Mathematical Excalibur

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Olympiad Corner

The 35th International Mathematical Olympiad was held in Hong Kong last summer. The following are the six problems given to the contestants. How many can you solve? (The country names inside the parentheses are the problem proposers.) - *Editors*

Problem 1. (France)
Let m and n be positive integers. Let $a_1, a_2, \ldots, a_m$ be distinct elements of $\{1, 2, \ldots, n\}$ such that whenever $a_i + a_j \le n$ for some i, j, $1 \le i \le j \le m$, there exists k, $1 \le k \le m$, with $a_i + a_j = a_k$. Prove that

$$\frac{a_1+a_2+\cdots+a_m}{m} \ge \frac{n+1}{2}$$

Problem 2. (Armenia/Australia)
ABC is an isosceles triangle with $AB = AC$. Suppose that

(i) M is the midpoint of BC and O is the point on the line AM such that OB is perpendicular to AB;

(ii) Q is an arbitrary point on the segment BC different from B and C;

(iii) E lies on the line AB and F lies on the line AC such that E, Q and F are distinct and collinear.

Prove that OQ is perpendicular to EF if and only if $QE = QF$.

(*continued on page 4*)

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The editors welcome contributions from all students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in TeX, MS Word and WordPerfect, are encouraged. The deadline for receiving material for the next issue is January 31, 1995. Send all correspondence to:

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Pigeonhole Principle

Kin-Yin Li

What in the world is the pigeonhole principle? Well, this famous principle states that if $n+1$ objects (pigeons) are taken from n boxes (pigeonholes), then at least two of the objects will be from the same box. This is clear enough that it does not require much explanation. A problem solver who takes advantage of this principle can tackle certain combinatorial problems in a manner that is more elegant and systematic than case-by-case. To show how to apply this principle, we give a few examples below.

Example 1. Suppose 51 numbers are chosen from 1, 2, 3, ..., 99, 100. Show that there are two which do not have any common prime divisor.

Solution. Let us consider the 50 pairs of consecutive numbers (1,2), (3,4), ..., (99,100). Since 51 numbers are chosen, the pigeonhole principle tells us that there will be a pair $(k, k+1)$ among them. Now if a prime number p divides $k+1$ and k, then p will divide $(k+1) - k = 1$, which is a contradiction. So, k and $k+1$ have no common prime divisor.

Example 2. Suppose 51 numbers are chosen from 1, 2, 3, ..., 99, 100. Show that there are two such that one divides the other.

Solution. Consider the 50 odd numbers 1, 3, 5, ..., 99. For each one, form a box containing the number and all powers of 2 times the number. So the first box contains 1, 2, 4, 8, 16, ... and the next box contains 3, 6, 12, 24, 48, ... and so on. Then among the 51 numbers chosen, the pigeonhole principle tells us that there are two that are contained in the same box. They must be of the form $2^m k$ and $2^n k$ with the same odd number k. So one will divide the other.

Note that the two examples look alike, however the boxes formed are quite different. By now, the readers must have observed that forming the right boxes is the key to success. Often a certain amount of experience as well as clever thinking are required to solve such problems. The additional examples below will help beginners become familiar with this useful principle.

Example 3. Show that among any nine distinct real numbers, there are two, say a and b, such that

$$0 < (a-b)/(1+ab) < \sqrt{2}-1.$$

Solution. The middle expression $(a-b)/(1+ab)$ reminds us of the formula for $\tan(x-y)$. So we proceed as follow. Divide the interval $(-\pi/2,\pi/2]$ into 8 intervals $(-\pi/2,-3\pi/8]$, $(-3\pi/8,-\pi/4]$, ..., $(\pi/4,3\pi/8]$, $(3\pi/8,\pi/2]$. Let the numbers be $a_1, a_2, \ldots, a_9$ and let $x_i = \arctan a_i$, $i = 1, 2, \ldots, 9$. By the pigeonhole principle, two of the x_i's, say x_j and x_k with $x_j > x_k$, must be in one of the 8 subintervals. Then we have $0 < x_j - x_k < \pi/8$, so $0 < \tan(x_j-x_k) = (a_j-a_k)/(1+a_ja_k) < \tan(\pi/8) = \sqrt{2} - 1$.

Example 4. Suppose a triangle can be placed inside a square of unit area in such a way that the center of the square is not inside the triangle. Show that one side of the triangle has length less than 1. (This example came from the *XLI Mathematical Olympiad in Poland.*)

Solution. Through the center C of the square, draw a line L_1 parallel to the closest side of the triangle and a second line L_2 perpendicular to L_1 at C. The lines L_1 and L_2 divide the square into four congruent quadrilaterals. Since C is not

(*continued on page 2*)

Pigeonhole Principle

(*continued from page 1*)

inside the triangle, the triangle can lie in at most two (adjacent) quadrilaterals. By the pigeonhole principle, two of the vertices of the triangle must belong to the same quadrilateral. Now the furthest distance between two points in the quadrilateral is the distance between two of its opposite vertices, which is at most 1. So the side of the triangle with two vertices lying in the same quadrilateral must have length less than 1.

Below we provide some exercises for the active readers.

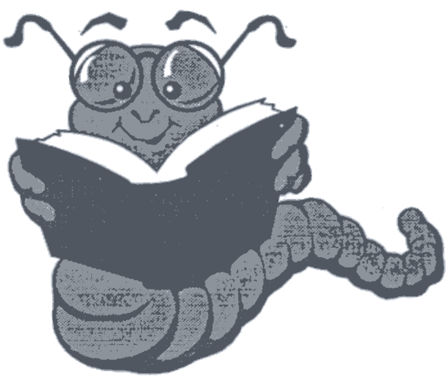
1. Eleven numbers are chosen from 1, 2, 3, ..., 99, 100. Show that there are two nonempty disjoint subsets of these eleven numbers whose elements have the same sum.

2. Suppose nine points with integer coordinates in the three dimensional space are chosen. Show that one of the segments with endpoints selected from the nine points must contain a third point with integer coordinates.

3. Show that among any six people, either there are three who know each other or there are three, no pair of which knows each other.

4. In every 16-digit number, show that there is a string of one or more consecutive digits such that the product of these digits is a perfect square. [*Hint:* The exponents of a factorization of a perfect square into prime numbers are even.] (This problem is from the *1991 Japan Mathematical Olympiad.*)

(*Answers can be found on page 3.*)



The Game of "Life"

Tsz-Mei Ko

The game of "Life" was first introduced by John Conway, a mathematician and a game hobbyist currently working at Princeton University. The game is played on an infinite chessboard, where each cell has eight neighboring cells. Initially, an arrangement of stones is placed on the board (the live cells) as the first generation. Each new generation is determined by two simple generic rules:

The Death Rule: Consider a live cell (occupied by a stone). If it has 0 or 1 live neighbors (among the eight neighboring cells), then it dies from isolation. If it has 4 or more live neighbors, then it dies from overcrowding. If it has 2 or 3 live neighbors, then it survives to the next generation.

The Birth Rule: Consider a dead (unoccupied) cell. If it has exactly 3 live neighbors, then it becomes a live cell (with a stone placed on it) in the next generation.

Here is an example. The six circles in Figure 1 indicate the live cells in the first generation. Those marked i and c will die due to isolation and overcrowding respectively (Death Rule). The empty cells marked b will become live cells in the next generation (Birth Rule). The second generation is shown in Figure 2.

What will happen in the third, fourth, and *n*th generation? Is there an initial generation that will grow infinitely?

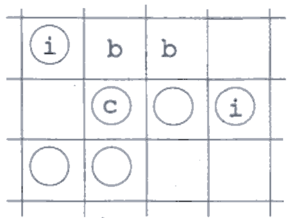


Figure 1

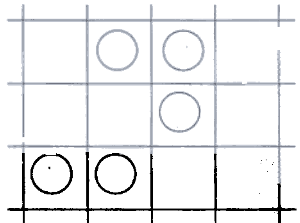
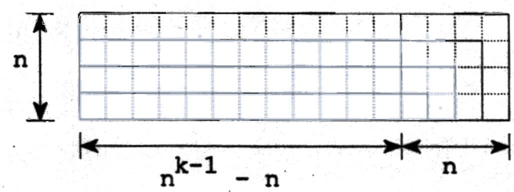


Figure 2

Proof Without Words

kth Power of a Natural Number n as the Sum of n Consecutive Odd Numbers (k = 2,3,...)



$$n^k = (n^{k-1} - n + 1) + (n^{k-1} - n + 3) + \cdots + (n^{k-1} - n + 2n - 1)$$

Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceeded by the solver's name, address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is *January 31st, 1995.*

Problem 1. The sum of two positive integers is 2310. Show that their product is not divisible by 2310.

Problem 2. Given N objects and $B(\geq 2)$ boxes, find an inequality involving N and B such that if the inequality is satisfied, then at least two of the boxes have the same number of objects.

Problem 3. Show that for every positive integer n, there are polynomials $P(x)$ of degree n and $Q(x)$ of degree n-1 such that $(P(x))^2 - 1 = (x^2-1)(Q(x))^2$.

Problem 4. If the diagonals of a quadrilateral in the plane are perpendicular, show that the midpoints of its sides and the feet of the perpendiculars dropped from the midpoints to the opposite sides lie on a circle.

Problem 5. (*1979 British Mathematical Olympiad*) Let $a_1, a_2, \ldots, a_n$ be n distinct positive odd integers. Suppose all the differences $|a_i - a_j|$ are distinct, $1 \leq i < j \leq n$. Prove that $a_1 + a_2 + \cdots + a_n \geq n(n^2+2)/3$.

Answers to Exercises in "Pigeonhole Principle"

1. The set of eleven numbers have $2^{11}-2 =$ 2046 nonempty subsets with less than eleven elements, and the maximal sum of the elements in any of these subsets is $91 + 92 + \cdots + 99 + 100 = 955$. So, by the pigeonhole principle, there are two nonempty subsets with the same sum. If they have common elements, then remove them from both subsets and we will get two nonempty disjoint subsets with the same sum.

2. For the nine points, each of the three coordinates is either even or odd. So, there are 2^3=8 parity patterns for the coordinates. By the pigeonhole principle, two of the nine points must have the same parity coordinate patterns. Then their midpoint must have integer coordinates.

3. Let the six people correspond to the six vertices of a regular hexagon. If two people know each other, then color the segment with the associated vertices red, otherwise blue. Solving the problem is equivalent to showing that a red triangle or a blue triangle exists.

Take any vertex. By the pigeonhole principle, of the five segments issuing from this vertex, three have the same color c. Consider the three vertices at the other ends of these segments and the triangle T with these vertices. If T has an edge colored c, then there is a triangle with color c. Otherwise, all edges of T are colored opposite to c. In both cases, there is a triangle with all edges the same color.

4. Let $d_1, d_2, \ldots, d_{16}$ be the digits of a 16-digit number. If one of the digits of the sixteen digits is either 0 or 1 or 4 or 9, then the problem is solved. So, we may assume each of the digits is 2, 3, 5, 6=2x3, 7 or 8=2^3. Let $x_0 = 1$ and x_i be the product of $d_1, d_2, \ldots, d_i$ for $i = 1, 2, \ldots, 16$. Now each $x_i = 2^{p_i} \times 3^{q_i} \times 5^{r_i} \times 7^{s_i}$ for $i = 0, 1, 2, \ldots, 16$. Each of the p_i, q_i, r_i, s_i is either even or odd. So there are $2^4 = 16$ possible parity patterns. By the pigeonhole principle, the p_i, q_i, r_i, s_i for two of the seventeen x_i's, say x_j and x_k with $j < k$, must have the same parity pattern. Then $d_{j+1} \times \cdots \times d_k = x_k/x_j$ is a perfect square.

Mathematical Application: Pattern Design

Roger Ng

Mathematics is by far the most powerful tool that human race has created. We invite articles which can share with us different areas of applications in mathematics. We wish that this column will inspire students to study mathematics. - *Editors*

In this first issue, I would like to introduce an interesting application which exemplifies the power of mathematics to define an artistic work in a formalised manner.

Take a look at your school uniform. It is made up of patches of fabrics. Before the fabric is cut, the overall shape and measurement of each patch must be drawn. Each patch is known as a pattern piece.

When a pattern is drawn, it must match the surface of a human body. Therefore, the pattern design process is in fact a surface unfolding problem.

What makes the pattern design process an artistic activity is the drawing of curves in the pattern. Each person has his/her own preference. That is why some brand manufacturers can produce better looking garments.

To see how mathematics can be applied, let us consider a specific problem in curve drawing. Take a look at your pant or your skirt. Do you see any smooth overlapping at the center front where you button up the garment? If the pattern is not drawn correctly, you should see a scissor-like crossing at the opening along the waist.

You may imagine that to button up your pant is equivalent to let two curves meet at the same point x. In mathematics, we define two types of continuity conditions, namely, C^0 and C^1 (Figure 1). C^0 means that the two curves meet at the point x, i.e., $f(x^-) = f(x^+)$. C^1 means that the two curves have the same slope at x, i.e., $f'(x^-) = f'(x^+)$.

There will be a smooth overlapping when a pant is buttoned up if both continuity conditions C^0 and C^1 are met. Thus clever fashion designers use a ruler to keep track of the slope $f'(x)$ (Figure 1). This technique dramatically improves the quality of a garment.

In the above example, we see how the continuity concept in mathematics can help a fashion designer to improve the smoothness of a pattern and thus to design nice-looking garments. In fact, there are many other such areas where mathematics can be useful.

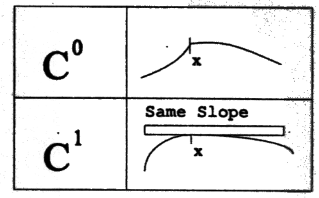


Figure 1

Olympiad Corner

(continued from page 1)

Problem 3. (Romania)
For any positive integer k, let $f(k)$ be the number of elements in the set $\{k+1, k+2, ..., 2k\}$ whose base 2 representation has precisely three 1s.

(a) Prove that, for each positive integer m, there exists at least one positive integer k such that $f(k)=m$.
(b) Determine all positive integers m for which there exists exactly one k with $f(k)=m$.

Problem 4. (Australia)
Determine all ordered pairs (m,n) of positive integers such that

$$\frac{n^3+1}{mn-1}$$

is an integer.

Problem 5. (United Kingdom)
Let S be the set of real numbers strictly greater than -1. Find all functions $f: S \to S$ satisfying the two conditions:

(i) $f(x+f(y)+xf(y)) = y + f(x) + yf(x)$ for all x and y in S;
(ii) $f(x)/x$ is strictly increasing on each of the intervals $-1 < x < 0$ and $0 < x$.

Problem 6. (Finland)
Show that there exists a set A of positive integers with the following property: For any infinite set S of primes there exist two positive integers $m \in A$ and $n \notin A$ each of which is a product of k distinct elements of S for some $k \geq 2$.

From Fermat Primes to Constructible Regular Polygons

Tsz-Mei Ko

Pierre de Fermat (1601-1665), an amateur mathematician, once guessed that all numbers in the form $2^{2^n}+1$ are prime numbers. If we try the first five n's ($n = 0, 1, 2, 3, 4$), they are in fact all primes:

	$2^{2^n}+1$
0	3
1	5
2	17
3	257
4	65537

It was later discovered by Leonhard Euler (1707-1783) in 1732 that the next Fermat number ($n = 5$) can be factored as

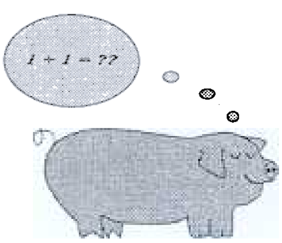
$$2^{2^5}+1 = 641 \times 6700417$$

and thus not a prime. The story would have ended here if without an ingenious discovery by Carl Friedrich Gauss (1777-1855).

In 1794, at the age of seventeen, Gauss found that a regular "p-gon" (a polygon with p sides), where p is a prime, is constructible (i.e., using only ruler and compass) if and only if p is a "Fermat prime" (a prime number in the form $2^{2^n}+1$). He proved this by considering the solutions of certain algebraic equations. (The interested reader may refer to the book, "What Is Mathematics?" written by Courant and Robbins, Oxford University Press.) The young Gauss was so overwhelmed by his discovery that he then decided to devote his life to mathematics. After his death, a bronze statue in memory of him standing on a regular 17-gon pedestal was erected in Brauschweig−the hometown of Gauss.

Which regular polygons are constructible? From Gauss's result, we know that the regular triangle, pentagon, 17-gon, 257-gon and 65537-gon are constructible. (How?) We also know that regular polygons with 7, 11, 13, 19, ··· sides are not constructible since they are primes but not Fermat primes. In addition, we know how to bisect an angle and thus regular polygons with 4, 8, 16, 32, ··· or 6, 12, 24, 48, ··· sides are also constructible. What about the others? Is a regular 15-gon constructible? The answer turns out to be yes since $1/15 = 2/5 - 1/3$ and thus we can divide a circle into 15 equal parts. What about a regular 9-gon? It can be proved that a regular 9-gon is not constructible. Can you find a general theorem on which regular polygons are constructible?

Are there any other constructible p-gons (where p is a prime) besides the five mentioned? This question is equivalent to asking whether there are any other Fermat primes. To date, no other Fermat number has been shown to be prime, and it is still not known whether there are more than five Fermat primes. Perhaps you can discover a new Fermat prime and make a note in the history of mathematics.



Right: A photo of the six members of the Hong Kong Team and one of the editors (far right) taken at the Shatin Town Hall after the closing ceremony of the 35th International Mathematical Olympiad.

From left to right are: Suen Yun-Leung, Chu Hoi-Pan, Tsui Ka-Hing, Wong Him-Ting, Ho Wing-Yip, Poon Wai-Hoi Bobby, and Li Kin-Yin.