Mathematical Excalibur

Volume 18, Number 4 *December 2013 – January 2014*

Olympiad Corner

Below are the problems of the Dutch Team Selection Test for IMO 2013.

Problem 1. Show that

$$\sum_{n=0}^{2013}\frac{4026!}{\left(n!(2013-n)!\right)^2}$$

is the square of an integer.

Problem 2. Let P be the intersection of the diagonals of a convex quadrilateral $ABCD$. Let X, Y and Z be points on the interior of AB, BC and CD respectively such that

$$\frac{AX}{XB}=\frac{BY}{YC}=\frac{CZ}{ZD}=2.$$

Suppose moreover that XY is tangent to the circumcircle of ΔBXY. Show that $\angle APD=\angle XYZ$.

Problem 3. Fix a sequence a_1, a_2, a_3, … of integers satisfying the following condition: for all prime numbers p and all positive integers k, we have

$$a_{pk+1}=pa_k-3a_p+13.$$

Determine all possible values of a_{2013}.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***February 25, 2014***.

For individual subscription for the next five issues for the 13-14 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

Dr. Kin-Yin LI, Math Dept., Hong Kong Univ. of Science and Technology, Clear Water Bay, Kowloon, Hong Kong

Fax: (852) 2358 1643
Email: makyli@ust.hk

PUMaC 2013

Andy Loo (Princeton University)

In the United States there are several annual math competitions organized by undergraduate students at different universities for high school enthusiasts, including the Harvard-MIT Math Tournament (HMMT), the Stanford Math Tournament (SMT), and, last but not least, the Princeton University Mathematics Competition (PUMaC). Started in 2006, PUMaC has grown into an international event in which high schoolers across America are joined by teams from as far away as Bulgaria and China on Princeton campus each year.

PUMaC 2013 was held on November 16, engaging over 600 participants, and I was honored to serve as Problem Tsar (academic coordinator who heads the problem writing team). The responsibility of creating, grading and defending the problems and solutions of a competition of such scale and repute gave me an inspiring learning experience.

The competition is split into Division A (more challenging) and Division B (for less experienced contestants). Each team consists of eight students. In the morning, each contestant takes two out of four one-hour answer-only individual tests (Algebra, Geometry, Combinatorics and Number Theory, eight problems each) of his/her choice, followed by the one-hour Team Round, where members of the same team may discuss and work together (each team enjoying a separate room!).

The top 10 performers on each individual test (possibly with nonempty intersection) qualify for the Individual Finals, a one-hour proof-based test with three problems. I personally feel that an average Individual Finals problem lies somewhere near an IMO problem 1 or 4 in terms of difficulty. Remarkably, in PUMaC 2013, two contestants got a perfect score on the Division A Individual Finals despite the time pressure! Also worth mentioning is the Power Round, which is a relatively long series of problems revolving around a central theme – knot theory in 2013 – released one week before the competition day for the teams to work on and turn in on competition day. (Teams may also enroll on a Power Round-only basis.) It usually takes frantic grading to determine the individual and team rankings in time for the award ceremony in the late afternoon, while mini-events such as Math (quiz) Bowl and Rubik's cube as well as a lecture by a Princeton professor keep the participants entertained.

I would like to discuss a few problems in PUMaC 2013, not necessarily because they are the hardest, but mostly because they bring out certain lessons of problem solving we can learn.

Individual Finals B1.

Let a_1 = 2013 and a_{n+1} = 2013^{a_n} for all positive integers n. Let b_1 = 1 and b_{n+1} = 2013^{2012b_n} for all positive integers n. Prove that $a_n > b_n$ for all positive integers n.

At first sight, one natural reaction to this problem would be to do induction. However, we would quickly realize that the assumption $a_n > b_n$ does not imply $a_{n+1} > b_{n+1}$, as it does not imply $2013^{a_n} > 2013^{2012b_n}$. Many contestants performed pages of tedious calculations in vain. Are we doomed? It turns out that a clever little tweak to the induction idea would lead us to a crisp and compact solution:

Instead of $a_n > b_n$, we shall prove $a_n \geq 2013b_n$ for all positive integers n. This is clearly true for n = 1. If $a_k \geq 2013b_k$ for

(continued on page 2)

some positive integer k, then

$$\begin{aligned} a_{k+1} &= 2013^{a_k} \\ &\geq 2013^{2013 b_k} \\ &= 2013^{b_k} \cdot 2013^{2012 b_k} \\ &\geq 2013 b_{k+1}. \end{aligned}$$

There is something intriguing about this seemingly easy proof: if we cannot even prove just the original result, how come we can miraculously prove a stronger result? The answer to this paradox lies in the nature of mathematical induction: when we use induction, our task is essentially to prove the original statement about an arbitrary positive integer but equipped with an additional tool – the assumption that the statement is true for the preceding positive integer(s). If the statement is strengthened, what we need to prove becomes more demanding but the inductive hypothesis that we can use also gets more powerful. In the case of this problem, since the recurrence relations are exponential, the upgrade of the inductive hypothesis outweighs the increase in difficulty of the desired result.

Individual Finals A1.

Prove that

$$\frac{1}{a^2+2}+\frac{1}{b^2+2}+\frac{1}{c^2+2}$$

$$\leq \frac{1}{6ab+c^2}+\frac{1}{6bc+a^2}+\frac{1}{6ca+b^2}$$

for any positive real numbers a, b and c satisfying $a^2 + b^2 + c^2 = 1$.

The usual first step in proving such a symmetric inequality is to use the given condition to *homogenize* the inequality, i.e. to make the terms carry equal degrees. Afterwards, various inequality theorems can be applied. Here we first write the left-hand side as

$$\frac{1}{3a^2+2b^2+2c^2}+\frac{1}{3b^2+2c^2+2a^2}+\frac{1}{3c^2+2a^2+2b^2}$$

and note that by the AM-GM inequality, $3a^2 + 3b^2 \geq 6ab$ and analogous inequalities hold. So

$$\frac{1}{6ab+c^2}+\frac{1}{6bc+a^2}+\frac{1}{6ca+b^2}$$

$$\geq \frac{1}{3a^2+3b^2+c^2}+\frac{1}{3b^2+3c^2+a^2}+\frac{1}{3c^2+3a^2+b^2}.$$

It suffices to prove the following inequality

$$\frac{1}{3x+2y+2z}+\frac{1}{3y+2z+2x}+\frac{1}{3z+2x+2y}$$

$$\leq \frac{1}{3x+3y+z}+\frac{1}{3y+3z+x}+\frac{1}{3z+3x+y}$$

where x, y and z are positive real numbers.

At this stage, one may resort to passionate expansion and then apply Muirhead's inequality and/or Schur's inequality, or, alternatively, factorization and completing the square.

But I wish to share a solution using the *majorization inequality* (see <u>*Math Excalibur*</u>, vol. 5, no. 5, p.2): Without loss of generality we may assume $x \geq y \geq z$. Then

$$(3x + 3y + z,\ 3y + 3z + x,\ 3z + 3x + y)$$

majorizes

$$(3x + 2y + 2z,\ 3y + 2z + 2x,\ 3z + 2x + 2y).$$

Due to the convexity of the function $f(t) = 1/t$, the desired inequality follows by the majorization inequality.

Readers may also be interested in an alternative solution involving calculus: First, by Muirhead's inequality (see *Mathematical Excalibur*, vol. 11, no. 1), we have

$$u^3v^3w + v^3w^3u + w^3u^3v$$

$$\geq u^3v^2w^2 + v^3w^2u^2 + w^3u^2v^2$$

for any positive u, v, w. Letting

$$u = t^{x-1/7},\ v = t^{y-1/7} \text{ and } w = t^{z-1/7}$$

where $0 < t < 1$, we get

$$t^{3x+3y+z-1} + t^{3y+3z+x-1} + t^{3z+3x+y-1}$$

$$\geq t^{3x+2y+2z-1} + t^{3y+2z+2x-1} + t^{3z+2x+2y-1}.$$

Now, integrating both sides with respect to t from 0 to 1, we obtain nothing but the desired inequality!

Lastly I encourage all readers to try out the following problem which only one out of the 123 contestants attempting Combinatorics A got right. This is really my favorite problem in PUMaC 2013 because I love eating sushi and find the setting very interesting:

Combinatorics A8.

Eight different pieces of sushi are placed evenly around a round table which can rotate about its center. Eight people sit evenly around the table. Each person has one favorite piece of sushi among the eight, and their favorites are all distinct. Sadly, they find that no matter how they rotate the table, there are never more than three people who have their favorite sushi in front of them simultaneously.

How many possible arrangements of the eight pieces of sushi are there? (Two arrangements that differ by a rotation are considered the same.)

In 1908, a classic Chinese newspaper article famously raised three questions for the country: When can China first send an individual athlete to the Olympic Games? When can China first send a delegation to the Olympic Games? When can China first host the Olympic Games?

In closing, I would also like to ask three questions: When can Hong Kong first take part in the Power Round of PUmaC? When can Hong Kong first send a team to Princeton to join the main competition of PUMaC? When can a university in Hong Kong first host a math competition run by undergraduates for secondary school students?

As Dr. Kin Li (editor of *Math Excalibur*) observes, Hong Kong students need more opportunities to participate in different competitions and broaden their horizons. They will also be able to experience a beautiful university, make friends with some of the most brilliant brains from around the world, and learn team spirit especially through the Power Round and Team Round. With optimism, I hope my three questions will find answers before long.

For further information and past papers, please visit PUMaC's website http://www.pumac.princeton.edu/

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***February 25, 2014.***

Problem 436. Prove that for every positive integer n, there exists a positive integer $p(n)$ such that the interval $[1, p(n)]$ can be divided into n pairwise disjoint intervals with each contains at least one integer and the sum of the integers in each of these intervals is the square of some integer.

Problem 437. Determine all real numbers x satisfying the condition that $\cos x$, $\cos 2x$, $\cos 4x$, …, $\cos 2^n x$, … are all negative.

Problem 438. Suppose $P(x)$ is a polynomial with integer coefficients such that for every integer n, $P(n)$ is divisible by at least one of the positive integers $a_1, a_2, \ldots, a_m$. Prove that there exists one of the a_i such that for all integer n, $P(n)$ is divisible by that a_i.

Problem 439. In acute triangle ABC, T is a point on the altitude AD (with D on side BC). Lines BT and AC intersect at E, lines CT and AB intersect at F, lines EF and AD intersect at G. A line ℓ passing through G intersects side AB, side AC, line BT, line CT at M, N, P, Q respectively.

Prove that $\angle MDQ = \angle NDP$.

Problem 440. There are n schools in a city. The i-th school will send C_i students to watch a performance at a field. It is <u>known</u> that $0 \le C_i \le 39$ for i=1, 2, …, n and $C_1+C_2+\cdots+C_n=1990$. The seats will be put in a rectangle arrangement with each row having 199 seats. Determine the least number of rows needed to satisfy the condition that all students from the same school must sit in the same row for all possibilities of the known conditions above.

****

Solutions

****

Problem 431. There are 100 people, composed of 2 people from 50 distinct nations, are seated in a round table. Two people sitting next to each other are <u>*neighbors*</u>.

Prove that it is possible to divide the 100 people in two groups of 50 people so that no 2 people from the same nation are in the same group and each person in a group has at most one neighbor in the group.

Solution. **Jeffrey HUI Pak Nam** (La Salle College, Form 6), **Math Group** (Carmel Alison Lam Foundation Secondary School) and **ZOLBAYAR Shagdar** (Orchlon International School, Ulaanbaatar, Mongolia).

Suppose these 100 people $V_1, V_2, \ldots, V_{100}$ are seated in a round table in clockwise order. For $n = 1,2,\ldots, 50$, call $\{V_{2n-1}, V_{2n}\}$ a <u>*partner pair*</u>. We color V_1 in black and color the person with the same nation as him, say V_r, in white. If V_r's partner is not yet colored, then we color V_r's partner, say V_s, in black (this completes the coloring of the partner pair $\{V_r, V_s\}$) and go on to color the person with the same nation as V_s in white. Repeat this process until we reach a V_r whose partner V_s was colored already, then $V_r=V_2$ and $V_s=V_1$ since the only partner pair not yet completing the coloring is $\{V_1,V_2\}$ with V_1 black and V_2 waiting to be colored. This gives the first cycle. Then we start to form another cycle with a remaining partner pair. Since there are 100 people, we will eventually stop. At the end, there are two groups with 50 black's and 50 white's and the required conditions are satisfied.

Problem 432. Determine all prime numbers p such that there exist integers a,b,c satisfying $a^2 + b^2 + c^2 = p$ and $a^4+b^4+c^4$ is divisible by p.

Solution. **Ioan Viorel CODREANU** (Secondary School Satulung, Maramures, Romania), **Jeffrey HUI Pak Nam** (La Salle College, Form 6), **KIM Minsuk Luke** (The South Island School, Hong Kong, Year 13), **Corneliu MĂNESCU-AVRAM** ("Henri Mathias Berthelot" Secondary School, Ploieşti, Romania), **Math Center** (Carmel Alison Lam Foundation Secondary School) and **O Kin Chit Alex** (G.T. (Ellen Yeung) College).

Without loss of generality, we may assume $a \ge b \ge c \ge 0$. Then

$$\begin{aligned}0 &\equiv (p - b^2 - c^2)^2 + b^4 + c^4\\ &\equiv (b^2 + c^2)^2 + b^4 + c^4 = 2(b^4 + b^2c^2 + c^4)\\ &= 2(b^2 - bc + c^2)(b^2 + bc + c^2) \pmod p.\end{aligned}$$

Next,

$$\begin{aligned}0 \le bc &\le b^2 - bc + c^2\\ &\le b^2 + bc + c^2\\ &\le a^2 + b^2 + c^2 = p.\end{aligned}$$

Since $a \ge b \ge c \ge 0$, if $bc=a^2$, then $a=b=c$ and p being prime implies $a = 1$ and $p = 3$. Otherwise $bc < a^2$ leads to $b^2 + bc + c^2 = 0$ or 1. If $b = 0$, then $a^2 = p$ contradicts p is prime. Then $c = 0$, $b = 1$ and $a^2 + 1 = p$, which leads to

$$0 \equiv a^4+b^4+c^4 = a^4+1 \equiv 2 \pmod p.$$

Then p=2 and a=b=1, c=0. Therefore, the only solutions are $p = 2$ or 3.

Problem 433. Let P_1, P_2 be two points inside ΔABC. Let $BC = a$, $CA = b$ and $AB = c$. For $i = 1,2$, let $P_iA = a_i$, $P_iB = b_i$ and $PC_i = c_i$. Prove that

$$aa_1a_2 + bb_1b_2 + cc_1c_2 \ge abc.$$

Solution. **Math Group** (Carmel Alison Lam Foundation Secondary School).

Let the complex numbers $\alpha, \beta, \gamma, \mu, v$ correspond to the points A, B, C, P_1, P_2 in the complex plane respectively. By expansion, we have

$$\frac{(\mu-\alpha)(v-\alpha)}{(\beta-\alpha)(\gamma-\alpha)} + \frac{(\mu-\beta)(v-\beta)}{(\alpha-\beta)(\gamma-\beta)} + \frac{(\mu-\gamma)(v-\gamma)}{(\alpha-\gamma)(\beta-\gamma)} = 1.$$

Then

$$\frac{a_1a_2}{cb} + \frac{b_1b_2}{ca} + \frac{c_1c_2}{ba}$$

$$= \left|\frac{(\mu-\alpha)(v-\alpha)}{(\beta-\alpha)(\gamma-\alpha)}\right| + \left|\frac{(\mu-\beta)(v-\beta)}{(\alpha-\beta)(\gamma-\beta)}\right| + \left|\frac{(\mu-\gamma)(v-\gamma)}{(\alpha-\gamma)(\beta-\gamma)}\right|$$

$$\ge \left|\frac{(\mu-\alpha)(v-\alpha)}{(\beta-\alpha)(\gamma-\alpha)} + \frac{(\mu-\beta)(v-\beta)}{(\alpha-\beta)(\lambda-\beta)} + \frac{(\mu-\gamma)(v-\gamma)}{(\alpha-\gamma)(\beta-\gamma)}\right|$$

$= 1$.

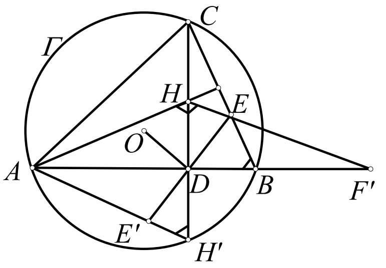
Multiplying both sides by abc, we get the desired result.

Problem 434. Let O and H be the circumcenter and orthocenter of ΔABC respectively. Let D be the foot of perpendicular from C to side AB. Let E be a point on line BC such that $ED \perp OD$. If the circumcircle of ΔBCH intersects line AB at F, then prove that points E, F, H are collinear.

Solution 1. **Jeffrey HUI Pak Nam** (La Salle College, Form 6) and **T. W. LEE** (Alumni of New Method College).

Let lines HE and AB intersect at F'. Let Γ be the circumcircle of ΔABC. Let H'

be the intersection of line CD and Γ different from C. Let E' be the intersection of lines DE and AH'.

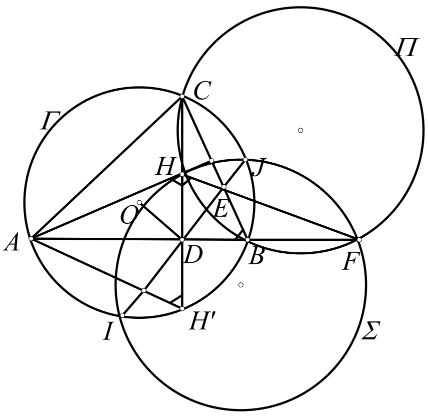


Observe that since $\angle H'DE' = \angle HDE$, $\angle AH'C = \angle ABC = 90^\circ - \angle BAH = \angle AHD$ implies $H'D=HD$ and the butterfly $H'CBAH'$ on Γ gives $E'D=ED$ as $OD\perp DE$, we have $\triangle H'E'D \cong \triangle HED$. Then

$$\angle F'HD = \angle EHD = \angle E'H'D = \angle AH'C = \angle DBC.$$

It follows $\angle CHF' = \angle CBF'$. Then F' is on the line AB and the circumcircle of $\triangle BCH$. Therefore, $F'=F$ and E, F, H are collinear.

Solution 2. **Jerry AUMAN, Georgios BATZOLIS** (*Mandoulides High School, Thessaloniki, Greece*) and **Jon GLIMMS** (Vancouver, Canada).



Let Π be the circle passing through C, H, B, F and let Γ be the circumcircle of $\triangle ABC$. Let line DE meet Γ at I and J. Since $OD\perp DE$, D bisects chord IJ. Next,

$$\angle DCF = \angle DBH = 90^\circ - \angle BAC = \angle DCA$$

implies D bisects AF. Hence $AIFJ$ is a parallelogram. Then $\angle IFJ = \angle IAJ$.

Let H' be the intersection point (different from C) of line CD and Γ. Then D bisects HH' (see solution 1 -- *Ed.*) and $IHJH'$ is a parallelogram. So $\angle IHJ = \angle IH'J$. Then

$$\angle IFJ + \angle IHJ = \angle IAJ + \angle IH'J = 180^\circ.$$

So I,F,J,H lies on a circle Σ.

Finally, the radical axis of Γ and Π is line BC, while the radical axis of Γ and Σ is line IJ. So the radical center of Γ, Π, Σ is the intersection of lines BC and IJ, which is E. Therefore, E is also on the radical axis of Π and Σ, which is line HF.

Comments: One can also solve via coordinate geometry by assigning lines AB and CD as the x-axis and y-axis respectively.

Other commended solvers: **Math Group** (Carmel Alison Lam Foundation Secondary School), **Vijaya Prasad NALLURI** (Retired Principal, AP Educational Service, India) and **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade'' Secondary School, Buzău, Romania).

Problem 435. Let $n > 1$ be an integer that is not a power of 2. Prove that there exists a permutation $a_1, a_2,\ldots, a_n$ of $1,2,\ldots, n$ such that

$$\sum_{k=1}^{n} a_k \cos\frac{2k\pi}{n} = 0.$$

Solution. **Jeffrey HUI Pak Nam** (La Salle College, Form 6) and **Math Center** (Carmel Alison Lam Foundation Secondary School).

For integer $n > 1$, let $c_k = \cos(2k\pi/n)$ for $k = 1, 2, \ldots, n$. We have $c_n = 1$, $c_k = c_{n-k}$ and

$$\sum_{k=1}^{n} c_k = \operatorname{Re}\sum_{k=1}^{n}\omega^k = \operatorname{Re}\frac{1-\omega^n}{1-\omega} = 0, \qquad (*)$$

where $\omega = e^{2\pi i/n}$.

Suppose $n = 2m+1$, where $m = 1,2,3,\ldots$. We have $c_1+c_2+\cdots+c_m = -1/2$ (using $c_n = 1$ and $c_k = c_{n-k}$). Hence

$$(2m+2)(c_1+c_2+\cdots+c_m) = -(m+1)c_{2m+1}.$$

Since $c_k = c_{2m+1-k}$, we have

$$(2m+1)c_1+2mc_2+\cdots+(m+2)c_m = (m+2)c_{m+1}+\cdots+2mc_{2m-1}+(2m+1)c_{2m}.$$

Subtracting the two displayed equations above and transposing all terms to the left, we get

$$\sum_{k=1}^{m} kc_k + \sum_{k=m+1}^{2m}(k+1)c_k + (m+1)c_{2m+1} = 0.$$

This solves the cases $n = 3,5,7,\ldots$.

Next, assuming the case n is true, we will show the case $2n$ is also true. Let $d_m = \cos(m\pi/n)$ for $m = 1,2,\ldots,2n$. The case n gives us an equation of the form

$$a_1d_2 + a_2d_4 + \cdots + a_nd_{2n} = 0, \qquad (**)$$

where $a_1, a_2, \ldots, a_{2n}$ is a permutation of $1,2,\ldots,n$.

Using (*), we have

$$d_1 + d_2 + \cdots + d_{2n} = \sum_{k=1}^{2n}\cos\frac{2k\pi}{2n} = 0$$

and

$$d_2 + d_4 + \cdots + d_{2n} = \sum_{k=1}^{n}\cos\frac{2k\pi}{n} = 0.$$

Subtracting these equations, we have $d_1+d_3+\cdots+d_{2n-1}=0$. For $k=1,3,\ldots,2n-1$, we have

$$d_{2n-k} = \cos((2n-k)\pi/n) = \cos(k\pi/n) = d_k.$$

Using this, $d_1+3d_3+\cdots+(2n-1)d_{2n-1} = d_{2n-1}+3d_{2n-3}+\cdots+(2n-1)d_1$. Adding the left and right sides, we get the equation $2n(d_1+d_3+\cdots+d_{2n-1}) = 0$. So

$$d_1+3d_3+\cdots+(2n-1)d_{2n-1} = 0. \qquad (***)$$

Finally, taking twice the equation in (**) and adding it to the equation in (***), we solve the case $2n$.

Comments: **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade'' Secondary School, Buzău, Romania) pointed out that Problem 435 is the same as Problem 26753 in the Romanian Mathematical Gazette (G.M.-B) and a solution was appeared in G.M-B, No. 10, 2013, pp. 468-469.

Olympiad Corner

(*Continued from page 1*)

Problem 4. Determine all positive integers $n \geq 2$ satisfying

$$i + j \equiv \binom{n}{i} + \binom{n}{j} \pmod{2}$$

for all i and j such that $0 \leq i \leq j \leq n$.

Problem 5. Let $ABCDEF$ be a cyclic hexagon satisfying $AB\perp BD$ and $BC=EF$. Let P be the intersection of lines BC and AD and let Q be the intersection of lines EF and AD. Assume that P and Q are on the same side of D and that A is on the opposite side. Let S be the midpoint of AD. Let K and L be the centres of the incircles of $\triangle BPS$ and $\triangle EQS$ respectively. Prove that $\angle KDL = 90^\circ$.