Mathematical Excalibur

Volume 16, Number 5 *March - April 2012*

Olympiad Corner

Below are the problems of the 2011 IMO Team Selection Contest from Estonia.

Problem 1. Two circles lie completely outside each other. Let A be the point of intersection of internal common tangents of the circles and let K be the projection of this point onto one of their external common tangents. The tangents, different from the common tangent, to the circles through point K meet the circles at M_1 and M_2. Prove that the line AK bisects angle M_1KM_2.

Problem 2. Let n be a positive integer. Prove that for each factor m of the number $1+2+\cdots+n$ such that $m \geq n$, the set $\{1,2,\cdots,n\}$ can be partitioned into disjoint subsets, the sum of the elements of each being equal to m.

Problem 3. Does there exist an operation * on the set of all integers such that the following conditions hold simultaneously:

(1) for all integers x, y and z,

$$(x*y)*z = x*(y*z);$$

(2) for all integers x and y,

$$x*x*y = y*x*x = y\ ?$$

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***May 11, 2012***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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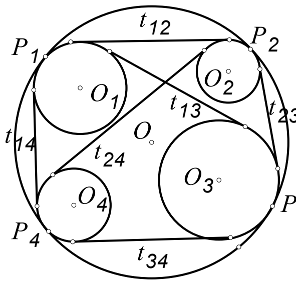
Casey's Theorem

Kin Y. Li

We recall *Ptolemy's theorem*, which asserts that for four noncollinear points A, B, C, D on a plane, we have

$$AB\cdot CD + AD\cdot BC = AC\cdot BD$$

if and only if $ABCD$ is a cyclic quadrilateral (cf *vol.* 2, *no.* 4 of Math Excalibur). In this article, we study a generalization of this theorem known as

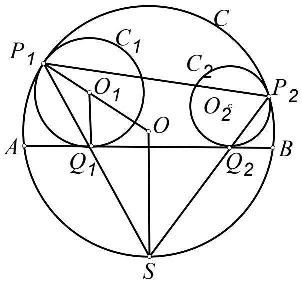


Casey's Theorem. If circles C_1, C_2, C_3, C_4 with centers O_1, O_2, O_3, O_4 are internally tangent to a circle C with center O at points P_1, P_2, P_3, P_4 in cyclic order respectively, then

$$t_{12}\cdot t_{34} + t_{14}\cdot t_{23} = t_{13}\cdot t_{24}, \qquad (*)$$

where t_{ik} denote the length of an external common tangent of circle C_i and C_k.

To prove this, consider the following figure.



Let line AB be an external common tangent to C_1, C_2 intersecting C_1 at Q_1, C_2 at Q_2. Let line P_1Q_1 intersect C at S. Let r_1, r be the respective radii of C_1, C. Then the isosceles triangles $P_1O_1Q_1$ and P_1OS are similar. So $O_1Q_1 \parallel OS$. Since $O_1Q_1 \perp AB$, so $OS \perp AB$, hence S is the midpoint of arc AB. Similarly, line P_2Q_2 passes through S. Now $\angle SQ_1Q_2 = \angle P_1Q_1A = ½\angle P_1O_1Q_1 = ½\angle P_1OS = \angle SP_2P_1$. Then ΔSQ_1Q_2 and ΔSP_2P_1 are similar. So

$$\frac{Q_1Q_2}{P_2P_1} = \frac{SQ_1}{SP_2} = \frac{SQ_2}{SP_1} = \sqrt{\frac{SQ_1\cdot SQ_2}{SP_1\cdot SP_2}} = \sqrt{\frac{OO_1\cdot OO_2}{OP_1\cdot OP_2}}$$

$$t_{12} = Q_1Q_2 = P_1P_2\frac{\sqrt{(r-r_1)(r-r_2)}}{r}. \quad (**)$$

The expressions for the other t_{ik}'s are similar. Since $P_1P_2P_3P_4$ is cyclic, by Ptolemy's theorem,

$$P_1P_2\cdot P_3P_4 + P_1P_4\cdot P_2P_3 = P_1P_3\cdot P_2P_4.$$

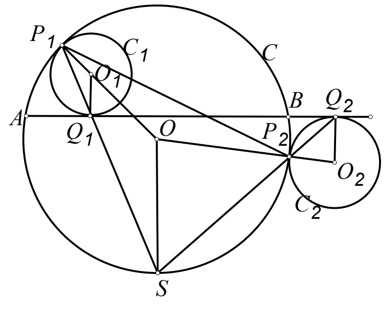
Multiplying all terms by

$$\frac{\sqrt{(r-r_1)(r-r_2)(r-r_3)(r-r_4)}}{r^2}$$

and using (**), we get (*).

Casey's theorem can be *extended* to cover cases some C_k's are externally tangent to C. For this, define t_{ik} more generally to be the length of the external (resp. internal) common tangent of circles C_i and C_k when the circles are on the same (resp. opposite) side of C.

In case C_k is externally tangent to C, consider the following figure. The proof is the same as before except the factor $r-r_k$ should be replaced by $r+r_k$.



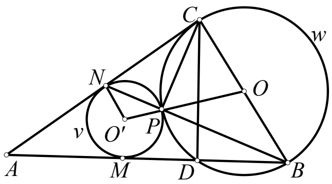
The converse of Casey's theorem and its extension are also true. However, the proofs are harder, longer and used inversion in some cases. For the curious readers, a proof of the converse can be found in *Roger A. Johnson*'s book Advanced Euclidean Geometry, published by Dover.

Next we will present some geometry problems that can be solved by Casey's theorem and its converse.

Example 1. (*2009 China Hong Kong Math Olympiad*) Let $\triangle ABC$ be a right-angled triangle with $\angle C=90^\circ$. CD is the altitude from C to AB, with D on AB. w is the circumcircle of $\triangle BCD$. v is a circle situated in $\triangle ACD$, it is tangent to the segments AD and AC at M and N respectively, and is also tangent to circle w.

(i) Show that $BD \cdot CN + BC \cdot DM = CD \cdot BM$.

(ii) Show that $BM = BC$.



Solution. (i) Think of B, C, D as circles with radius 0 externally tangent to w. Then $t_{BD} = BD$, $t_{Cv} = CN$, $t_{BC} = BC$, $t_{Dv} = DM$, $t_{CD} = CD$ and $t_{Bv} = BM$. By Casey's theorem, (*) yields

$$BD \cdot CN + BC \cdot DM = CD \cdot BM.$$

(ii) Let circles v and w meet at P. Then $\angle BPC=90^\circ$. Let O and O' be centers of circles w and v. Then O, P, O' are collinear. So

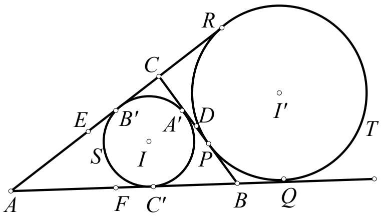
$$\angle PNC+\angle PCN = ½(\angle PO'N+\angle POC) = ½(360°-\angle O'NC-\angle OCN) = 90°.$$

So $\angle NPC = 90^\circ$. Hence, B, P, N are collinear. By the power-of-a-point theorem, $BM^2=BP\cdot BN$. Also $\angle C=90^\circ$ and $CP\perp BN$ imply $BC^2 = BP\cdot BN$. Therefore, $BM=BC$.

Example 2. (*Feuerbach's Theorem*) Let D,E,F be the midpoints of sides AB, BC, CA of ΔABC respectively.

(i) Prove that the inscribed circle S of ΔABC is tangent to the (nine-point) circle N through D, E, F.

(ii) Prove that the described circle T on side BC is also tangent to N.



Solution. (1) We consider D, E, F as circles of radius 0. Let A', B', C' be the points of tangency of S to sides BC, CA, AB respectively.

First we recall that the two tangent segments from a point to a circle have the same length. Let $AB' = x = C'A$, $BC' = y = A'B$, $CA' = z = B'C$ and $s = (a+b+c)/2$, where $a=BC$, $b=CA$, $c=AB$. From $y+x = BA = c$, $z+y = CB = a$ and $x+z = AC = b$, we get $x = (c+b-a)/2 = s-a$, $y=s-b$, $z=s-c$. By the midpoint theorem, $t_{DE} = DE = ½BA = c/2$ and

$$t_{FS}= FC' = |FB-BC'| = |(c/2)-y| = |c-2(s-b)|/2 = |b-a|/2.$$

Similarly, $t_{EF} = a/2$, $t_{DS}= |c-b|/2$, $t_{FD} = b/2$ and $t_{ES}=|a-c|/2$. Without loss of generality, we may assume $a\leq b\leq c$. Then

$$t_{DE} \cdot t_{FS} + t_{EF} \cdot t_{DS} = c(b-a)/4+a(c-b)/4 = b(c-a)/4 = t_{FD} \cdot t_{ES}.$$

By the converse of Casey's theorem, we get S is tangent to the circle N through D,E,F.

(2) Let I' be the center of T, let P,Q,R be the points of tangency of T to lines BC, AB, CA respectively. As in (1), $t_{DE}= c/2$.

To find t_{FT}, we need to know BQ. First note $AQ=AR$, $BP=BQ$ and $CR=CP$. So $2AQ=AQ+AR=AB+BP+CP+AC=2s$. So $AQ=s/2$. Next $BQ=AQ-AB=s-c$. Hence, $t_{FT}=FQ=FB+BQ = (c/2)+(s-c) = (b+a)/2$. Similarly, $t_{ET}= (a+c)/2$. Now $t_{DT}= DP = DB-BP = DB-BQ = (a/2) - (s-c) = (c-b)/2$. Then

$$t_{FD} \cdot t_{ET} + t_{EF} \cdot t_{DT} = b(a+c)/4+a(c-b)/4 = c(b+a)/4 = t_{DE} \cdot t_{FT}.$$

By the converse of Casey's theorem, we get T is tangent to the circle N through D,E,F.

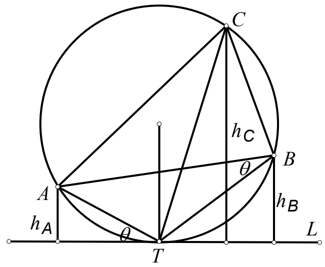
Example 3. (*2011 IMO*) Let ABC be an acute triangle with circumcircle Γ. Let L be a tangent line to Γ, and let L_a, L_b and L_c be the line obtained by reflecting L in the lines BC, CA and AB, respectively. Show that the circumcircle of the triangle determined by the lines L_a, L_b and L_c is tangent to the circle Γ.

Solution. (*Due to* ***CHOW Chi Hong****, 2011 Hong Kong IMO team member*)

Below for brevity, we will write $\angle A$, $\angle B$, $\angle C$ to denote $\angle CAB$, $\angle ABC$, $\angle BCA$ respectively.

Lemma. In the figure below, L is a tangent line to Γ, T is the point of tangency. Let h_A, h_B, h_C be the length of the altitudes from A, B, C to L respectively. Then

$$\sqrt{h_A}\sin\angle A+\sqrt{h_B}\sin\angle B=\sqrt{h_C}\sin\angle C.$$



Proof. By Ptolemy's theorem and sine law,

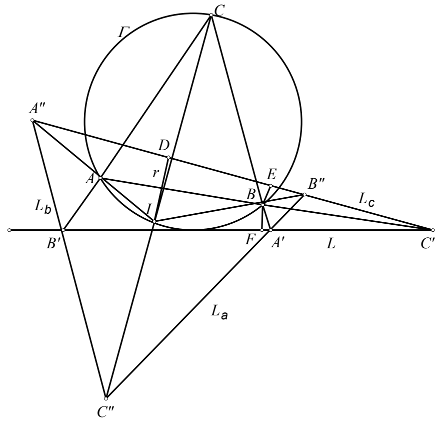
$$AT\cdot BC +BT\cdot CA = CT\cdot BC \quad \text{(or}$$

$$AT \sin \angle A + BT \sin \angle B = CT \sin \angle C).$$

Let θ be the angle between lines AT and L as shown. Then $AT = h_A / \sin\theta = h_A(2k/AT)$, where k is the circumradius of ΔABC. Solving for AT (then using similar argument for BT and CT), we get

$$AT=\sqrt{2kh_A},BT=\sqrt{2kh_B},CT=\sqrt{2kh_C}.$$

Substituting these into (*), the result follows. This finishes the proof of the lemma.



For the problem, let $L_a\cap L=A'$, $L_b\cap L = B'$, $L_c\cap L=C'$, $L_a\cap L_b=C''$, $L_b\cap L_c=A''$, $L_c\cap L_a=B''$. Next

$$\angle A''C''B''=\angle A''B'A'-\angle C''A'B' = 2\angle CB'A'-(180°-2\angle CA'B') = 180°-2\angle C.$$

Similarly, $\angle A''B''C'' = 180° -2\angle B$ and $\angle B''A''C'' = 180° -2\angle A$. (***)

Consider $\Delta A'C'B''$. Now $A'B$ bisects $\angle B'A'B''$ and $C'B$ bisects $\angle A'C'B''$. So B is the excenter of $\Delta A'C'B''$ opposite C'. Hence $B''B$ bisects $\angle A''B''C''$. Similarly, $A''A$ bisects $\angle B''A''C''$ and $C''C$ bisects $\angle B''C''A''$. Therefore, they intersect at the incenter I of $\Delta A''B''C''$.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***May 11, 2012.***

Problem 391. Let $S(x)$ denote the sum of the digits of the positive integer x in base 10. Determine whether there exist distinct positive integers a, b, c such that $S(a+b)<5$, $S(b+c)<5$, $S(c+a)<5$, but $S(a+b+c)>50$ or not.

Problem 392. Integers $a_0, a_1, \cdots, a_n$ are all greater than or equal to -1 and are not all zeros. If

$$a_0+2a_1+2^2a_2+\cdots+2^na_n = 0,$$

then prove that $a_0+a_1+a_2+\cdots+a_n>0$.

Problem 393. Let p be a prime number and $p \equiv 1 \pmod 4$. Prove that there exist integers x and y such that

$$x^2 - py^2 = -1.$$

Problem 394. Let O and H be the circumcenter and orthocenter of acute $\triangle ABC$. The bisector of $\angle BAC$ meets the circumcircle Γ of $\triangle ABC$ at D. Let E be the mirror image of D with respect to line BC. Let F be on Γ such that DF is a diameter. Let lines AE and FH meet at G. Let M be the midpoint of side BC. Prove that $GM \perp AF$.

Problem 395. One frog is placed on every vertex of a $2n$-sided regular polygon, where n is an integer at least 2. At a particular moment, each frog will jump to one of the two neighboring vertices (with more than one frog at a vertex allowed).

Find all n such that there exists a jumping of these frogs so that after the moment, all lines connecting two frogs at different vertices do not pass through the center of the polygon.

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Solutions

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Problem 386. Observe that $7+1=2^3$ and $7^7+1= 2^3\times113\times911$. Prove that for $n = 2, 3, 4,\ldots$, in the prime factorization of $A_n = 7^{7^n}+1$, the sum of the exponents is at least $2n+3$.

Solution. **Mathematics Group** (Carmel Alison Lam Foundation Secondary School) and **William PENG**.

The case $n = 0$ is given. Suppose the result is true for n. Let $x = A_n-1$. Then $A_{n+1} = x^7+1 = (x+1)P = A_nP$, where $P = x^6 - x^5 + x^4 - x^3 + x^2 - x + 1$. Comparing P with $(x+1)^6$, we find

$$\begin{aligned} P &= (x+1)^6 - 7x(x^4+2x^3+3x^2+2x+1) \\ &= (x+1)^6 - 7x(x^2+x+1)^2. \end{aligned}$$

Now $7x=7^{2m}$, where $m=(7^n+1)/2$. Then $P = [(x+1)^3+7^m(x^2+x+1)][(x+1)^3-7^m(x^2+x+1)]$. Next, $x > 7^m \geq 7$, $x^2+x+1 > (x+1)^2$ and

$$(x+1)^3-7^m(x^2+x+1) > (x+1)^2(x+1-7^m) > 1.$$

So P is the product of at least 2 more primes. Therefore, the result is true for $n+1$.

Problem 387. Determine (with proof) all functions $f : [0,+\infty) \to [0,+\infty)$ such that for every $x \geq 0$, we have $4f(x) \geq 3x$ and $f(4f(x) - 3x) = x$.

Solution. **Mathematics Group** (Carmel Alison Lam Foundation Secondary School) and **William PENG**.

We can check $f(x) = x$ is a solution. Assume there is another solution such that $f(c) \neq c$ for some $c \geq 0$. Let $x_0 = f(c)$, $x_1 = c$ and

$$x_{n+2} = 4x_n - 3x_{n+1} \quad \text{for } n = 0,1,2,\ldots.$$

From the given conditions, we can check by math induction that $x_n = f(x_{n+1}) \geq 0$ for $n = 0,1,2,\ldots$. Since $z^2+3z-4 = (z-1)(z+4)$, we see $x_n = \alpha + (-4)^n\beta$ for some real α and β. Taking $n = 0$ and 1, we get $f(c) = \alpha+\beta$ and $c = \alpha - 4\beta$. Then $\beta = (f(c)-c)/5 \neq 0$.

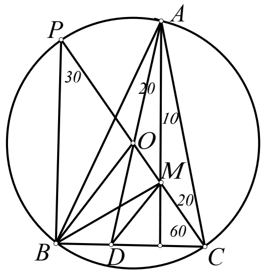
If $\beta > 0$, then $x_{2k+1} = \alpha+(-4)^{2k+1}\beta \to -\infty$ as $k \to \infty$, a contradiction. Similarly, if $\beta < 0$, then $x_{2k} = \alpha+(-4)^{2k}\beta \to -\infty$ as $k \to \infty$, yet another contradiction.

Other commended solvers: **CHAN Yin Hong** (St. Paul's Co-educational College) and **YEUNG Sai Wing** (Hong Kong Baptist University, Math, Year 1).

Problem 388. In $\triangle ABC$, $\angle BAC=30°$ and $\angle ABC=70°$. There is a point M lying inside $\triangle ABC$ such that $\angle MAB = \angle MCA = 20°$. Determine $\angle MBA$ (with proof).

Solution 1. **CHOW Chi Hong** (Bishop Hall Jubilee Schol) and **AN-anduud Problem Solving Group** (Ulaanbaatar, Mongolia).

Extend CM to meet the circumcircle Γ of $\triangle ABC$ at P.



Then we have $\angle BPC = \angle BAC = 30°$ and $\angle PBC = 180° - \angle BPC - \angle BCM = 90°$. So line CM passes through center O of Γ.

Let lines AO and BC meet at D. Then $\angle AOB = 2\angle ACB = 160°$. Now $OA = OB$ implies $\angle OAB = 10°$. Then $\angle MAO = 10° = \angle MAC$ and $\angle ADC = 180° - 100° = 80° = \angle ACD$. These imply AM is the perpendicular bisector of CD. Then $MD = MC$. This along with $OB = OC$ and $\angle BOC = 60$ imply $\triangle OCB$ and $\triangle MCD$ are equilateral, hence $BOMD$ is cyclic. Then $\angle DBM = \angle DOM = 2\angle OAC = 40°$. So $\angle MBA = \angle ABC - \angle DBM = 30°$.

Solution 2. **CHAN Yin Hong** (St. Paul's Co-educational College), **Mathematics Group** (Carmel Alison Lam Foundation Secondary School), **O Kin Chit Alex** (G.T.(Ellen Yeung) College) and **Mihai STOENESCU** (Bischwiller, France).

Let $x = \angle MBA$. Applying the sine law to $\triangle ABC$, $\triangle ABM$, $\triangle AMC$ respectively, we get

$$\frac{AB}{AM} = \frac{\sin(20°+x)}{\sin x}, \quad \frac{AB}{AC} = \frac{\sin 80°}{\sin 70°}, \quad \frac{AC}{AM} = \frac{\sin 30°}{\sin 20°}.$$

Multiplying the last 2 equations, we get

$$\frac{\sin(20°+x)}{\sin x} = \frac{AB}{AM} = \frac{\sin 80°}{\sin 70°} \cdot \frac{\sin 30°}{\sin 20°}. \quad (\dagger)$$

Multiplying

$$\frac{\sin 80°}{\sin 40°} = 2\cos 40° = \frac{\sin 50°}{\sin 30°},$$

$$\frac{\sin 40°}{\sin 20°} = 2\cos 20° = \frac{\sin 70°}{\sin 30°},$$

we see (†) can be simplified to $\sin(20°+x)/\sin x = \sin 50°/\sin 30°$. Since the left side is equal to $\sin 20° \cot x + \cos 20°$, which is strictly decreasing (hence injective) for x between 0° to 70°, we must have $x=30°$.

Comments: One can get a similar equation as (†) directly by using the trigonometric form of Ceva's theorem.

Other commended solvers: **CHEUNG Ka Wai** (Munsang College (Hong Kong Island)), **NG Ho Man** (La Salle

College, Form 5), **Bobby POON** (St. Paul's College), **St. Paul's College Mathematics Team**, **Aliaksei SEMCHANKAU** (Secondary School No.41, Minsk, Belarus) and **ZOLBAYAR Shagdar** (9th grader, Orchlon International School, Ulaanbaatar, Mongolia),

Problem 389. There are 80 cities. An airline designed flights so that for each of these cities, there are flights going in both directions between that city and at least 7 other cities. Also, passengers from any city may fly to any other city by a sequence of these flights. Determine the least k such that no matter how the flights are designed subject to the conditions above, passengers from one city can fly to another city by a sequence of at most k flights.

(*Source*: 2004 *Turkish MO*.)

***Solution*. William PENG**.

Below we denote the number of elements in a set S by $|S|$.

To show $k\ge 27$, take cities $A_1, A_2, \cdots, A_{28}$. For i=1,2,⋯,27, design flights between A_i and A_{i+1}. For the remaining 52 cities, partition them into pairwise disjoint subsets $Y_0, Y_1, \cdots, Y_9$ so $|Y_0|=6=|Y_9|$ and the other $|Y_k|=5$. Let $Z_0=\{A_1,A_2\}\cup Y_0$, $Z_9=\{A_{27},A_{28}\}\cup Y_9$ and for $1\le m\le 8$, let $Z_m=\{A_{3m},A_{3m+1},A_{3m+2}\}\cup Y_m$. Then design flights between each pair of cities in Z_m for $1\le m\le 8$. In this design, from A_1 to A_{28} requires 27 flights.

Assume $k > 27$. Then there would exist two cities A_1 and A_{29} the shortest connection between them would involve a sequence of 28 flights from cities A_i to A_{i+1} for i=1,2,…,28. Due to the shortest condition, each of A_1 and A_{29} has flights to 6 other cities not in $B=\{A_2,A_3,\ldots,A_{28}\}$. Each A_i in B has flights to 5 other cities not in $C = \{A_1,A_2,\ldots,A_{29}\}$.

Next for each A_i in $\{A_1,A_4,A_7,A_{10},A_{13},A_{16},A_{19},A_{22},A_{25},A_{29}\}$, let X_i be the set of cities not in C that have a flight to A_i. We have $|X_1|\ge 6$, $|X_{29}|\ge 6$ and the other $|X_i|\ge 5$. Now every pair of X_i's is disjoint, otherwise we can shorten the sequence of flights between A_1 and A_{29}. However, the union of C and all the X_i's would yield at least 29+6×2+ 5×8 =81 cities, contradiction. So $k = 27$.

Problem 390. Determine (with proof) all ordered triples (x, y, z) of positive integers satisfying the equation

$$x^2y^2 = z^2(z^2 - x^2 - y^2).$$

Solution. **CHEUNG Ka Wai** (Munsang College (Hong Kong Island)), **Ioan Viorel CODREANU** (Satulung Secondary School, Maramure, Romania) and **Aliaksei SEMCHANKAU** (Secondary School No.41, Minsk, Belarus).

Lemma. The system $a^2-b^2=c^2$ and $a^2+b^2=w^2$ has no solution in positive integers.

Proof. Assume there is a solution. Then consider a solution with minimal a^2+b^2. Due to minimality, gcd(a,b)=1. Also $2a^2 = w^2+c^2$. Considering (mod 2), we see $w+c$ and $w-c$ are even. Then $a^2=r^2+s^2$, where $r=(w+c)/2$ and $s=(w-c)/2$.

Let d=gcd(a,r,s). Then d divides a and $r+s=w$. Since $a^2+b^2=w^2$, d divides b. As gcd(a,b)=1, we get d=1. By the theorem on Pythagorean triples, there are relatively prime positive integers m,n with $m>n$ such that $\{r,s\}=\{m^2-n^2, 2mn\}$ and $a=m^2+n^2$. Now $b^2 = (w^2-c^2)/2 = 2rs = 4mn(m^2-n^2)$ implies b is an even integer, say b=2k. Then $k^2=mn(m+n)(m-n)$. As gcd(m,n)=1, we see $m, n, m+n, m-n$ are pairwise relatively prime integers. Hence, there exist positive integers d,e,f,g such that $m=d^2$, $n=e^2$, $m+n=f^2$ and $m-n=g^2$. Then $d^2-e^2=g^2$ and $d^2+e^2=f^2$, but

$$d^2+e^2=m+n < 4mn(m^2-n^2)=b^2 < a^2+b^2,$$

contradicting a^2+b^2 is minimal. The lemma is proved.

Now for the problem, the equation may be rearranged as $z^4-(x^2+y^2)z^2-x^2y^2=0$. If there is a solution (x,y,z) in positive integers, then considering discriminant, we see $x^4 + 6x^2y^2 + y^4 = w^2$ for some integer w. This can be written as $(x^2-y^2)^2+2(2xy)^2 = w^2$. Also, we have $(x^2-y^2)^2 + (2xy)^2 = (x^2+y^2)^2$. Letting $c = |x^2-y^2|$, $b = 2xy$ and $a = x^2+y^2$. Then we have $c^2+b^2=a^2$ and $c^2+2b^2=w^2$ (or $a^2+b^2=w^2$). This contradicts the lemma above. So there is no solution.

Other commended solvers: **Mathematics Group** (Carmel Alison Lam Foundation Secondary School).

Olympiad Corner

(continued from page 1)

Problem 4. Let a, b, c be positive real numbers such that $2a^2+b^2=9c^2$. Prove that

$$\frac{2c}{a}+\frac{c}{b}\ge\sqrt{3}.$$

Problem 5. Prove that if n and k are positive integers such that $1 < k < n-1$, Then the binomial coefficient $\binom{n}{k}$ is divisible by at least two different primes.

Problem 6. On a square board with m rows and n columns, where $m \le n$, some squares are colored black in such a way that no two rows are alike. Find the biggest integer k such that for every possible coloring to start with one can always color k columns entirely red in such a way that no two rows are still alike.

Casey's Theorem

(continued from page 2)

We have $\angle IAB = \angle AA''C' + \angle AC'A'' = ½(\angle B'A''C' + \angle B'C'A'') = ½\angle A'B'C''$ and similarly $\angle IBA = ½\angle B'A'C''$. So

$$\begin{aligned}\angle AIB &= 180^\circ - \angle IA''B'' - \angle IB''A''\\ &= 180^\circ - ½\angle C''A''B'' - ½\angle C''B''A''\\ &= 90^\circ + ½\angle A''C''B''\\ &= 90^\circ + ½(180^\circ - 2\angle C) \text{ by (***)}\\ &= 180^\circ - \angle ACB.\end{aligned}$$

Hence, I lies on Γ.

Let D be the foot of the perpendicular from I to $A''B''$, then $ID=r$ is the inradius of $\Delta A''B''C''$. Let E, F be the feet of the perpendiculars from B to $A''B''$, $B'A'$ respectively. Then $BE = BF = h_B$.

Let $T(X)$ be the length of tangent from X to Γ, where X is outside of Γ. Since $\angle A''B''I = ½\ \angle A''B''C'' = 90^\circ - \angle B$ by (***), we get

$$\begin{aligned}T(B'') &= \sqrt{B''B\cdot B''I}\\ &= \sqrt{\frac{BE}{\sin(90^\circ-\angle B)}\cdot\frac{ID}{\sin(90^\circ-\angle B)}}\\ &= \frac{\sqrt{h_B r}}{\cos B}.\end{aligned}$$

Let R be the circumradius of $\Delta A''B''C''$. Then

$$\begin{aligned}T(B'')\cdot C''A'' &= \frac{\sqrt{h_B r}}{\cos B}2R\sin(180^\circ-2\angle B)\\ &= 4R\sqrt{r}\sqrt{h_B}\sin B.\end{aligned}$$

Similarly, we can get expressions for $T(A'')\cdot B''C''$ and $T(C'')\cdot A''B''$. Using the lemma, we get

$$T(A'')\cdot B''C'' + T(B'')\cdot C''A'' = T(C'')\cdot A''B''.$$

By the converse of Casey's theorem, we have the result.