. We will show $a = 1$.

Assume $a > 1$. Then every prime divisors of a is at least $p+2$. Also $c = a+p-1$ ($> p$) is a divisor of n. Since

$$\gcd(c,a) = \gcd(c-a,a) = \gcd(p-1,a) = 1,$$

this implies $c=p^r$ with $r \ge 2$. Then $d = a+p^2-1$ ($> p^2$) is also a divisor of n. Similarly,

$$\gcd(d,a) = \gcd(d-a,a) = \gcd(p^2-1,a) = 1.$$

So $d=p^s$ with $s\ge 3$. Finally, $p^r-p = c-p = a-1 = d-p^2$, which is divisible by p^2, while p^r-p is not. Therefore, $a = 1$.

It is easy to check all $n=p^m$ with p an odd prime and m a positive integer indeed satisfy the condition.

Olympiad Corner

(continued from page 1)

Problem 3. (Cont.) A non-empty subset T of S is <u>bad</u> if whenever $x, y\in T$ and $x<y$, the ration y/x is not a power of a prime number. We agree that a singleton subset of S is both good and bad. Let k be the largest possible size of a good subset of S. Prove that k is also the smallest number of pairwise-disjoint bad subsets whose union is S.

Problem 4. Let $ABCDEF$ be a convex hexagon of area 1, whose opposite sides are parallel. The lines AB, CD and EF meet in pairs to determine the vertices of a triangle. Similarly, the lines BC, DE and FA meet in pairs to determine the vertices of another triangle. Show that the area of one of these two triangles is at least 3/2.

Euler's Planar Graph Formula

(continued from page 2)

<u>*Solution.*</u> Call $\{a,b\}$ a <u>*hook*</u> if a, b are two consecutive edges on the boundary of some face of H. Call a hook $\{a,b\}$ <u>*traversible*</u> if the arrowheads on a and b are both counterclockwise or both clockwise.

Note every hook is part of the boundary of a unique face. Let E be the number of edges on H and h be the number of hooks on H. As each edge on H is a part of 4 hooks, we get $h = 2E$.

Next at every vertex v, $d(v) \ge 3$. By the given condition on the vertices, there must be at least 2 traversible hooks through every vertex. Let V be the number of vertices on H, then there are at least $2V$ traversible hooks on H.

Let h_+ and h_- be the number of traversible and non-traversible hooks respectively on H. Then $h_+ \ge 2V$.

In every face where the boundary arrowheads do not form a cycle, there are at least two changes in directions on the boundary, which result in at least two non-traversible hooks. Let F be the number of faces on H. Let f_+ be the number of faces the boundary arrowheads form cycles. Let $f_-=F-f_+$. Then $h_- \ge 2f_-$.

By Euler's formula, $V - E + F = 2$. Then

$$\begin{aligned}2f_+ &= 2F - 2f_-\\ &= (4 + 2E - 2V) - 2f_-\\ &\ge 4 + h - h_+ - 2f_-\\ &= 4 + h_- - 2f_- \ge 4,\end{aligned}$$

which implies $f_+ \ge 2$. This gives the desired conclusion.