Mathematical Excalibur

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Olympiad Corner

The following are the six problems from the two-day Final Selection Exam for the 1994 Hong Kong Mathematical Olympiad Team. Would you like to try these problems to see if you could have qualified to be a Hong Kong team member? - *Editors*

Instructions (the same instructions were given on both days): *Answer all three questions. Each question carries 35 points. Time allowed is 4½ hours.*

<u>First Day</u>

Question 1. In a triangle ΔABC, $\angle C=2\angle B$. P is a point in the interior of ΔABC satisfying that $AP = AC$ and $PB = PC$. Show that AP trisects the angle $\angle A$.

Question 2. In a table-tennis tournament of 10 contestants, any two contestants meet only once. We say that there is a winning triangle if the following situation occurs: ith contestant defeated jth contestant, jth contestant defeated kth contestant, and kth contestant defeated ith contestant. Let W_i and L_i be respectively the number of games won and lost by the ith contestant. Suppose $L_i + W_j \geq 8$ whenever the ith contestant wins

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Editors: Cheung, Pak-Hong, Curr. Studies, HKU
Ko,Tsz-Mei, EEE Dept, HKUST
Leung, Tat-Wing, Appl. Math Dept, HKPU
Li, Kin-Yin, Math Dept, HKUST
Ng, Keng Po Roger, ITC, HKPU

Artist: Yeung, Sau-Ying Camille, Fine Arts Dept, CU

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The editors welcome contributions from all students. With your submission, please include your name, address, school, email address, telephone and fax numbers (if available). Electronic submissions, especially in TeX, MS Word and WordPerfect, are encouraged. The deadline for receiving material for the next issue is March 31, 1995. Send all correspondence to:

Dr. Li, Kin-Yin
Department of Mathematics
Hong Kong University of Science and Technology
Clear Water Bay, Kowloon
Hong Kong

Fax: 2358-1643
Email: makyli@uxmail.ust.hk

Fractal Game of Escape

Roger Ng

Consider the following scenario. John, a secret agent, is being held captive in terrorists' headquarters. He has found an escape route, and knows it follows the quadratic equation $z_{n+1} = z_n^2 + c$ if the floor map is encoded as a complex z-plane (i.e., each point (x,y) is represented by a complex number $x+yi$). However, John does not know the value of the complex constant c. John only knows that he should start from the origin with $z_0 = 0 + 0i$. For which values of c, will John have not even a chance for a successful escape?

To help John to answer the above question, it is natural to first try $c = 0$ and see what will happen. The recursion becomes $z_{n+1} = z_n^2$ and thus $z_n = 0$ for all n. That is, John will be going nowhere but staying at the origin!

If we try other values of c, there are three possible outcomes: (1) the sequence z_n converges to a fixed point; (2) the sequence z_n repeats in a finite cycle of points and thus becomes a periodic sequence; or (3) the sequence z_n diverges from the origin, i.e., John may have a chance to escape successfully.

The above story is a dramatization for the definition of a fractal called the Mandelbrot set. (The word "fractal" was coined by Benoit Mandelbrot to describe sets with self-similarity, i.e., they look the same if you magnify a portion of them.) The Mandelbrot set can be defined as the set of complex numbers c for which the sequence $z_{n+1} = z_n^2 + c$ is bounded (i.e., does not diverge) when the starting point z_0 is the origin (0,0). Figure 1 shows the asymptotic behaviour of z_n for real c's that generate bounded sequences (i.e., outcomes 1 and 2). The number of points on a vertical line indicates the period of the asymptotic sequence. Figure 2 shows the values for c (the black area) that would keep z_n bounded, i.e., the Mandelbrot set.

Now if we modify our story slightly–assume that John knows the constant c but not the starting point z_0, this will lead us to the definition of Julia sets–named after the mathematician Gaston Julia (1893-1978). For any given complex number c, some initial points z_0 generate divergent sequences $z_{n+1} = z_n^2 + c$ while others generate nondivergent sequences. The Julia set is the boundary that separates the set of "diverging" starting points from the set of "nondiverging" starting points.

Here is a simple example. For $c = 0$, the equation is $z_{n+1} = z_n^2$. If the starting point lies within a distance of 1 from the origin, the subsequent points will get closer and closer to the origin. If the intial point is more than a distance of 1 from the origin, the subsequent points will get farther and farther away from the origin. The unit circle separates these two sets of starting points. This boundary is the Julia set corresponding to $c = 0$.

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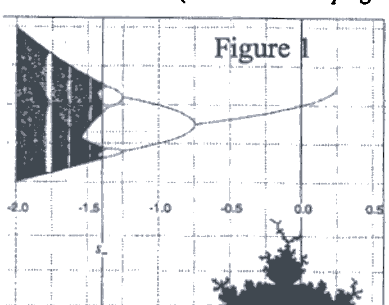

Figure 1


Figure 2
The Mandelbrot Set

Fractal Game of Escape

(*continued from page 1*)

By varying c, we will obtain an infinite number of different pictures of Julia sets. Some examples are shown in the figures on this page. However, no matter what c is, we observe that there are basically two major types of Julia sets. Either all the points z_0 are connected in one piece, or these points are broken into a number of pieces (in fact, an infinite number of pieces to form something called a Cantor set).

We may ask ourselves an interesting question. For which values of c, will the corresponding Julia set be connected? This seems to be a very hard problem. It seems that we need to look at all Julia sets to find out which one is connected, and it would take an eternity to compile this huge amount of data. But mathematicians John Hubbard and Adrien Douady found a quick way to carry out this task. They proved that a Julia set is connected if the sequence $z_{n+1} = z_n^2 + c$ is bounded when the starting point z_0 is the origin (0,0). That is, if c belongs to the Mandelbrot set, then its corresponding Julia set will be connected! Thus the Mandelbrot set is known as the table of contents for all Julia sets.

Besides this interesting relationship and the fascinating pictures, the Julia set and many other fractals provide us insight into many physical phenomenon. As an example, the Julia Set is directly related to the equipotential field lines of an electrostatic circular metal rod. The interested reader may refer to the book "Chaos and Fractals: New Frontiers of Science," written by H.O. Peitgen, H. Jürgens, and D. Saupe (Springer Verlag, 1992).

Due to the self-similarity of fractals, one usually needs only a few lines of computer programming to generate a fractal image. (Would you like to try?) There is also a free computer software FRACTINT (developed by the Stone Soup Group) that can generate many popular fractal images. If you would like to get a copy of this computer software, send a stamped self-addressed envelope and a PC-formatted high-density diskette to the author at the following address: *Roger Ng, Institute of Textile and Clothing, Hong Kong Polytechnic University, Hung Hom, Kowloon.* There are over a hundred fractal images for your investigation.

Pythagorean Triples

Kin-Yin Li

In geometry, we often encounter triangles whose sides are integers. Have you ever thought about how to produce many nonsimilar triangles of this kind *without guessing*? For this, we first define *Pythagorean triples* to be triples (a, b, c) of positive integers satisfying $a^2 + b^2 = c^2$. For example, (3, 4, 5) and (5, 12, 13) are Pythagorean triples. Clearly, if $a^2 + b^2 = c^2$, then $(ad)^2 + (bd)^2 = (cd)^2$ for any positive integer d. So, solutions of $a^2 + b^2 = c^2$ with a, b, c relatively prime (i.e., having no common prime divisors) are important. These are called *primitive solutions.* Below we will establish a famous theorem giving all primitive solutions.

Theorem. If u, v are relatively prime positive integers, $u > v$ and one is odd, the other even, then $a = u^2 - v^2$, $b = 2uv$, $c = u^2 + v^2$ give a primitive solution of $a^2 + b^2 = c^2$. Conversely, every primitive solution is of this form, with a possible permutation of a and b.

For example, $u = 2$, $v = 1$ corresponds to $a = 3$, $b = 4$, $c = 5$. Now let us try to see why the theorem is true. For the first statement, simple algebra shows $a^2 + b^2 = u^4 + 2u^2v^2 + v^4 = c^2$. If two of a, b, c have a common prime divisor p, then the equation will imply all three have p as a common divisor and $p \neq 2$. It will also follow that $(c - a)/2 = u^2$ and $(c + a)/2 = v^2$ are integers with p as a common divisor. This will contradict u, v being relatively prime. So a, b, c must be relatively prime.

For the second statement, we introduce modulo arithmetic. If r, s are integers having the same remainder upon division by a positive integer m, then we say *r is congruent to s modulo m* and let us denote this by $r \equiv s \pmod{m}$. For example, $r \equiv 0$ or $1 \pmod{2}$ depending on whether r is even or odd. From the definition, we see that congruence is an equivalence relation between r and s. Also, if $r \equiv s \pmod{m}$ and $r' \equiv s' \pmod{m}$, then $r + r' \equiv s + s' \pmod{m}$, $r - r' \equiv s - s' \pmod{m}$, $rr' \equiv ss' \pmod{m}$ and $r^k \equiv s^k \pmod{m}$ for any positive integer k.

In working with squares, modulo 4 is often considered. This comes from the observation that $r^2 \equiv 0$ or $1 \pmod{4}$ depending on r is even or odd. Now, if $a^2 + b^2 = c^2$, then $a^2 + b^2 \equiv 0$ or $1 \pmod{4}$.

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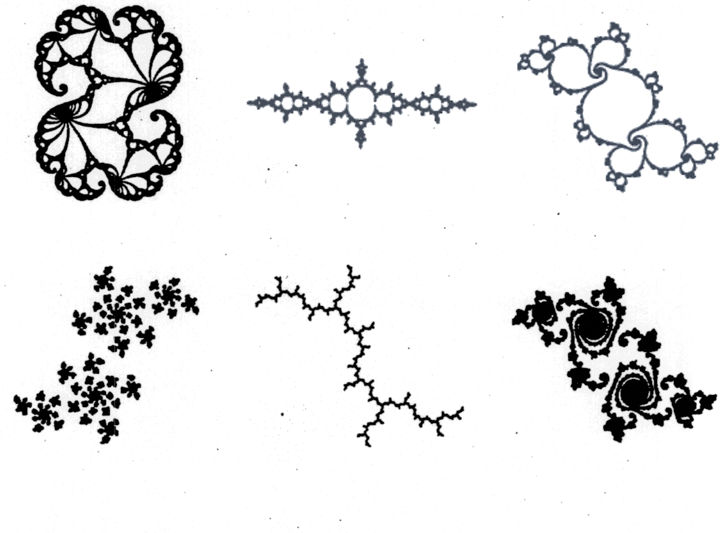
Julia Sets for Various Values of *c*



Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* Solutions to the following problems should be submitted by *March 31, 1995.*

Problem 6. For quadratic polynomials $P(x) = ax^2 + bx + c$ with real coefficients satisfying $|P(x)| \le 1$ for $-1 \le x \le 1$, find the maximum possible values of b and give a polynomial attaining the maximal b coefficient.

Problem 7. If positive integers a, b, c satisfy $a^2 + b^2 = c^2$, show that there are at least three noncongruent right triangles with integer sides having hypotenuses all equal to c^3.

Problem 8. (1963 Moscow Mathematical Olympiad) Let $a_1 = a_2 = 1$ and $a_n = (a_{n-1}^2 + 2)/a_{n-2}$ for $n = 3, 4, \cdots$. Show that a_n is an integer for $n = 3, 4, \cdots$.

Problem 9. On sides AD and BC of a convex quadrilateral $ABCD$ with $AB < CD$, locate points F and E, respectively, such that $AF/FD = BE/EC = AB/CD$. Suppose EF when extended beyond F meets line BA at P and meets line CD at Q. Show that $\angle BPE = \angle CQE$.

Problem 10. Show that every integer $k > 1$ has a multiple which is less than k^4 and can be written in base 10 with at most four different digits. [*Hint:* First consider numbers with digits 0 and 1 only.] (This was a problem proposed by Poland in a past IMO.)

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Solutions

Problem 1. The sum of two positive integers is 2310. Show that their product is not divisible by 2310.

Solution: W. H. FOK, Homantin Government Secondary School.

Let x, y be two positive integers such that $x + y = 2310$. Suppose xy is divisible by 2310, then $xy = 2310n$ for some positive integer n. We get $x + (2310n/x) = 2310$. So $x^2 - 2310x + 2310n = 0$. It follows the discriminant $\Delta = 2310^2 - 4(2310n) = 2^2 \times 3 \times 5 \times 7 \times 11 \times (1155 - 2n)$ must be a perfect square. Then for some positive integer k, $1155 - 2n = 3 \times 5 \times 7 \times 11 \times k^2 = 1155k^2 \ge 1155$, which is a contradiction. So xy is not divisible by 2310.

Comments: A similar problem appeared in the magazine *Quantum*, Sept./Oct. 1993, p. 54, published by Springer-Verlag.

Other commended solvers: **AU Kwok Nin** (Tsung Tsin College), **HO Wing Yip** (Clementi Secondary School), **POON Wai Hoi Bobby** (St. Paul's College) and **SZE Hoi Wing** (St. Paul's Co-ed College).

Problem 2. Given N objects and $B(\ge 2)$ boxes, find an inequality involving N and B such that if the inequality is satisfied, then at least two of the boxes have the same number of objects.

Solution: POON Wai Hoi Bobby, St. Paul's College.

Denote the number of objects in the kth box by N_k. Suppose no two boxes have the same number of objects. Then $N = N_1 + N_2 + \cdots + N_B \ge 0 + 1 + 2 + \cdots + (B-1) = B(B-1)/2$. So if $N < B(B-1)/2$, then at least two of the boxes have the same number of objects.

Other commended solvers: **CHAN Wing Sum** (HKUST), **W. H. FOK** (Homantin Government Secondary School), and **HO Wing Yip** (Clementi Secondary School).

Problem 3. Show that for every positive integer n, there are polynomials $P(x)$ of degree n and $Q(x)$ of degree n-1 such that $(P(x))^2 - 1 = (x^2-1)(Q(x))^2$.

Solution: POON Wai Hoi Bobby, St. Paul's College.

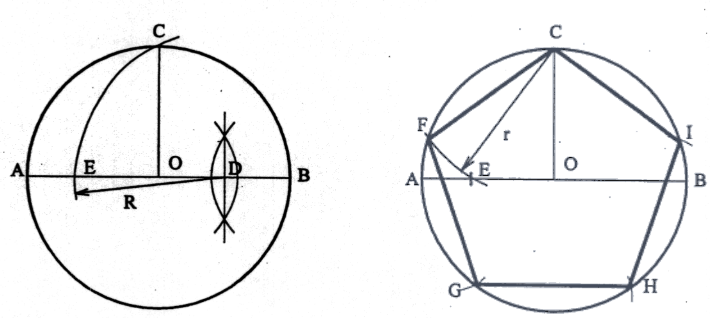
For $k = 1, 2, \cdots$, define $P_k(x)$, $Q_k(x)$ by $P_1(x) = x$, $Q_1(x) = 1$, $P_{k+1}(x) = xP_k(x) + (x^2-1)Q_k(x)$ and $Q_{k+1}(x) = P_k(x) + xQ_k(x)$. We can check that the degree of P_n is n and the degree of Q_n is n-1 by showing inductively that $P_n(x) = 2^{n-1}x^n + \cdots$ and $Q_n(x) = 2^{n-1}x^{n-1} + \cdots$. For the problem, when $n = 1$, $P_1(x)^2 - 1 = x^2 - 1 = (x^2-1)Q_1(x)^2$. Suppose the case $n = k$ holds. Then

$$\begin{aligned} P_{k+1}(x)^2 - 1 &= [xP_k(x) + (x^2-1)Q_k(x)]^2 - 1 \\ &= (x^2-1)[P_k(x)^2 + 2xP_k(x)Q_k(x) \\ &\quad + (x^2-1)Q_k(x)^2] + P_k(x)^2 - 1 \\ &= (x^2-1)[P_k(x)^2 + 2xP_k(x)Q_k(x) \\ &\quad + (x^2-1)Q_k(x)^2] + (x^2-1)Q_k(x)^2 \\ &= (x^2-1)Q_{k+1}(x)^2. \end{aligned}$$

Comments: The solvers mainly observed that if we substitute $x = \cos\theta$, then $P_k(\cos\theta) = \cos k\theta$ and $Q_k(\cos\theta) = \sin k\theta / \sin\theta$. The recurrence relations for P_{k+1} and Q_{k+1} are just the usual identities for $\cos(k\theta + \theta)$ and $\sin(k\theta + \theta)$. The polynomials P_k, Q_k are

(*continued on page 4*)

Construction Without Words: Inscribe a Regular Pentagon in a Unit Circle



Side length $= 2\sin 36° = \dfrac{\sqrt{10 - 2\sqrt{5}}}{2} = \sqrt{1 + \left(\dfrac{\sqrt{5}-1}{2}\right)^2}$.

How would you construct a regular 17-gon inscribed in a given circle?

Problem Corner

(*continued from page 3*)

called *Chebychev polynomials* and have many interesting properties.

We thank Professor Andy Liu (University of Alberta, Canada) for informing us that his colleague Professor Murray Klamkin located this problem in Goursat-Hedrick's "A Course in Mathematical Anaysis", vol. 1, p. 32, published by Ginn and Company in 1904. Professor Klamkin has a calculus solution, first showing Q divides P', then obtaining $Q = nP'$ and solving a differential equation in P to get $P(x) = \cos(n \arccos x)$. Professor Liu also forwarded an alternative recurrence approach by Byung-Kyu Chun, a Korean-Canadian secondary school student. He observed that $P_n(x) = 2xP_{n-1}(x) - P_{n-2}(x)$ and $Q_n(x) = 2xQ_{n-1}(x) - Q_{n-2}(x)$ and showed by simultaneous induction that $P_n(x)P_{n-1}(x) - x = (x^2-1)Q_n(x)Q_{n-1}(x)$ and $P_n(x)^2 - 1 = (x^2-1)Q_n(x)^2$.

Other commended solver: **HO Wing Yip** (Clementi Secondary School).

Problem 4. If the diagonals of a quadrilateral in the plane are perpendicular, show that the midpoints of its sides and the feet of the perpendiculars dropped from the midpoints to the opposite sides lie on a circle.

Solution: Independent solution by **W. H. FOK** (Homantin Government Secondary School) and **POON Wai Hoi Bobby** (St. Paul's College).

Let $ABCD$ be a quadrilateral such that AC is perpendicular to BD. Let E, F, G, H be the midpoints of AB, BC, CD, DA, respectively. By the midpoint theorem, EH, BD, FG are parallel to each other and so are EF, AC, HG. Since AC and BD are perpendicular, $EFGH$ is a rectangle. Hence E, F, G, H are concyclic.

Let M be the foot of the perpendicular from E to CD, then $\angle EMG = \angle EFG = 90^0$. So E, F, M, G, H lie on a circle. Similarly, the other feet of perpendiculars are on the same circle.

Problem 5. (*1979 British Mathematical Olympiad*) Let $a_1, a_2, ..., a_n$ be n distinct positive odd integers. Suppose all the differences $|a_i - a_j|$ are distinct, $1 \le i < j \le n$. Prove that $a_1 + a_2 + \cdots + a_n \ge n(n^2+2)/3$.

Solution: Independent solution by **Julian CHAN Chun Sang** (Lok Sin Tong Wong Chung Ming Secondary School), **W. H. FOK** (Homantin Government Secondary School) and **HO Wing Yip** (Clementi Secondary School).

Without loss of generality, suppose $a_1 < a_2 < \cdots < a_n$. For $k = 2, 3, ..., n$, since the differences are distinct, $a_k = a_1 + (a_2 - a_1) + \cdots + (a_k - a_{k-1}) \ge 1 + (2 + 4 + \cdots + 2(k-1)) = 1 + k^2 - k$. Summing from $k = 1$ to n, we get $a_1 + a_2 + \ldots + a_n \ge n(n^2+2)/3$.

Comments: Ho Wing Yip proved the result by induction on n, which did not require the formula for summing k^2 in the last step.

Pythagorean Triples

(*continued from page 2*)

So, if a, b, c are also relatively prime, then one of a or b is odd and the other is even. Let us say a is odd and b is even. Then c is odd and it follows $m = (c-a)/2$ and $n = (c+a)/2$ are positive integers. Note a (= $m-n$) and c (= $m+n$) relatively prime implies m, n cannot have a common prime divisor. Now considering the prime factorization of $(b/2)^2$, which equals mn, it follows that both m and n are perfect squares with no common prime divisors. Let us say $m = u^2$ and $n = v^2$. Then $a = u^2 - v^2$, $b = 2uv$ and $c = u^2 + v^2$.

Example 1. Show that there are exactly three right triangles whose sides are integers while the area is twice the perimeter as numbers. (This was a problem on the 1965 Putnam Exam, a North American Collegiate Competition.)

Solution: For such a triangle, the sides are of the form $a = (u^2 - v^2)d$, $b = 2uvd$ and $c = (u^2 + v^2)d$, where u, v are relatively prime, $u > v$, one is odd, the other even and d is the greatest common divisor of the three sides. The condition $ab/2 = 2(a+b+c)$ expressed in terms of u, v, d can be simplified to $(u-v)vd = 4$. It follows that $u - v$ being odd must be 1. Then $v = 1, 2$ or 4; $u = 2, 3$ or 5; $d = 4, 2$ or 1 corresponding to the 12-16-20, 10-24-26 and 9-40-41 triangles.

Example 2. Show that there are infinitely many points on the unit circle such that the distance between any two of them is rational. (This was essentially a problem in the 1975 International Mathematical Olympiad).

Solution: Let $A = (-1, 0)$, $B = (1, 0)$ and O be the origin. Consider all points P such that $AP = 2(u^2-v^2)/(u^2+v^2)$ and $BP = 4uv/(u^2+v^2)$, where u, v are as in the theorem. Since $AP^2 + BP^2 = AB^2$, all such P's are on the unit circle. Using similar triangles, we find the coordinates of P is (x,y), where $x = (AP^2/2) - 1$ and $y = \pm AP \cdot BP/2$ are both rational. Let $\theta = \angle BOP = 2\angle BAP$. Then $\cos(\theta/2) = (1+x)/AP$ and $\sin(\theta/2) = |y|/AP$ are rational. Finally, for two such points P and P', $PP' = 2|\sin(\theta-\theta')/2| = 2|\sin(\theta/2)\cos(\theta'/2) - \cos(\theta/2)\sin(\theta'/2)|$ is rational.

Example 3. Find all positive integral solutions of $3^x + 4^y = 5^z$. (cf. W. Sierpinski, On the Equation $3^x + 4^y = 5^z$ (Polish), Wiadom. Mat.(1955/56), pp. 194-5.)

Solution. We will show there is exactly one solution set, namely $x = y = z = 2$. To simplify the equation, we consider modulo 3. We have $1 = 0 + 1^y \equiv 3^x + 4^y = 5^z \equiv (-1)^z \pmod 3$. It follows that z must be even, say $z = 2w$. Then $3^x = 5^z - 4^y = (5^w + 2^y)(5^w - 2^y)$. Now $5^w + 2^y$ and $5^w - 2^y$ are not both divisible by 3, since their sum is not divisible by 3. So, $5^w + 2^y = 3^x$ and $5^w - 2^y = 1$. Then, $(-1)^w + (-1)^y \equiv 0 \pmod 3$ and $(-1)^w - (-1)^y \equiv 1 \pmod 3$. Consequently, w is odd and y is even. If $y > 2$, then $5 \equiv 5^w + 2^y = 3^x \equiv 1$ or $3 \pmod 8$, a contradiction. So $y = 2$. Then $5^w - 2^y = 1$ implies $w = 1$ and $z = 2$. Finally, we get $x = 2$.

Olympiad Corner

(*continued from page 1*)

the jth contestant. Prove that there are exactly 40 winning triangles in this tournament.

Question 3. Find all the non-negative integers x, y, and z satisfying that $7^x + 1 = 3^y + 5^z$.

<u>Second Day</u>

Question 4. Suppose that $yz + zx + xy = 1$ and x, y, and $z \ge 0$. Prove that $x(1-y^2)(1-z^2) + y(1-z^2)(1-x^2) + z(1-x^2)(1-y^2) \le 4\sqrt{3}/9$.

Question 5. Given that a function $f(n)$ defined on natural numbers satisfies the conditions: $f(n) = n - 12$ if $n > 2000$, and $f(n) = f(f(n+16))$ if $n \le 2000$.

(a) Find $f(n)$.

(b) Find all solutions to $f(n) = n$.

Question 6. Let m and n be positive integers where m has d digits in base ten and $d \le n$. Find the sum of all the digits (in base ten) of the product $(10^n - 1)m$.