Mathematical Excalibur

Volume 17, Number 1 *May-August 2012*

Olympiad Corner

Below are the problems of the 2012 International Math Olympiad.

Problem 1. Given triangle ABC the point J is the centre of the excircle opposite the vertex A. This circle is tangent to the side BC at M, and to the lines AB and AC at K and L, respectively. The lines LM and BJ meet at F, and the lines KM and CJ meet at G. Let S be the point of intersection of the lines AF and BC, and let T be the point of intersection of the lines AG and BC. Prove that M is the midpoint of ST.

(The *excircle* of ABC opposite the vertex A is the circle that is tangent to the line segment BC, to the ray AB beyond B, and to the ray AC beyond C.)

Problem 2. Let $n \ge 3$ be an integer, and let $a_2,a_3,\ldots,a_n$ be positive real numbers such that $a_2a_3\cdots a_n = 1$. Prove that

$$(1+a_2)^2(1+a_3)^3\cdots(1+a_n)^n > n^n.$$

Problem 3. The *liar's guessing game* is a game played between two players A and B. The rules of the game depend on two positive integers k and n which are known to both players.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***September 20, 2012***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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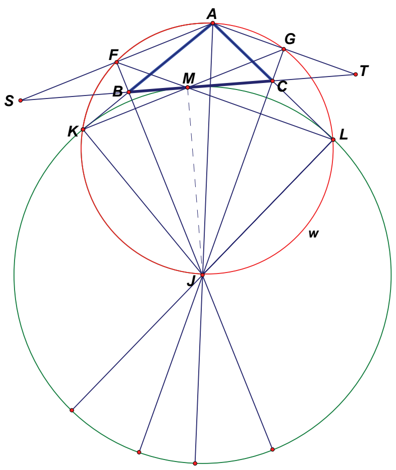
IMO 2012 (Leader Perspective)

Tat-Wing Leung

As leader, I arrived Mar del Plata, Argentina (the IMO 2012 site) four days earlier than the team. Despite cold weather, jet lag and delay of luggage, I managed to get myself involved in choosing the problems for the contest. Once the "easy" pair was selected, the jury did not have much choice but to choose problems of possibly other topics for the "medium" and the "difficult" pairs. The two papers of the contest were then set. We had to decide the various official versions and the marking scheme of the contest. After that, I just had to wait for the contestants to finish the contest and get myself involved in the coordination to decide the points obtained by our team. Here I would like to discuss the problems. (Please see Olympiad Corner for the statements of the problems.)



Problem 1. Really problem 1 is quite easy, merely a lot of angle chasings and many angles of 90° (tangents) and similar triangles, etc, and no extra lines or segments needed to be constructed. First note that $\angle AKJ = \angle ALJ = 90°$, hence A,K,L,J lie on the circle ω with diameter AJ. The idea is to show that F and G also lie on the same circle. Looking at angles around B, we see that $4\angle MBJ + 2\angle ABC = 360°$. Thus $\angle MBJ = 90° - ½\angle ABC$. Also, $\angle BMF = \angle CML = ½\angle ACB$ (as $CM=CL$). Then $\angle LFJ = \angle MBJ - \angle BMF = ½\angle BAC$. Thus $\angle LFJ = \angle LAJ$. Hence, F lies on ω. By the same token, so is G. Now AB and SB are symmetric with respect to the external bisector of $\angle ABC$, so is BK and BM. Now $SM = SB+BM = AB+BK = AK$. Similarly, $TM=AL$. So $SM=TM$.

It is relatively easy to tackle the problem using coordinate geometry. For instance, we can let the excircle be the unit circle with J=(0,0), M=(0,1), BC is aligned so that B=(b,1) and C=(c,1). Coordinates of other points are then calculated to verify the required property. But one must be really careful if he tries to use coordinate method. It was somehow decided that if a contestant cannot get a full solution using coordinate method, then he will be "seriously penalized"!

Problem 2. As it turned out, this problem caused quite a bit of trouble and many students didn't know how to tackle the problem at all. More sophisticated inequalities such as Muirhead do not work, since the expression is not "homogeneous". The Japanese leader called the problem a disaster. There were trivial questions such as "why is there no a_1?" A more subtle issue is how to isolate $a_2,a_3,\ldots,a_n$.

Clearly$(1+a_2)^2 \ge 2^2a_2$ by the AM-GM inequality. But how about $(1+a_3)^3$? Indeed the trick is to apply AM-GM inequality to get for k=2 to $n-1$,

$$(1+a_{k+1})^{k+1} = \left(\frac{1}{k}+\cdots+\frac{1}{k}+a_{k+1}\right)^{k+1}$$
$$\ge \left((k+1)\sqrt[k+1]{\frac{a_{k+1}}{k^k}}\right)^{k+1} = \frac{(k+1)^{k+1}a_{k+1}}{k^k}.$$

By multiplying the inequalities, the constants cancelled out and we get the final inequality. That the inequality is strict is trivial using the conditions of AM equals GM. The above inequality can also be used as the inductive step of proving the equivalent inequality

$$(1+a_2)^2(1+a_3)^3\cdots(1+a_n)^n > n^n a_2a_3\cdots a_n.$$

Problem 3. Comparing with problem 6, I really found this problem harder to approach! Nevertheless there were still 8 contestants who completely solved the problem. Among them three were from the US team. That was an amazing achievement!

We can deal with this combinatorial probabilistic problem as follows. Ask repeatedly if x is 2^k. If A answers *no* $k+1$ times in a row, then the answer is honest and $x\neq 2^k$. Otherwise B stops asking about 2^k at the first time answer *yes*. He then asks, for each $i=1,2,\ldots,k$, if the binary representation of x has a 0 in the i-th digit. Whatever the answer is, they are all inconsistent with a certain number y in the set $\{0,1,2,\cdots, 2^k-1\}$. The answer *yes* to 2^k is also inconsistent with y. Hence $x\neq y$. Otherwise the last $k+1$ answers are not honest and that is impossible. So we find y and it can be eliminated. Or we can eliminate corresponding numbers with nonzero digits at higher end. Notice we may need to do some re-indexing and asking more questions about the *indices* of the numbers subsequently. With these questions, we can reduce the size of the set that x lies until it lies in a set of size 2^k.

Part 2 makes use of a function so that using the function, A can devise a strategy (to lie or not to lie, but lying not more than k times consecutively) so that no extra information will be provided to B and hence B cannot eliminate anything for sure. Due to limit of space, I cannot provide all details here.

It was decided that part 1 answered correctly alone was worth 3 points and part 2 alone worth 5 points. But altogether a problem is worth at most 7 points. So 3 + 5 = 7! At the end it really did not matter. After all, not too many students did the problem right.

The problem is noted to be related to the Lovasz Local Lemma. See N. Alon et al, <u>*The Probabilistic Methods*</u>, Wiley, 1992. In the book it seems that there is an example that deals with similar things. One may check how the lemma and the problem are related!

Problem 4. Despite being regarded as an easy problem, this problem is not at all easy. It is much more involved than expected. Also this problem eventually caused more trouble because of the disputes about the marking scheme. First, by putting $a=b=c=0$, one gets $f(0)=0$. By putting $b=-a$ and $c=0$, one gets $f(a) = f(-a)$. More importantly, by putting $c=-(a+b)$ and solving $f(a+b) = f(-(a+b))$ as a quadratic equation of $f(a)$ and $f(b)$, one gets

$$f(a+b) = f(a) + f(b) \pm 2\sqrt{f(a)f(b)}.$$

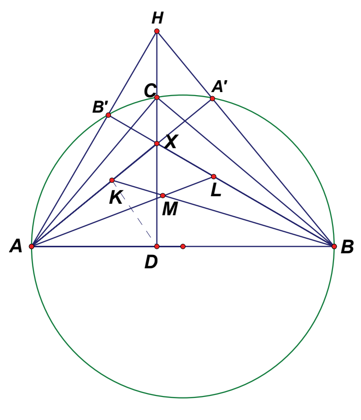
Putting $a=b$ and $c=-2a$ into the original equation, one gets $f(2a)=0$ or $f(2a)=4f(a)$. Now the problem becomes getting all possible solutions from these two relations. Using the two conditions, one checks that there are four types of solution:

(i) $f_1(x)\equiv 0$, (ii) $f_2(x) = kx^2$,

(iii) $f_3(x) = \begin{cases} 0, x \text{ even} \\ k, x \text{ odd} \end{cases}$ and

(iv) $f_4(x) = \begin{cases} 0, x\equiv 0 \pmod 4 \\ k, x\equiv \pm 1 \pmod 4 \\ 4k, x\equiv 2 \pmod 4 \end{cases}.$

The "k" in the solutions is essentially $f(1)$. Indeed if $f(1)=0$, then $f(2)=0$, one then show by induction $f(x)=0$ for all x. (Or by showing $f(x)$ is periodic of period 1.) Now if $f(1)=k$, using the condition $f(2a)=0$, one can show again by induction $f(x)$ is k for x odd and is 0 for x even. Now if $f(1)=k$ and $f(2)=4k$, then $f(4)=0$ or $16k$. In the first case we get a function with period 4 and arrive at the solution $f_4(x)$. In the second case we get $f_2(x)$. (One needs to verify the details.) By checking the values of a, b and c mod 2 or 4, or other possible forms, one can check the solutions are indeed valid. Eventually if a contestant claimed that all the solutions are easy to check, but without checking, one point would be deducted. If a contestant says nothing about the solutions satisfy the functional equation and check nothing, then two points would be deducted!



Problem 5. The following solution was obtained by one of our team members.

Extend AX to meet the circumcircle of ABC at A', likewise extend BX to meet the circle at B'. Now extend AB' and BA' to meet at H, which is exactly the orthocentre of ABX and it lies on the extension of DC.

Since $BK^2 = BC^2 = BD\cdot BA$, we have $\Delta ABK \sim \Delta KBD$, so $\angle BKD = \angle BAK = \angle BHD$, which implies B, D, K, H concyclic. So $\angle BKH = \angle BDH = 90°$. This implies $HK^2 = BH^2 - BK^2 = BH^2 - BD\cdot BA = BH^2 - BA'\cdot BH = HA'\cdot HB$. Similarly $HL^2 = HB'\cdot HA$. But $HA'\cdot HB = HB'\cdot HA$. Hence $HK=HL$. Using similar arguments as above, we have $\angle ALH = 90°$ ($=\angle BKH$.) Along with $HK=HL$, we see $\Delta MKH \cong \Delta MLH$. Therefore, $MK=ML$.

Problem 6. Clearing denominators of

$$\frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + \ldots + \frac{n}{3^{a_n}} = 1,$$

one gets $x_1+2x_2+\cdots+nx_n=3^a$, where $x_1,x_2,\ldots,x_n$ are non-negative integer powers of 3. Taking mod 2, one gets $n(n+1)/2 \equiv 1 \pmod 2$. This is the case only when $n \equiv 1, 2 \pmod 4$. The hard part is to prove the converse also holds. The cases $n=1$ or 2 are easy. By trials, for $n=5$, $(a_1,\ldots,a_5)=(2,2,2,3,3)$ works. The official solution gave a systematic analysis of how to obtain solutions by using identities $1/2^a=1/2^{a+1}+1/2^{a+1}$ and $w/3^a=u/3^{a+1}+v/3^{a+1}$, where $u+v=3w$. For $n=4k+1\geq 5$, one can arrive at the solution $a_1=2=a_3$, $a_2=k+1$, $a_{4k} = k+2 = a_{4k+1}$ and $a_m = [m/4]+3$ for $4\leq m<4k$. Similarly, for $n=4k+2\geq 6$, one can arrive at the solution $a_1 = 2$, $a_2 = k+1$, $a_3 = a_4 = 3$, $a_{4k+1} = k+2 = a_{4k+2}$ and $a_m = [(m-1)/4]+3$ for $4<m\leq 4k$. One can check these are indeed solutions by math induction on k. In the inductive steps of both cases, just notice a_2, a_{n-1}, a_n are increased by 1 so to balance the new a_{n+1}, a_{n+2}, a_{n+3}, a_{n+4} terms.

This reminds me of the 1978 *USAMO* problem: an integer n is called <u>*good*</u> if we can write $n=a_1+a_2+\cdots+a_k$, where $a_1,a_2,\ldots,a_k$ are positive integers (not necessarily distinct) satisfying

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k} = 1.$$

Given 33 to 73 are good, prove that all integer greater than 33 are good. The idea there is to show if n is good, then $2n+8$ and $2n+9$ are good by dividing both sides of the above equation by 2 and adding the terms 1/4+1/4 and 1/3+1/6 respectively.

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***September 20, 2012.***

Problem 396. Determine (with proof) all functions $f:\mathbb{R}\to\mathbb{R}$ such that for all real numbers x and y, we have

$$f(x^2+xy+f(y))=(f(x))^2+xf(y)+y.$$

Problem 397. Suppose in some set of 133 distinct positive integers, there are at least 799 pairs of relatively prime integers. Prove that there exist a,b,c,d in the set such that $\gcd(a,b)=\gcd(b,c)=\gcd(c,d)=\gcd(d,a)=1$.

Problem 398. Let k be positive integer and m an odd integer. Show that there exists a positive integer n for which the number n^n-m is divisible by 2^k.

Problem 399. Let ABC be a triangle for which $\angle BAC=60°$. Let P be the point of intersection of the bisector of $\angle ABC$ and the side AC. Let Q be the point of intersection of the bisector of $\angle ACB$ and the side AB. Let r_1 and r_2 be the radii of the incircles of triangles ABC and APQ respectively. Determine the radius of the circumcircle of triangle APQ in terms of r_1 and r_2 with proof.

Problem 400. Determine (with proof) all the polynomials $P(x)$ with real coefficients such that for every rational number r, the equation $P(x)=r$ has a rational solution.

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Solutions

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Problem 391. Let $S(x)$ denote the sum of the digits of the positive integer x in base 10. Determine whether there exist distinct positive integers a, b, c such that $S(a+b)<5$, $S(b+c)<5$, $S(c+a)<5$, but $S(a+b+c)>50$ or not.

Solution. **AN-anduud Problem Solving Group** (Ulaanbaatar, Mongolia), **CHEUNG Ka Wai** (Munsang College (Hong Kong Island)), **LI Jianhui** (CNEC Christian College, F.5), **LO Shing Fung** (Carmel Alison Lam Foundation Secondary School), **Andy LOO** (St. Paul's Co-educational College), **YUEN Wai Kiu** (St. Francis' Canossian College) and **ZOLBAYAR Shagdar** (9th grader, Orchlon International School, Ulaanbaatar, Mongolia).

Yes, we can try a=5,555,554,445 and b=5,554,445,555 and c=4,445,555,555. Then

$S(a+b)=S(11{,}110{,}000{,}000)=4$,
$S(b+c)=S(10{,}000{,}001{,}110)=4$,
$S(c+a)=S(10{,}001{,}110{,}000)=4$.

Finally,
$S(a+b+c)=S(15{,}555{,}555{,}555)=51$.

Other commended solvers: **Alice WONG** (Diocesan Girls' School), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Problem 392. Integers $a_0, a_1, \cdots, a_n$ are all greater than or equal to -1 and are not all zeros. If

$$a_0+2a_1+2^2a_2+\cdots+2^na_n=0,$$

then prove that $a_0+a_1+a_2+\cdots+a_n>0$.

Solution. **AN-anduud Problem Solving Group** (Ulaanbaatar, Mongolia), **Kevin LAU** (St. Paul's Co-educational College, S.3), **Simon LEE** (Carmel Alison Lam Foundation Secondary School), **Harry NG Ho Man** (La Salle College, Form 5), **SHUM Tsz Hin** (City University of Hong Kong), **Alice WONG** (Diocesan Girls' School), **ZOLBAYAR Shagdar** (9th grader, Orchlon International School, Ulaanbaatar, Mongolia),**Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

For all the conditions to hold, $n\neq 0$. We will prove by mathematical induction. For n=1, if $a_0+2a_1=0$, then the conditions on a_0 and a_1 imply a_0 is an even positive integer. So $a_0+a_1=a_0/2>0$. Suppose the case $n=k$ is true. For the case $n=k+1$, the given equation implies a_0 is even, hence $a_0\geq 0$. So $a_0=2b$, with b a nonnegative integer. Then dividing the equation by 2 on both sides, we get that $(b+a_1)+2a_2+\cdots+2^ka_{k+1}=0$. From the cases $n=k$ and n=1 (in cases $a_2=\cdots=a_{k+1}=0$), we get $a_0+a_1+a_2+\cdots+a_n\geq(b+a_1)+a_2+\cdots+a_n>0$, ending the induction.

Problem 393. Let p be a prime number and $p\equiv 1 \pmod 4$. Prove that there exist integers x and y such that

$$x^2-py^2=-1.$$

Solution. **AN-anduud Problem Solving Group** (Ulaanbaatar, Mongolia), **Kevin LAU** (St. Paul's Co-educational College, S.3), **Simon LEE** (Carmel Alison Lam Foundation Secondary School), **Andy LOO** (St. Paul's Co-educational College), **Corneliu MĂNESCU-AVRAM** (Dept of Math, Transportation High School, Ploiesti, Romania), **Alice WONG** (Diocesan Girls' School) and **ZOLBAYAR Shagdar** (9th grader, Orchlon International School, Ulaanbaatar, Mongolia).

Let (m,n) be the fundamental solution (i.e. the least positive integer solution) of the Pell's equation $x^2-py^2=1$ (see <u>*Math Excal*</u>., *vol*. 6, *no*. 3, *p*.1). Then

$$m^2-n^2\equiv m^2-pn^2=1 \pmod 4.$$

Then m is odd and n is even. Since

$$\frac{m-1}{2}\cdot\frac{m+1}{2}=p\left(\frac{n}{2}\right)^2$$

and $(m-1)/2$, $(m+1)/2$ are consecutive integers (hence relatively prime), either

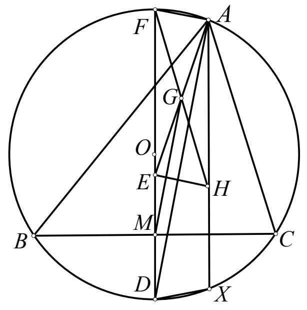
$$\frac{m-1}{2}=pu^2,\ \frac{m+1}{2}=v^2,\ n=2uv$$

or $$\frac{m-1}{2}=u^2,\ \frac{m+1}{2}=pv^2,\ n=2uv$$

for some positive integers u and v. In the former case, $v^2-pu^2=1$ with $0<v\leq v^2=(m+1)/2<m$ and $0<u=n/(2v)<n$. This contradicts the minimality of (m,n). So the latter case must hold, i.e. $u^2-pv^2=-1$.

Problem 394. Let O and H be the circumcenter and orthocenter of acute $\triangle ABC$. The bisector of $\angle BAC$ meets the circumcircle Γ of $\triangle ABC$ at D. Let E be the mirror image of D with respect to line BC. Let F be on Γ such that DF is a diameter. Let lines AE and FH meet at G. Let M be the midpoint of side BC. Prove that $GM\perp AF$.

Solution **1. AN-anduud Problem Solving Group** (Ulaanbaatar, Mongolia), **Kevin LAU** (St. Paul's Co-educational College, S.3), **MANOLOUDIS Apostolos** (4° Lyk. Korydallos, Piraeus, Greece), **Mihai STOENESCU** (Bischwiller, France), **ZOLBAYAR Shagdar** (9th grader, Orchlon International School, Ulaanbaatar, Mongolia), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).



As AD bisects $\angle BAC$, D is the midpoint of arc BC. Hence, FD is the perpendicular bisector of BC. Thus, (1) $FE \parallel AH$. Let line AH meet Γ again at X. Since

$\angle BCX = \angle BAX = 90^\circ - \angle ABC = \angle BCH$,

H is the mirror image of X with respect to BC. Therefore, $\angle HED = \angle XDE = \angle AFE$. Thus, (2) $AF \parallel HE$. By (1) and (2), $AFEH$ is a parallelogram. Hence, G is the midpoint of AE. As M is also the midpoint of DE, we get $GM \parallel AD$. Since DF is the diameter of Γ, $AD \perp AF$, hence $GM \perp AF$.

***Solution* 2. Andy LOO** (St. Paul's Co-educational College).

Place the figure on the complex plane and let the circumcircle of $\triangle ABC$ be the unit circle centered at the origin. Denote the complex number representing each point by the respective lower-case letter. Without loss of generality we may assume $a = 1$ and that the points A, B and C lie on the circle in anticlockwise order. Let $b = u^2$ and $c = v^2$, where $|u| = |v| = 1$. Then $d = uv$ and hence $f = -uv$. Next, E is the mirror image of D with respect to BC means

$$\frac{e-b}{c-b} = \overline{\left(\frac{d-b}{c-b}\right)},$$

giving $e = u^2 - uv + v^2$. By the Euler line theorem, $h = a+b+c = 1+u^2+v^2$. Now G on lines AE and FH means

$$\frac{g-a}{e-a} = \frac{\overline{g}-\overline{a}}{\overline{e}-\overline{a}} \text{ and } \frac{g-f}{h-f} = \frac{\overline{g}-\overline{f}}{\overline{h}-\overline{f}}.$$

Solving these simultaneously for G, we get $g = (u^2-uv+v^2+1)/2$. Also, $m = (b+c)/2 = (u^2+v^2)/2$.

To show $GM \perp AF$, it suffices to prove that $(m-g)/(f-a)$ is an imaginary number. Indeed, $\frac{m-g}{f-a} = \frac{1}{2}\cdot\frac{1-uv}{1+uv}$ and

$$\overline{\left(\frac{m-g}{f-a}\right)} = \frac{1}{2}\cdot\frac{1-\frac{1}{u}\cdot\frac{1}{v}}{1+\frac{1}{u}\cdot\frac{1}{v}} = \frac{1}{2}\cdot\frac{uv-1}{uv+1} = -\frac{m-g}{f-a}$$

as desired.

Other commended solvers: **Simon LEE** (Carmel Alison Lam Foundation Secondary School), and **Alice WONG** (Diocesan Girls' School).

Problem 395. One frog is placed on every vertex of a $2n$-sided regular polygon, where n is an integer at least 2. At a particular moment, each frog will jump to one of the two neighboring vertices (with more than one frog at a vertex allowed).

Find all n such that there exists a jumping of these frogs so that after the moment, all lines connecting two frogs at different vertices do not pass through the center of the polygon.

Solution. **Kevin LAU** (St. Paul's Co-educational College, S.3), **Simon LEE** (Carmel Alison Lam Foundation Secondary School), **LI Jianhui** (CNEC Christian College, F.5) and **Andy LOO** (St. Paul's Co-educational College).

If $n \equiv 2 \pmod 4$, say $n=4k+2$, then label the $2n=8k+4$ vertices from 1 to $8k+4$ in clockwise direction. For $j \equiv 1$ or $2 \pmod 4$, let the frog at vertex j jump in the clockwise direction. For $j \equiv 3$ or $4 \pmod 4$, let the frog at vertex j jump in the counter-clockwise direction. After the jump, the frogs are at vertices 2, 6, …, $8k+2$ and 3,7, …, $8k+3$. No two of these vertex numbers have a difference of the form $2 \pmod 4$. So no line through two different vertices with frogs will go through the center.

If $n \not\equiv 2 \pmod 4$, then assume there is such a jump. We may exclude the cases all frogs jump clockwise or all frogs jump counter-clockwise, which clearly do not work. Hence, in this jump, there is a frog, say at vertex i, jumps in the counter-clockwise direction, then the frog at vertex $i+m(n-2) \pmod{2n}$ must jump in the same direction as the frog at vertex i for $m=1,2,\ldots$.

If n is odd, then $\gcd(n-2,2n) = 1$. So there are integers a and b such that $a(n-2) + b(2n) = 1$. For every integer q in $[1,2n]$, letting $m = (q-i)a$, we have $i+m(n-2) \equiv q \pmod{2n}$. This means all frogs jump in the counter-clockwise direction, which does not work.

If n is divisible by 4, then $\gcd(n-2,2n) = 2$. So there are integers c and d such that $c(n-2)+d(2n)=2$. Letting $m=nc/2$, we have $i+m(n-2) \equiv i+n \pmod{2n}$. Then frogs at vertices i and $i+n$ jump in the counter-clockwise direction and the line after the jump passes through the center, contradiction.

Therefore, the answer is $n \equiv 2 \pmod 4$.

Other commended solvers: **Alice WONG** (Diocesan Girls' School).

Olympiad Corner

(continued from page 1)

Problem 3. (Cont.) At the start of the game A chooses integers x and N with $1 \le x \le N$. Player A keeps x secret, and truthfully tells N to B. Player B now tries to obtain information about x by asking player A questions as follows: each question consists of B specifying an arbitrary set S of positive integers (possibly one specified in some previous question), and asking A whether x belongs to S. Player B may ask as many such questions as he wishes. After each question, player A must immediately answer it with *yes* or *no*, but is allowed to lie as many times as she wants; the only restriction is that, among any $k+1$ consecutive answers, at least one answer must be truthful.

After B has asked as many questions as he wants, he must specify a set X of at most n positive integers. If x belongs to X, then B wins; otherwise, he loses. Prove that:

1. If $n \ge 2^k$, then B can guarantee a win.
2. For all sufficiently large k, there exists an integer $n \ge 1.99^k$ such that B cannot guarantee a win.

Problem 4. Find all functions $f\colon Z \to Z$ such that, for all integers a, b, c that satisfy $a+b+c=0$, the following equality holds:

$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a).$$

(Here Z denotes the set of integers.)

Problem 5. Let ABC be a triangle with $\angle BCA = 90°$, and let D be the foot of the altitude from C. Let X be a point in the interior of the segment CD. Let K be the point on the segment AX such that $BK=BC$. Similarly, let L be the point on the segment BX such that $AL=AC$. Let M be the point of intersection of AL and BK. Show that $MK=ML$.

Problem 6. Find all positive integers n for which there exist non-negative integers $a_1, a_2, \ldots, a_n$ such that

$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \ldots + \frac{1}{2^{a_n}} = \frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + \ldots + \frac{n}{3^{a_n}} = 1.$$