Mathematical Excalibur

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Olympiad Corner

Below are the problems of the Final Selection Test for the 2012 Croatian IMO Team.

Problem 1. Determine all functions f: $\mathbb{R}\to\mathbb{R}$ such that for all real numbers x and y holds

$$f(x^2+f(y)) = (f(x)+y^2)^2.$$

(*Tonći Kokan*)

Problem 2. Along the coast of an island there are 20 villages. Each village has 20 fighters. Every fighter fights all the fighters from all the other villages. No two fighters have equal strength and the stronger fighter wins the fight.

We say that the village A is *stronger* than the village B if in at least k fights among the fighters from A and B a fighter from the village A wins. It turned out that every village is stronger than its neighbour (in the clockwise direction).

Show that the maximal possible k is 290.

(*Moscow Olympiad 2003, modified*)

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***May 10, 2013***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

Dr. Kin-Yin LI, Math Dept., Hong Kong Univ. of Science and Technology, Clear Water Bay, Kowloon, Hong Kong

Fax: (852) 2358 1643
Email: makyli@ust.hk

The Inequality of A. Oppenheim

Prof. Marcel Chirita, Bucharest, Romania

In this note we establish conditions solving the problems of A. Oppenheim and O. Bothema, then we solve some problems. Below, we let a,b,c,S,s,R,r denote the sides BC, CA, AB, area, semiperimeter, circumradius, inradius of a triangle ABC respectively. In [1], two problems are stated as follow:

Problem 1. (*O. Bothema*) For $\triangle ABC$, give conditions on real numbers x,y,z so

$$yza^2+zxb^2+xyc^2 \le R^2(x+y+z)^2 \quad (1)$$

with equality if and only if

$$\frac{x}{\sin 2A}=\frac{y}{\sin 2B}=\frac{z}{\sin 2C}. \quad (2)$$

Problem 2. (*A.Oppenheim*) For $\triangle ABC$, give conditions on real numbers x,y,z so

$$xa^2+yb^2+zc^2 \ge 4S\sqrt{xy+yz+zx} \quad (3)$$

with equality if and only if

$$\frac{x}{-a^2+b^2+c^2}=\frac{y}{a^2-b^2+c^2}=\frac{z}{a^2+b^2-c^2}. \quad (4)$$

The author will solve problem 2, then use it to solve problem 1. It is easy to see these problems are false for some x,y,z. For example, if one of x,y,z is negative, problems 1 and 2 may be false.

Theorem. For $\triangle ABC$, if $x+y>0$, $y+z>0$, $z+x>0$ and $xy+yz+zx>0$, then (3) and (4) hold.

Proof. Let $k=4\sqrt{xy+yz+zx}$. Using $c^2=a^2+b^2-2ab\cos C$ and $S=\frac{1}{2}ab\sin C$, we can rewrite (3) as

$$2(x+z)a^2+2(y+z)b^2 \ge (4z\cos C+k\sin C)ab.$$

By the *AM-GM* inequality, the left side is greater than or equal to

$$4\sqrt{(x+z)(y+z)}ab=\sqrt{16z^2+k^2}ab,$$

which is greater than or equal to the right side by the Cauchy-Schwarz inequality. So (3) is true. Equality holds (from *AM-GM* and Cauchy-Schwarz) if and only if

$$\frac{a^2}{y+z}=\frac{b^2}{z+x}=\frac{c^2}{x+y}.$$

Let t be this ratio. Then $a^2=t(y+z)$, $b^2=t(z+x)$, $c^2=t(x+y)$. So $-a^2+b^2+c^2=2tx$, $a^2-b^2+c^2=2ty$ and $a^2+b^2-c^2=2tz$. This gives (4) and steps can be reversed. Using the cosine law, we can see (4) is equivalent to

$$\frac{xa}{\cos A}=\frac{yb}{\cos B}=\frac{zc}{\cos C}.$$

From (4), we see x,y,z can be all positive or one negative and two positive.

To solve problem 1, in place of x,y,z, we use x/a^2, y/b^2, z/c^2, which also satisfy the conditions of the theorem. Then (3) is

$$x+y+z \ge 4S\sqrt{\frac{xy}{a^2b^2}+\frac{yz}{b^2c^2}+\frac{zx}{c^2a^2}}.$$

Using the formula $S=abc/(4R)$ (which is from $S=\frac{1}{2}ab\sin C$ and $c/(\sin C)=2R$), the last inequality becomes

$$x+y+z \ge \frac{1}{R}\sqrt{xyc^2+yza^2+zxb^2},$$

which is equivalent to (1). For equality case, observe that using the cosine law and $a/(\sin A)=2R$,

$$\frac{x/a^2}{-a^2+b^2+c^2}=\frac{x}{2a^2bc\cos A}=t\frac{x}{\sin 2A},$$

where $t=1/(2Rabc)$. This gives (2).

Next we give many applications of these inequalities.

Example 1 If we take $x=y=z$ in (3), then we get $a^2+b^2+c^2 \ge 4S\sqrt{3}$, which dated back to Ionescu (1897), later to Weitzenböck (1919) and Carlitz (1961).

Example 2 If we take $x=a^2$, $y=b^2$ and $z=c^2$ in (3), then we get

$$a^4+b^4+c^4 \ge 4S\sqrt{a^2b^2+b^2c^2+c^2a^2}.$$

Since Heron's formula gives

$$2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)=16S^2$$

(continued on page 2)

and expanding $(a^2-b^2)^2 + (b^2-c^2)^2 + (c^2-a^2)^2 \geq 0$ leads to

$$a^2b^2+b^2c^2+c^2a^2 \leq a^4+b^4+c^4,$$

it follows immediately that

$$a^2b^2+b^2c^2+c^2a^2 \geq 16S^2$$

and hence $a^4+b^4+c^4 \geq 16S^2$.

Example 3 (*a*) If $x = 9$, $y = 5$ and $z = -3$ in (3), then we get $9a^2+5b^2-3c^2 \geq 4S\sqrt{3}$.

(b) If x=27, y=27 and z=−13 in (3), then we get $27a^2+27b^2-13c^2 \geq 12S\sqrt{3}$.

(c) If x=3, y=−1 and z=15 in (3), then we get $3a^2-b^2+15c^2 \geq 12S\sqrt{3}$.

These were exercises proposed in [6] and [9].

Example 4 If we consider $x=bc/a$, $y=ca/b$ and $z=ab/c$ in (3), then we have

$$3abc \geq 4\sqrt{a^2+b^2+c^2}\,S.$$

Taking into account that $4RS=abc$ and $ab+bc+ca \leq a^2+b^2+c^2$, we have $ab+bc+ca \leq a^2+b^2+c^2 \leq 9R^2$.

Example 5 If we consider $x=bc$, $y=ca$ and $z=ab$ in (3), then we have

$$abc(a+b+c) \geq 4S\sqrt{abc(a+b+c)},$$

which implies $abc(a+b+c) \geq 16S^2$. Using $S = ½(a+b+c)r = sr$, we get $abc \geq 8sr^2$. Using abc=$4RS$=$4Rsr$, we have $R \geq 2r$.

Example 6 Let $x > 0$. If we consider $2x-1, \dfrac{2}{x}-1$ and 1, then we can easily check that they satisfy the conditions in the theorem. So (3) yields

$$(2x-1)a^2+\left(\frac{2}{x}-1\right)b^2+c^2 \geq 4S\sqrt{3}.$$

This was a proposed exercise of B. Suceavă in [9].

Example 7 If we consider

$$x=\frac{s-a}{a^2}, y=\frac{s-b}{b^2}, z=\frac{s-c}{c^2}$$

in (3), then we get

$$s \geq 4S\sqrt{\frac{(s-a)(s-b)}{a^2b^2}+\frac{(s-b)(s-c)}{b^2c^2}+\frac{(s-c)(s-a)}{c^2a^2}}.$$

Squaring both sides and applying the *AM-GM* inequality on the right side, we get

$$s^2 \geq 48S^2\sqrt[3]{\frac{(s-a)^2(s-b)^2(s-c)^2}{a^4b^4c^4}},$$

which is equivalent to

$$a^4b^4c^4s^6 \geq 48^3S^6(s-a)^2(s-b)^2(s-c)^2.$$

Using abc=$4RS$=$4Rsr$ on the left and Heron's formula on the right, we can simplify this to $sR^2 \geq 12\sqrt{3}r^3$.

Example 8 Consider

$$x=\frac{s-a}{a}, y=\frac{s-b}{b}, z=\frac{s-c}{c}.$$

Then

$$xa^2+yb^2+zc^2 = a(s-a)+b(s-b)+c(s-c)$$

$$=\frac{2ab+2bc+2ca-(a^2+b^2+c^2)}{2}.$$

From [3], we have $ab+bc+ca$=s^2+r^2+4Rr and $a^2+b^2+c^2 = 2(s^2-r^2-4Rr)$. Putting these into the above equation, we get

$$xa^2+yb^2+zc^2 = 2r^2+8Rr.$$

Recall by cosine law

$$\frac{(s-a)(s-b)}{ab}=\frac{c^2-a^2-b^2+2ab}{4ab}$$

$$=\frac{1-\cos C}{2}$$

$$=\sin^2\frac{C}{2}.$$

Using this and similar equations, we have

$$4S\sqrt{xy+yz+zx}$$

$$=4S\sqrt{\frac{(s-a)(s-b)}{ab}+\frac{(s-b)(s-c)}{bc}+\frac{(s-c)(s-a)}{ca}}$$

$$=4S\sqrt{\sin^2\frac{C}{2}+\sin^2\frac{A}{2}+\sin^2\frac{B}{2}}$$

$$\geq 4S\sqrt{\frac{3}{4}}=2S\sqrt{3},$$

where the last inequality follows by applying Jensen's inequality to $f(x) = \sin^2(x/2)$ on $[0,\pi/2]$. Thus, (3) yields

$$r^2+4Rr \geq S\sqrt{3}.$$

Example 9 If instead of x, y, z, we replace them by $\dfrac{yz}{a^2}, \dfrac{xy}{b^2}, \dfrac{zx}{c^2}$ in (3), then we get after calculations that

$$xa^2+yb^2+zc^2 \leq R^2\frac{(xy+yz+zx)^2}{xyz}.$$

Example 10 If instead of x, y, z, we consider yz, zx, xy, then (1) and (3) yield the following inequality

$$4S\sqrt{xyz(x+y+z)} \leq a^2yz+b^2zx+c^2xy$$
$$\leq (x+y+z)^2R^2,$$

which is the subject of the article "On an inequality in a triangle" from GM 8 in 1984 by Prof. Virgil Nicula.

Example 11 If instead of x,y,z, we consider

$$\frac{p}{q+r},\ \frac{q}{r+p},\ \frac{r}{p+q},$$

where $p,q,r > 0$, then (3) yields

$$\frac{p}{q+r}a^2+\frac{q}{r+p}b^2+\frac{r}{p+q}c^2 \geq 2S\sqrt{3}.$$

This is problem *E3150* proposed by G. Tsintsifas in the American Math. Monthly in 1988.

Example 12 If instead of x,y,z, we consider

$$\frac{b}{a}m,\ \frac{c}{b}n,\ \frac{a}{c}p,$$

where $m, n, p > 0$, then (3) yields

$$mab+nbc+pca \geq 4S\sqrt{\frac{c}{a}mn+\frac{a}{b}np+\frac{b}{c}mp}.$$

By the AM-GM inequality, we have

$$\frac{c}{a}mn+\frac{a}{b}np+\frac{b}{c}mp \geq 3\sqrt[3]{m^2n^2p^2}.$$

Combining the last two inequalities, we get

$$mab+nbc+pca \geq 4S\sqrt{3\sqrt[3]{m^2n^2p^2}}.$$

If we take $m = n = p = 1$, then we get

$$ab+bc+ca \geq 4S\sqrt{3},$$

which is due to V. E. Olhov, see [7] and [8] in the bibliography on page 4.

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***May 10, 2013.***

Problem 416. If $x_1 = y_1 = 1$ and for $n > 1$,

$$x_n = -3x_{n-1} - 4y_{n-1} + n$$

and $$y_n = x_{n-1} + y_{n-1} - 2,$$

then find x_n and y_n in terms of n only.

Problem 417. Prove that there does not exist a sequence $p_0, p_1, p_2, \ldots$ of prime numbers such that for all positive integer k, p_k is either $2p_{k-1}+1$ or $2p_{k-1}-1$.

Problem 418. Point M is the midpoint of side AB of acute $\triangle ABC$. Points P and Q are the feet of perpendicular from A to side BC and from B to side AC respectively. Line AC is tangent to the circumcircle of $\triangle BMP$. Prove that line BC is tangent to the circumcircle of $\triangle AMQ$.

Problem 419. Let $n \ge 4$. M is a subset of $\{1,2,\ldots,2n-1\}$ with n elements. Prove that M has a nonempty subset, the sum of all its elements is divisible by $2n$.

Problem 420. Find (with proof) all positive integers x and y such that $2x^2y+xy^2+8x$ is divisible by xy^2+2y.

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Solutions

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Problem 411. A and B play a game on a square board divided into 100×100 squares. Each of A and B has a checker. Initially A's checker is in the lower left corner square and B's checker is in the lower right corner square. They take turn to make moves. The rule is that each of them has to move his checker one square up, down, left or right within the board and A goes first. Prove that no matter how B plays, A can always move his checker to meet B's checker eventually.

Solution. **Jon GLIMMS** (Vancouver, Canada) and **ZOLBAYAR Shagdar** (Orchlon International School, Ulaanbaatar, Mongolia).

Suppose the squares are unit length. A can apply the following strategy. After B made the n-th move, let $R(n)$ denote the rectangle bounded by the squares in the same row or same column as one of the two squares containing the checkers. Let $a(n)$ be the length (i.e. long side) and $b(n)$ be the width (i.e. short side) of $R(n)$. As $R(0)$ is consisted of the lowest row squares, $a(0)=100$ and $b(0)=1$. Following the rules, A can always make a move to decrease the length of $R(n)$. After B made $n+1$ moves, $a(n+1)+b(n+1)$ will either be $a(n)+b(n)$ or $a(n)+b(n)-2$. In particular, $a(n)+b(n)$ is always odd, non-increasing and $a(n) > b(n)$. Since the side of the board is finite, eventually $a(n)+b(n)$ must decrease to 3 and A can move his checker to meet B's checker in the next move.

Other commended solvers: **CHEUNG Ka Wai** (Munsang College (Hong Kong Island)) and **F5D** (Carmel Alison Lam Foundation Secondary School).

Problem 412. $\triangle ABC$ is equilateral and points D, E, F are on sides BC, CA, AB respectively. If

$$\angle BAD + \angle CBE + \angle ACF = 120^\circ,$$

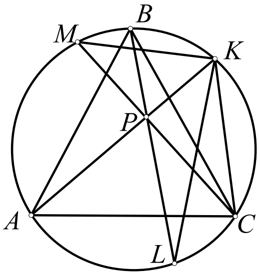
then prove that $\triangle BAD$, $\triangle CBE$ and $\triangle ACF$ cover $\triangle ABC$.
(*Source*: 2006 *Indian Math Olympiad Team Selection Test*)

Solution. **Jon GLIMMS** (Vancouver, Canada) and **William PENG**.

Assume P is in $\triangle ABC$ not covered by $\triangle BAD$, $\triangle CBE$ and $\triangle ACF$. Then $\angle BAD < \angle BAP$, $\angle CBE < \angle CBP$ and $\angle ACF < \angle ACP$. Adding these, we have

$$120^\circ < \angle BAP + \angle CBP + \angle ACP.$$

Now P cannot be the circumcenter of $\triangle ABC$ (otherwise $\angle BAP + \angle CBP + \angle ACP = 90^\circ$ would contradict the inequality above). So PA, PB, PC are not all equal. Suppose $PA > PB$. Let rays AP, BP, CP intersect the circumcircle of $\triangle ABC$ at points K, L, M respectively.



Since $\angle BAP = \angle KLP$ and $\angle ABP = \angle LKP$, $\triangle ABP$ and $\triangle LKP$ are similar. Then $PA > PB$ implies $PL > PK$ and so $\angle BAP = \angle KLP < \angle LKP$. We get

$$\begin{aligned}&\angle BAP + \angle CBP + \angle ACP\\ &= \angle KLP + \angle CKL + \angle AKM\\ &< \angle LKP + \angle CKL + \angle AKM\\ &< \angle BKC = 120^\circ,\end{aligned}$$

which contradicts the inequality above.

Other commended solvers: **KWAN Chung Hang** (Sir Ellis Kadoorie Secondary School (West Kowloon)) and **Cyril LETROUIT** (Lycée Jean-Baptiste Say, Paris, France).

Problem 413. Determine (with proof) all integers $n\ge3$ such that there exists a positive integer M_n satisfying the condition for all n positive numbers $a_1, a_2, \ldots, a_n$, we have

$$\frac{a_1+a_2+\cdots+a_n}{\sqrt[n]{a_1a_2\cdots a_n}} \le M_n\left(\frac{a_2}{a_1}+\frac{a_3}{a_2}+\cdots+\frac{a_n}{a_{n-1}}+\frac{a_1}{a_n}\right).$$

(*Source*: 2005 *Chinese Taipei Math Olympiad Team Selection Test*)

Solution. **F5D** (Carmel Alison Lam Foundation Secondary School) and **Paolo PERFETTI** (Math Dept, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy).

For $n=3$, let $a_1, a_2, a_3 > 0$ and

$$x = \frac{a_2}{a_1} + \frac{a_3}{a_2} + \frac{a_1}{a_3}.$$

Suppose $a_3 \ge a_1, a_2$. Then $x > a_2/a_1, a_3/a_2, a_1/a_3$. So $a_2 > a_3/x$ and $a_1 > a_2/x > a_3/x^2$. Hence,

$$\frac{a_1+a_2+a_3}{\sqrt[3]{a_1a_2a_3}} \le \frac{3a_3}{\sqrt[3]{\frac{a_3}{x^2}\frac{a_3}{x}a_3}} = 3x.$$

So we can take $M_3=3$. For $n>3$, assume there is such M_n. Let $a_1 = c$, $a_2 = c^2, \ldots, a_n = c^n$. Then

$$M_n \ge \frac{c+c^2+\cdots+c^n}{\sqrt[n]{c^{n(n+1)/2}}}\left((n-1)c+\frac{1}{c^{n-1}}\right)^{-1}$$

$$\ge \frac{c^n}{c^{(n+1)/2}}\frac{1}{c((n-1)+c^{-n})} = \frac{c^{(n-3)/2}}{n-1+c^{-n}}.$$

As $c\to\infty$, $c^{(n-3)/2}/(n-1+c^{-n})\to\infty$. Then M_n cannot be finite, contradiction.

Problem 414. Let p be an odd prime number and $a_1, a_2, \ldots, a_{p-1}$ be positive integers not divisible by p. Prove that there exist integers $b_1, b_2, \ldots, b_{p-1}$, each equals 1 or −1 such that

$$a_1b_1+a_2b_2+\cdots+a_{p-1}b_{p-1}$$

is divisible by p.

Solution. **Jon GLIMMS** (Vancouver, Canada).

For $k = 1, 2, \ldots, p-1$, we will prove the numbers of the form $a_1c_1 + a_2c_2 + \cdots + a_kc_k$ (where each c_i is 0 or 1) when divided by p will yield at least $k + 1$ different remainders. For $k = 1$, we are given that $a_1 \not\equiv 0 \pmod p$.

Suppose a case $k < p-1$ is true. For the case k+1, if the numbers $a_1c_1+a_2c_2+\cdots+a_kc_k$ when divided by p yield at least k+2 different remainders, then the case k+1 is also true. Otherwise, there are numbers $m_1, m_2, \ldots, m_{k+1}$ of the form $a_1c_1+a_2c_2+\cdots+a_kc_k$ when divided by p yield exactly k+1 different remainders. Considering (mod p), we see m_1+a_{k+1}, $m_2+a_{k+1}, \ldots, m_{k+1}+a_{k+1}$ also have k+1 different remainders.

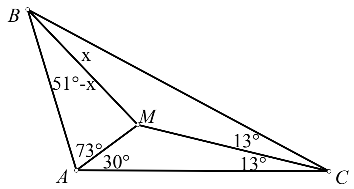
Assume these two groups of k+1 remainders are the same. Then we get $m_1+m_2+\cdots+m_{k+1} \equiv (m_1+a_{k+1}) + (m_2+a_{k+1}) + \cdots +(m_{k+1}+a_{k+1}) \pmod p$. This implies $(k+1)a_{k+1}\equiv 0 \pmod p$, which is not possible as $k+1<p$ and a_{k+1} is not divisible by p. Hence, there must be at least k+2 different remainders among the two groups. So the case k+1 is true.

Let $S=a_1+a_2+\cdots+a_{p-1}$. Since $\gcd(2,p) = 1$, there is an integer r such that $2r \equiv S \pmod p$. From the case $k = p - 1$ above, we see there is $a_1c_1 + a_2c_2 + \cdots + a_{p-1}c_{p-1} \equiv r \pmod p$. Let $b_i = 1 - 2c_i$, then $b_i = \pm 1$ and $a_1b_1 + a_2b_2 + \cdots + a_{p-1}b_{p-1} \equiv S-2r \equiv 0 \pmod p$.

Other commended solvers: **F5D** (Carmel Alison Lam Foundation Secondary School).

Problem 415. (*Due to MANOLOUDIS Apostolos, Piraeus, Greece*) Given a triangle ABC such that $\angle BAC = 103^\circ$ and $\angle ABC = 51^\circ$. Let M be a point inside $\triangle ABC$ such that $\angle MAC = 30^\circ$ and $\angle MCA = 13^\circ$. Find $\angle MBC$ with proof.

Solution. **F5D** (Carmel Alison Lam Foundation Secondary School), **KWAN Chung Hang** (Sir Ellis Kadoorie Secondary School (West Kowloon)), **Adrian Iain LAM** (St. Paul's College), **Vijaya Prasad NALLURI** (Retired Principal, AP Educational Service, Andhra Pradesh, India), **Alex Kin-Chit O** (G.T. (Ellen Yeung) College), **Titu ZVONARU** (Comănești, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).



Let $x = \angle MBC$. By the trigonometric form of Ceva's theorem, we have

$$\frac{\sin 13^\circ}{\sin 13^\circ}\,\frac{\sin 73^\circ}{\sin 30^\circ}\,\frac{\sin x}{\sin(51^\circ - x)} = 1.$$

Then $2\sin 73^\circ = \dfrac{\sin 51^\circ \cos x - \cos 51^\circ \sin x}{\sin x}$

$$= \sin 51^\circ \cot x - \cos 51^\circ.$$

Using $\sin 73^\circ = \cos 17^\circ$, we get

$$\cot x = (2\cos 17^\circ+\cos 51^\circ)/\sin 51^\circ. \quad (*)$$

Since cot is strictly decreasing on $(0^\circ, 51^\circ)$, there is at most one such x. Now we have

$$\begin{aligned}2\sin y \cos y &= \sin 2y\\ &= \sin(3y-y)\\ &= \sin 3y \cos y - \cos 3y \sin y.\end{aligned}$$

Dividing by $\sin y$ leads to

$$2\cos y = \sin 3y \cot y - \cos 3y.$$

Solving for $\cot y$ and setting y=17°, we get

$$\cot 17^\circ = (2\cos 17^\circ+\cos 51^\circ)/\sin 51^\circ.$$

Therefore, $x = 17^\circ$.

Other commended solvers: **Christian Pratama BUNAIDI (**University of Tarumanagara, Jakarta, Indonesia), **CHEUNG Ka Wai** (Munsang College (Hong Kong Island)), **Prithwijit DE** (HBCSE, Mumbai, India), **Uma GIRISH** (Vidya Mandir Senior Secondary School, Chennai, India), **KWOK Man Yi** (S2, Baptist Lui Ming Choi Secondary School), **Cyril LETROUIT** (Lycée Jean-Baptiste Say, Paris, France), **Mihai STOENESCU** (Bischwiller, France) and **ZOLBAYAR Shagdar** (Orchlon International School, Ulaanbaatar, Mongolia).

Olympiad Corner

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Problem 3. Trapzoid $ABCD$ with a longer base AB is inscribed in the circle k. Let A_0, B_0 be respectively the midpoints of segments BC, CA. Let N be the foot of the altitude from the point C to AB, and G the centroid of the trangle ABC. Circle k_1 goes through A_0 and B_0 and touches the circle k in the point X, different than C. Prove that the points D, G, N and X are collinear.

(*IMO Shortlist 2011, modified*)

Problem 4. For a given positive integer k let $S(k)$ denote the sum of all numbers from the set $\{1,2\ldots,k\}$ relatively prime to k. Let m be a positive integer and n an odd positive integer. Prove that there exist positive integers x and y such that m divides x and $2S(x) = y^n$.

(*Columbia 2008*)

The Inequality of A. Oppenheim

(continued from page 2)

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