Mathematical Excalibur

Volume 18, Number 3 *November 2013*

Olympiad Corner

Below are the problems of the North Korean Team Selection Test for IMO 2013.

Problem 1. The incircle of a non-isosceles triangle ABC with the center I touches the sides BC, CA, AB at A_1, B_1, C_1 respectively. The line AI meets the circumcircle of ABC at A_2. The line B_1C_1 meets the line BC at A_3 and the line A_2A_3 meets the circumcircle of ABC at A_4 ($\neq A_2$). Define B_4, C_4 similarly. Prove that the lines AA_4, BB_4, CC_4 are concurrent.

(continued on page 4)

IMO 2016 Logo Design Competition

Hong Kong will host the 57th International Mathematical Olympiad (IMO) in July 2016. The Organising Committee now holds the IMO 2016 Logo Design Competition and invites all secondary school students in Hong Kong to submit logo designs for the event. Your design may win you $7,000 book coupons and become the official logo of IMO 2016! For details, please visit

www.imohkc.org.hk.

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***December 21, 2013***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Sequences

Kin Y. Li

Sequence problems occur often in math competitions. Below we will look at some of these problems involving limits in their solutions.

Example 1. (1980 *British Math Olympiad*) Find all real a_0 such that the sequence defined by $a_{n+1}=2^n-3a_n$ for n=0,1,2,… satisfies $a_0<a_1<a_2<\cdots$.

Solution. We have

$$\begin{aligned} a_{n+1} &= 2^n-3a_n = 2^n-3\times 2^{n-1}+3^2a_{n-1}\\ &=\cdots = 2^n-\sum_{j=1}^{n}(-3)^j2^{n-j}+(-3)^{n+1}a_0\\ &=\frac{2^{n+1}}{5}+\left(a_0-\frac{1}{5}\right)(-3)^{n+1}.\end{aligned}$$

If a_0=1/5, then it is good. If $a_0\neq 1/5$, then since $(2/3)^n$ goes to 0 as $n\to\infty$, so $a_n/3^n$ will have the same sign as $(a_0-1/5)(-1)^n$ when n is large. Hence, $a_n<a_{n+1}$ will not hold, contradiction.

Example 2. (1971-1972 *Polish Math Olympiad*) Prove that when n tends to infinity, the sum of the digits of 1972^n in base 10 will go to infinity.

Solution. Let a_i be the i-th digit of 1972^n from right to left in base 10. For $1\le k\le n/4$, we claim that among $a_{k+1}, a_{k+2},\ldots, a_{4k}$, at least one of them is nonzero.

Assume not. Then let

$$C = a_1+a_2\times 10+\cdots+a_k\times 10^{k-1}.$$

We have 1972^n-C divisible by 10^{4k}. Since $4k\le n$, so C is divisible by $2^{4k}=16^k>10^k>C$, contradiction.

From the claim, we get at least one digit in each of the following m+1 groups of digits will not be zero

$$\begin{aligned} &a_2, a_3, a_4,\\ &a_5, a_6, a_7,\ldots, a_{16},\\ &\ldots\\ &a_{j+1,}a_{j+2}, a_{j+3},\ldots, a_{4j},\end{aligned}$$

where $n/16<j=4^m\le n/4$. The digit sum of 1972^n is at least $m+1>(\log_4 n)-1$. So, the digit sum of 1972^n goes to infinity.

Example 3. Let a_1, a_2, a_3, … be a sequence of positive numbers. Prove that there exists infinitely many n such that $1+a_n>2^{1/n}a_{n-1}$.

Solution. Assume not. Then there is a M such that for all $n>M$, we have $1+a_n\le 2^{1/n}a_{n-1}$. Since $(1+1/n)^n\ge 2$, we have

$$a_n\le 2^{1/n}a_{n-1}-1\le((n+1)/n)a_{n-1}-1. \quad (*)$$

We claim that for $k\ge M$,

$$a_k\le(k+1)\left(\frac{a_M+1}{M+1}-\sum_{j=M+1}^{k+1}\frac{1}{j}\right).$$

The case $k=M$ is true as the right side is a_M. Suppose case k is true. By (*),

$$\begin{aligned} a_{k+1}&\le\frac{k+2}{k+1}a_k-1=\frac{k+2}{k+1}a_k-\frac{k+2}{k+2}\\ &\le(k+2)\left(\frac{a_M+1}{M+1}-\sum_{j=M+1}^{k+2}\frac{1}{j}\right).\end{aligned}$$

This concludes the induction. As $k\to\infty$, the above sum of $1/j$ goes to infinity, hence some $a_{k+1}<0$, contradiction.

Example 4. (2007 *Chinese Math Olympiad*) Let $\{a_n\}_{n\ge1}$ be a bounded sequence satisfying

$$a_n<\sum_{k=n}^{2n+2006}\frac{a_k}{k+1}+\frac{1}{2n+2007},\quad n=1,2,3,\ldots.$$

Prove that $a_n<1/n$ for $n=1,2,3,\ldots$.

Solution. Let $b_n=a_n-1/n$. Then for $n\ge1$,

$$b_n<\sum_{k=n}^{2n+2006}\frac{b_k}{k+1}. \quad (*)$$

It suffices to show $b_n<0$. Since a_n is bounded, so there is a constant M such that $b_n<M$. For $n>100{,}000$, we have

$$\begin{aligned} b_n&<\sum_{k=n}^{2n+2006}\frac{b_k}{k+1}<M\sum_{k=n}^{2n+2006}\frac{1}{k+1}\\ &=M\sum_{k=n}^{[3n/2]}\frac{1}{k+1}+M\sum_{k=[3n/2]+1}^{2n+2006}\frac{1}{k+1}\\ &<\frac{M}{2}+M\frac{2006+n/2}{1+3n/2}<\frac{6}{7}M.\end{aligned}$$

Repeating this m times, if $n>100{,}000$, then $b_n<(6/7)^mM$. Letting $m\to\infty$, we get $b_n\le 0$ for $n>100{,}000$. Using (*), we see if for $n\ge N+1$, we have $b_n<0$, then $b_N<0$. This gives $b_n<0$ for $n\ge 1$.

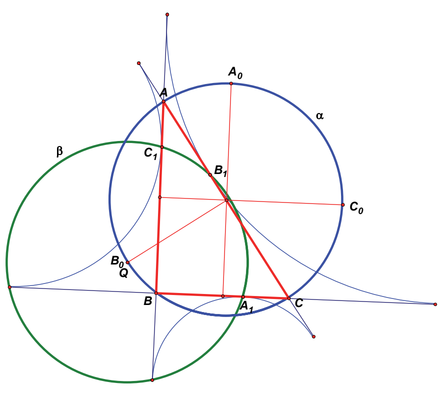
IMO 2013 - Leader Report(II)

Leung Tat-Wing

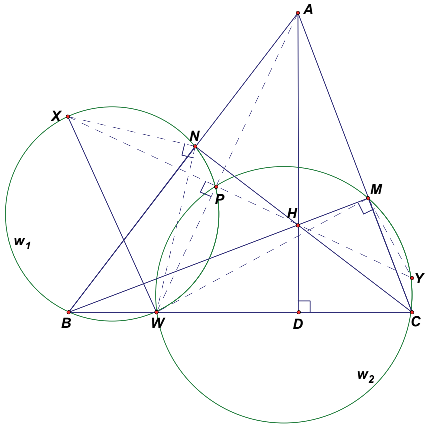
We will continue with our discussion on the IMO 2013 problems, which can be found in the Olympiad Corner of the last issue of *Math Excalibur*.

Problem 3: The problem was selected in the very last minute of the Jury meetings. Indeed another geometry problem concerning properties of hexagons was initially selected as a member of the hard pair. It was however discovered the problem was similar to an USAMO problem. I myself also recalled several similar problems. So the problem was rejected and replaced by this problem 3. After the selection process, it was announced both problem 3 and 6 come from Russia, indeed a problem similar to problem 4 was also found in a Russian geometry problem book. Truly the Russians are masters of posing problems!

Despite being a difficult problem (solved by 40 contestants), problem 3 is indeed a pure geometry problem and can be solved by pure synthetic geometry method. Indeed denote the circumcircles of ABC and $A_1B_1C_1$ by α and β respectively and let Q be the centre of the circumcircle of $A_1B_1C_1$. Let A_0 be the midpoint of arc BC containing A, and define B_0 and C_0 respectively. Then one can check $A_0B_1=A_0C_1$ and A,A_0,B_1,C_1 concyclic. (Likewise $B_0C_1=B_0C_1$ and B,B_0,C_1,A_1 concyclic; $C_0A_1=C_0B_1$ and C,C_0,B_1,A_1 concyclic.) One then consider the largest angle of $A_1B_1C_1$, say B_1, and if Q is on α, then Q must coincide with B_0, and hence $\angle B$=90°, not easy though!



Problem 4: There are more than 19 different solutions and surely there are more. It is possible to solve the problem using complicated angle chasings and/or coordinate geometry. But of course the basic or most natural approach is to look at the radical axis of the two circles. The following proof is given by Lau Chun Ting, a team member of ours.



Suppose ω_1 and ω_2 meet at another point P ($\neq W$). Since $\angle WPX=\angle WPY=90°$, so X, P and Y are collinear. To show H lies on XY, (X, Y, H collinear), it suffices to show $\angle HPW$ = 90°. Suppose now AH meets BC at D. Now B, N, M, C are concyclic (since $\angle BNC=\angle BMC$ = 90°), we have $AN\times AB=AM\times AC$. So the powers of the point A with respect to the circles ω_1 and ω_2 are the same, that means A lies on the radical axis WP, or A, P, W collinear (radical axis theorem). Now note that H, M, C, D are also concyclic, hence $AH\times AD=AM\times AC$ (quite a few concyclic conditions). As before

$$AM\times AC=AN\times AB=AP\times AW,$$

we get $AP\times AW=AH\times AD$. Therefore, W, P, H, D are concyclic and we get $\angle HPW$ = 90°, as required.

Using coordinate attack, we may let $A=(a_1,a_2)$, $B=(-b,0)$, $C=(c,0)$ and $W=(0,0)$. By computing slopes and equations of lines, (complicated but still manageable), one eventually gets the coordinates of X, H and Y. Hence can verify X, H and Y collinear by calculating slopes of XH and HY.

Problem 5: For problem of this kind, one can try many things to obtain partial results. But the essential (crucial) part of this problem is actually how to make use of condition 3. Indeed if this condition is released, then the function $f(x) = bx^2$, with $b \geq 1$, will satisfy the first and second condition. Now see what we can get by putting different values of x and y into the equations. For examples, put $x = a$ and $y = 1$, one gets $a\,f(1) = f(a)\,f(1) \geq f(a) = a$, hence $f(1) \geq 1$. We let $f(1) = c \geq 1$. By induction, one can then show $f(n) \geq nc$, for all natural numbers n. So in particular $f(n)$ is positive. Now we show $f(x)$ is strictly increasing. Indeed if $f(x+\Delta x) \leq f(x)$ for some positive rational numbers x and Δx, then

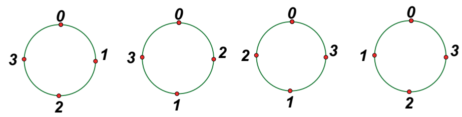
$$f(x) \geq f(x+\Delta x) \geq f(x)+f(\Delta x),$$

therefore $f(\Delta x) \leq 0$. However, we also have $f(n)\,f(\Delta x) \geq f(n\Delta x)$. Now since $f(\Delta x) \leq 0$, so we must also have $f(n\Delta x) \leq 0$ for all n, however surely we can find n so that $n\Delta x$ is a natural number and $f(n\Delta x)$ is positive, a contradiction. Using the same argument, we can show $f(x) > 0$ for all positive rational numbers. One then proceeds to show $f(1) = 1$. Hence $f(x) = x$ for all positive rationals. I am not going to produce all the details here. Suffices to say, we often need to expand a positive rational number in terms of a, say for a rational number $b < a$, it is of the form

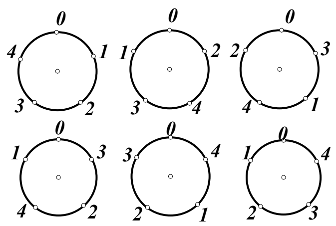
$$k_0+\frac{k_1}{a}+\frac{k_2}{a^2}+\cdots \quad \text{(finite sum)},$$

some kind of a-adic expansion!

Problem 6: Problem 6 is even harder than problem 3, only 7 contestants solved it. A nice point of the problem is that it links a geometric fact (intersecting chords) to a certain number property, and the relation is an exact relation (M=N+1). For n=3, the beautiful labellings are given below (we always label 0 at the top).



The pairs of positive integers satisfying the stated property are (1,1), (1,2) and (2,1). For n=4, to complete the list of integers with the stated property, we just have to consider those x and y satisfying $x+y$=4. Indeed we get two more pairs (1,3) and (3,1). Indeed the six beautiful labellings are



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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***December 21, 2013.***

Problem 431. There are 100 people, composed of 2 people from 50 distinct nations. They are seated in a round table. Two people sitting next to each other are *neighbors*.

Prove that it is possible to divide the 100 people in two groups of 50 people so that no 2 people from the same nation are in the same group and each person in a group has at most one neighbor in the group.

Problem 432. Determine all prime numbers p such that there exist integers a,b,c satisfying $a^2+b^2+c^2=p$ and $a^4+b^4+c^4$ is divisible by p.

Problem 433. Let P_1, P_2 be two points inside ΔABC. Let $BC=a$, $CA=b$ and $AB=c$. For $i=1,2$, let $P_iA=a_i$, $P_iB=b_i$ and $P_iC=c_i$. Prove that

$$aa_1a_2+bb_1b_2+cc_1c_2\ge abc.$$

Problem 434. Let O and H be the circumcenter and orthocenter of ΔABC respectively. Let D be the foot of perpendicular from C to side AB. Let E be a point on line BC such that $ED\perp OD$. If the circumcircle of ΔBCH intersects side AB at F, then prove that points E, F, H are collinear.

Problem 435. Let $n>1$ be an integer that is not a power of 2. Prove that there exists a permutation $a_1, a_2,\ldots, a_n$ of $1,2,\ldots, n$ such that

$$\sum_{k=1}^{n} a_k\cos\frac{2k\pi}{n}=0.$$

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Solutions

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Problem 426. Real numbers a, b, x, y satisfy the property that for all positive integers n, $ax^n+by^n=1+2^{n+1}$. Determine (with proof) the value of x^a+y^b.

Solution. **Ángel PLAZA** (Universidad de Las Palmas de Gran Canaria, Spain).

Considering the generating functions of the left and right sides of $ax^n+by^n=1+2^{n+1}$, we have

$$\sum_{n=1}^{\infty}ax^nz^{n-1}+\sum_{n=1}^{\infty}by^nz^{n-1}=\sum_{n=1}^{\infty}z^{n-1}+\sum_{n=1}^{\infty}2^{n+1}z^{n-1}.$$

For $|z|<\min\{1/2,1/|x|,1/|y|\}$, using the geometric series formula, we have

$$\frac{ax}{1-xz}+\frac{by}{1-yz}=\frac{1}{1-z}+\frac{4}{1-2z}.$$

The right side is a rational function of z. By the uniqueness of the partial fraction decomposition, either $ax=1$, $x=1$, $by=4$, $y=2$ or $ax=4$, $x=2$, $by=1$, $y=1$. In both cases, $x^a+y^b=1^1+2^2=5$.

Other commended solvers: **CHAN Long Tin** (Cambridge University, Year 1), **CHEUNG Ka Wai** (Munsang College (Hong Kong Island)), **Jeffrey HUI Pak Nam** (La Salle College, Form 6), **KIM Minsuk Luke** (The South Island School, Hong Kong, Year 13), **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, Form 2), **LO Wang Kin**(Wah Yan College, Kowloon), **Math Group** (Carmel Alison Lam Foundation Secondary School), **Alice WONG Sze Nga** (Diocesan Girls' School, Form 6) and **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Problem 427. Determine all (m,n,k), where m, n, k are integers greater than 1, such that $1!+2!+\cdots+m!=n^k$.

Solution. **Ioan Viorel CODREANU** (Secondary School Satulung, Maramures, Romania), **CHEUNG Ka Wai** (Munsang College (Hong Kong Island)), **Jeffrey HUI Pak Nam** (La Salle College, Form 6), **KIM Minsuk Luke** (The South Island School, Hong Kong, Year 13), **LO Wang Kin** (Wah Yan College, Kowloon), **Corneliu MĂNESCU-AVRAM** ("Henri Mathias Berthelot" Secondary School, Ploieşti, Romania School, Ploieşti, Romania), **Math Group** (Carmel Alison Lam Foundation Secondary School) and **William PENG.**

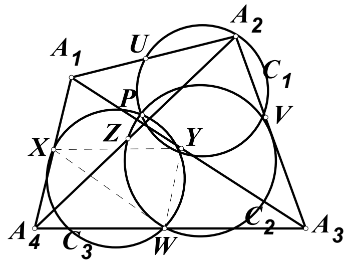
Let $S(m)=1!+2!+\cdots+m!$. Then $S(2)=3$, $S(3)=9=3^2$, $S(4)=33=3\times 11$, $S(5)=153=3^2\times 17$, $S(6)=873=3^2\times 97$, $S(7)=5913=3^4\times 73$, $S(8)=46233=3^2\times 11\times 467$.

For $m>8$, since $9!\equiv 0 \pmod{3^3}$, so $S(m)\equiv S(8)\equiv 0 \pmod{3^2}$ and $S(m)\equiv S(8)\not\equiv 0 \pmod{3^3}$. These imply that if $S(m)=n^k$ and $k>1$, then $k=2$.

Since $S(4)=33\equiv 3 \pmod 5$, $S(m)\equiv 3 \pmod 5$. Now $n^2\equiv 0, 1, 4 \pmod 5$. So $S(m)\ne n^2$. We have the only solution is $(m,n,k)=(3,3,2)$.

Problem 428. Let $A_1A_2A_3A_4$ be a convex quadrilateral. Prove that the nine point circles of $\Delta A_1A_2A_3$, $\Delta A_2A_3A_4$, $\Delta A_3A_4A_1$ and $\Delta A_4A_1A_2$ pass through a common point.

Solution. **HOANG Nguyen Viet** (Hanoi, Vietnam), **Jeffrey HUI Pak Nam** (La Salle College, Form 6), **Corneliu MĂNESCU-AVRAM** ("Henri Mathias Berthelot" Secondary School, Ploieşti, Romania School, Ploieşti, Romania), **Apostolis MANOLOUDIS, Math Group** (Carmel Alison Lam Foundation Secondary School) and **Alice WONG Sze Nga** (Diocesan Girls' School, Form 6).



Let C_1, C_2, C_3, C_4 be the nine point circles of $\Delta A_1A_2A_3$, $\Delta A_2A_3A_4$, $\Delta A_3A_4A_1$, $\Delta A_4A_1A_2$ respectively. Let U, V, W, X, Y, Z be the midpoints of A_1A_2, A_2A_3, A_3A_4, A_4A_1, A_1A_3, A_2A_4 respectively. Let C_1 and C_3 intersect at Y and P (in case C_1, C_3 are tangent, P will be the same as Y). We claim P is on C_2. For that it suffices to show P,V,W,Z are concyclic.

By the midpoint theorem, $XY=\frac12 A_4A_3=WA_3$ and $XW=\frac12 A_1A_3=YA_3$. So we have (1) $WXYA_3$ is a parallelogram. Similarly, (2) $YUVA_3$ and (3) $WZVA_3$ are also parallelograms. Now (4) P,U,V,Y are on C_1 and (5) P,X,W,Y are on C_3. We have

$$\begin{aligned}\angle VPW&=\angle YPV+\angle YPW\\&=\angle YUV+\angle YXW \quad\text{by (4), (5)}\\&=\angle YA_3V+\angle YA_3W \quad\text{by (2), (1)}\\&=\angle VA_3W\\&=\angle VZW \quad\text{by (3).}\end{aligned}$$

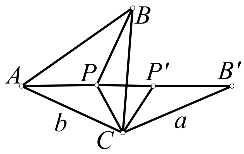
So P is on C_2. Similarly, P is on C_4.

Other commended solvers: **William FUNG, Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade'' Secondary School, Buzău, Romania).

Problem 429. Inside ΔABC, there is a point P such that $\angle APB = \angle BPC = \angle CPA$. Let $PA = u$, $PB = v$, $PC = w$, $BC = a$, $CA = b$ and $AB = c$. Prove that

$$(u+v+w)^2 \le ab+bc+ca - \left(\sqrt{a(b+c-a)} - \sqrt{b(c+a-b)}\right)^2.$$

Solution. **LO Wang Kin** (Wah Yan College, Kowloon).



Rotate ΔABC about C by 60° away from A. Let the images of B, P be B', P' respectively. As $\angle PCP' = 60° = \angle BCB'$, so $\Delta PCP'$ and $\Delta BCB'$ are equilateral. As $\angle B'PC = \angle CPA = 120°$, A, P, P', B' are collinear. So $AB' = AP+PP'+P'B' = u+w+v$. By the cosine law, $AB'^2 = a^2+b^2-2ab\cos(C+60°)$.

After expansion and cancellation, the right side of the desired inequality becomes

$$a^2+b^2-ab+2\sqrt{ab(b+c-a)(c+a-b)}.$$

Now

$$\sqrt{ab(b+c-a)(c+a-b)} = ab\sqrt{2\left(1-\frac{a^2+b^2-c^2}{2ab}\right)} = ab\sqrt{2(1-\cos C)}.$$

Using these, the right side minus the left side of the desired inequality is

$$ab(-1+2\sqrt{2(1-\cos C)}+2\cos(C+60°))$$
$$= ab(-1+2\sqrt{2(1-\cos C)}+\cos C-\sqrt{3}\sin C)$$
$$= 2ab\sqrt{1-\cos C}(\sqrt{2}-\sqrt{1+\cos(C-60°)}) \ge 0$$

and we are done.

Other commended solvers: **CHEUNG Ka Wai** (Munsang College (Hong Kong Island)), **T. W. LEE** (Alumni of New Method College), **Math Group** (Carmel Alison Lam Foundation Secondary School) and **Alice WONG Sze Nga** (Diocesan Girls' School, Form 6).

Problem 430. Prove that among any $2n+2$ people, there exist two of them, say A and B, such that there exist n of the remaining $2n$ people, each either knows both A and B or does not know A nor B. Here, x *knows* y does not necessarily imply y *knows* x.

Solution. **Jeffrey HUI Pak Nam** (La Salle College, Form 6) and **Math Group** (Carmel Alison Lam Foundation Secondary School).

Take a person P out of the $2n+2$ people. Suppose among the remaining $2n+1$ people, he knows k of them and does not know $2n+1-k$ of them. Among these $2n+1$ people, there are ${}_{2n+1}C_2 = n(2n+1)$ pairs. Call a pair *good* if P knows both of them or does not know both of them, *bad* if P knows one, but not both. By the AM-GM inequality, there are at most $\lceil k(2n+1-k)\rceil \le \lceil (n+½)^2\rceil = n^2+n$ bad pairs. Adding up all the bad pairs for all $2n+2$ people, the number is at most $(2n+2)(n^2+n) = 2n(n+1)^2$. There are ${}_{2n+2}C_2=(n+1)(2n+1)$ pairs altogether. Since the average

$$\frac{2n(n+1)^2}{(n+1)(2n+1)} = \frac{2n(n+1)}{2n+1} < n+1,$$

some pair $\{A, B\}$ will be a bad pair for at most n of the remaining $2n$ people. Then at least n other people will call $\{A,B\}$ a good pair and we are done.

Olympiad Corner

(Continued from page 1)

Problem 2. Let $a_1, a_2,\ldots, a_k$ be numbers such that $a_i\in\{0,1,2,3\}$, i=1 to k and $z = (x_k, x_{k-1},\ldots,x_1)_4$ be a base 4 expansion of $z \in \{0, 1, 2, \ldots, 4^k-1\}$. Define A as follows:

$A=\{z \mid p(z)=z, z=0,1,2,\ldots,4^k-1\}$, where

$$p(z) = \sum_{i=1}^{k} a_i x_i 4^{i-1}.$$

Prove that $|A|$ is a power of 2. ($|X|$ denotes the number of elements in X).

Problem 3. Find all $a,b,c\in\mathbb{Z}$, $c \ge 0$ such that $(a^n+2^n) \mid b^n+c$ for all positive integers n, where $2ab$ is non-square.

Problem 4. Positive integers 1 to 9 are written in each square of a 3×3 table. Let us define an operation as follows: Take an arbitrary row or column and replace these numbers a, b, c with either non-negative numbers $a-x$, $b-x$, $c+x$ or $a+x$, $b-x$, $c-x$, where x is a positive number and can vary in each operation.

1) Does there exist a series of operations such that all 9 numbers turn out to be equal from the following initial arrangement a) ?, b) ?

1	2	3
4	5	6
7	8	9

a)

2	8	5
9	3	4
6	7	1

b)

2) Determine the maximum value which all 9 numbers turn out to be equal to after some steps.

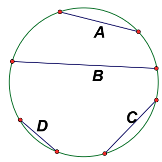
Problem 5. The incircle ω of a quadrilateral $ABCD$ touches AB, BC, CD, DA at E, F, G, H, respectively. Choose an arbitrary point X on the segment AC inside ω. The segments XB, XD meet ω at I, J respectively. Prove that FJ, IG, AC are concurrent.

Problem 6. Show that $x^3+x+a^2=y^2$ has at least one pair of positive integer solution (x,y) for each positive integer a.

IMO 2013–Leader Report (II)

(continued from page 2)

The problem is how to connect the geometry and the number theory information. In general, how to get started? I can only describe it roughly from the official solution. Call three chords *aligned* if one of them separates the other two. For more than three chords, they are aligned if any three of them aligned.



In the figure the chords A, B and C are aligned (the line formed by B separated the two chords A and C; while B, C and D are not aligned (none of the lines formed by B, C or D separates the other two chords). Now call a chord a *k-chord* if the sum of its two endpoints is k (the chord may be degenerated into a point of value k). The crucial observation is: in a beautiful labeling, the k-chords are aligned for any k. To prove this claim, one proceeds by induction. Indeed the only case is when there are three chords not aligned and such that one of the chords has endpoints 0 and n. After the claim is proved, one proceeds again using delicate induction arguments to show $M=N+1$. Indeed the beautiful labellings are eventually divided into classes. Elements of the first class are as before in the induction step. Elements of the second class correspond precisely with the pairs of positive integers satisfying $x+y=n$ and $\gcd(x,y)=1$, (which correspond exactly to the elements $\{x \mid 1 \le x \le n, \gcd(x,y) = 1\}$ with size $\varphi(n)$. Tough!