Mathematical Excalibur

Volume 1, Number 3 May - June, 1995

Olympiad Corner

The Seventh Asian Pacific Mathematics Olympiad was held on March 18, 1995. The five problems given in this contest are listed below for you to try. Time allowed was four hours. - *Editors*

Question 1. Determine all sequences of real numbers $a_1, a_2, \cdots, a_{1995}$ which satisfy:

$$2\sqrt{a_n - (n-1)} \geq a_{n+1} - (n-1)$$

for $n = 1, 2, \cdots, 1994$, and

$$2\sqrt{a_{1995} - 1994} \geq a_1 + 1.$$

Question 2. Let $a_1, a_2, \cdots, a_n$ be a sequence of integers with values between 2 and 1995 such that:

i) any two of the a_i's are relatively prime.

ii) each a_i is either a prime or a product of different primes.

Determine the smallest possible value of n to make sure that the sequence will contain a prime number.

Question 3. Let $PQRS$ be a cyclic quadrilateral (i.e., P, Q, R, S all lie on a circle) such that the segments PQ and RS are not parallel. Consider the set of circles through P and Q, and the set of circles through R and S. Determine the set A of points of tangency of circles in these two sets.

(*continued on page 4*)

Acknowledgment: Thanks to Martha A. Dahlen, Technical Writer, HKUST, for her comments.

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email address, telephone and fax numbers (if available). Electronic submissions, especially in TeX, MS Word and WordPerfect, are encouraged. The deadline for receiving material for the next issue is June 10, 1995. Send all correspondence to:

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Similar Triangles via Complex Numbers

Kin-Yin Li

Similar triangles are familiar to students who studied geometry. Here we would like to look at an algebraic way of describing similar triangles by complex numbers. Recall that every point Z on the coordinate plane corresponds to a complex number $z = r(\cos\theta + i\sin\theta)$, where $r = |z|$ and $\theta = \arg z$ are the polar coordinates of z. (From now on, we will use capital letters for points and small letters for the corresponding complex numbers.)

In general, there are two possible cases for similar triangles. Two triangles are said to be *directly similar* if one can be obtained by translating and rotating the other on the plane, then scaling up or down. (Note a triangle is not directly similar to its reflection unless it is isosceles or equilateral.) Suppose $\Delta Z_1Z_2Z_3$ is directly similar to $\Delta W_1W_2W_3$. Then $Z_2Z_1/Z_3Z_1 = W_2W_1/W_3W_1$ and $\angle Z_2Z_1Z_3 = \angle W_2W_1W_3$. These two equations are equivalent to $|z_2-z_1|/|z_3-z_1| = |w_2-w_1|/|w_3-w_1|$ and $\arg((z_2-z_1)/(z_3-z_1)) = \arg((w_2-w_1)/(w_3-w_1))$, which say exactly that

$$\frac{z_2 - z_1}{z_3 - z_1} = \frac{w_2 - w_1}{w_3 - w_1}.$$

Reversing steps, we see that the equation implies the triangles are directly similar. For the case $\Delta Z_1Z_2Z_3$ directly similar to the reflection of $\Delta W_1W_2W_3$, the equation is

$$\frac{z_2 - z_1}{z_3 - z_1} = \frac{\overline{w_2} - \overline{w_1}}{\overline{w_3} - \overline{w_1}}$$

because $\overline{w_1}, \overline{w_2}, \overline{w_3}$ provide a reflection of w_1, w_2, w_3.

Let $\Delta W_1W_2W_3$ be the equilateral triangle with vertices at $1, \omega, \omega^2 (= \overline{\omega})$, where $\omega = (-1 \pm i\sqrt{3})/2$ is a cube root of unity. We observe that $w_1+\omega w_2+\omega^2 w_3 = 1+\omega^2+\omega^4 = 0$. One can show that this equation is satisfied by any equilateral triangle in general. A triangle $\Delta Z_1Z_2Z_3$ is equilateral if and only if $(z_3-z_1)/(z_2-z_1) = (w_3-w_1)/(w_2-w_1) = -\omega^2$. (Note that $-\omega^2 = \pm(\cos 60° + i\sin 60°)$.) This equation can be simplified to $z_1+\omega z_2+\omega^2 z_3 = 0$ by utilizing $1+\omega+\omega^2=0$. Therefore, a triangle $\Delta Z_1Z_2Z_3$ is equilateral if and only if $z_1+\omega z_2+\omega^2 z_3 = 0$. Here $\omega = (-1 + i\sqrt{3})/2$ when Z_1, Z_2, Z_3 are in counterclockwise direction and $\omega = (-1 - i\sqrt{3})/2$ when Z_1, Z_2, Z_3 are in clockwise direction.

Example 1. (Napolean Triangle Theorem) Given ΔABC. Draw equilateral triangles DBA, ECB, FAC on the opposite sides of AB, BC, CA as ΔABC, respectively. Let G, H, I be the centroids of ΔDBA, ΔECB, ΔFAC, respectively. Show that ΔGHI is equilateral.

Solution. Since $d + \omega b + \omega^2 a = 0$, $e + \omega c + \omega^2 b = 0$, $f + \omega a + \omega^2 c = 0$ and $\omega^3 = 1$, we have

$$\begin{aligned} &g + \omega h + \omega^2 i \\ &= (a+d+b)/3+\omega(b+e+c)/3+\omega^2(c+f+a)/3 \\ &= [(d+\omega b+\omega^2 a) + \omega(e+\omega c+\omega^2 b) \\ &\quad + \omega^2(f+\omega a+\omega^2 c)]/3 = 0. \end{aligned}$$

Example 2. Given an acute triangle $A_1A_2A_3$, let H_1, H_2, H_3 be the feet of the altitudes dropped from A_1, A_2, A_3, respectively. Show that each of the triangles $A_1H_2H_3$, $A_2H_3H_1$, $A_3H_1H_2$ is similar to $\Delta A_1A_2A_3$.

Solution. Set up coordinates so that $A_1 = (0,0)$, $A_2 = (t,0)$ and $A_3 = (x,y)$, i.e., $a_1 = 0$, $a_2 = t$, $a_3 = x+iy$. Observe that $A_1H_2 = A_1A_2 \cos \angle A_1 = tx/\sqrt{x^2+y^2}$. Thus $h_2 = (tx/\sqrt{x^2+y^2})(a_3/|a_3|) = tx(x+iy)/(x^2+y^2)$. Also, $h_3 = x$. Now

$$\frac{h_2 - a_1}{h_3 - a_1} = \frac{t(x+iy)}{x^2+y^2} = \frac{t}{x-iy} = \frac{\overline{a_2} - \overline{a_1}}{\overline{a_3} - \overline{a_1}}.$$

So, in fact, $\Delta A_1H_2H_3$ is similar to (the reflection of) $\Delta A_1A_2A_3$. By changing indices, we also get similarity for the other two triangles.

(*continued on page 4*)

From The Editors' Desk:

This is the last issue for the 94-95 academic year. Thanks for all the supports, comments, suggestions, and especially the elegant solutions for the Problem Corner. We will give out a few book prizes to show our appreciation. We are also planning a Best Paper Award for articles to be submitted in the next academic year. Details will be given in the September issue. Meanwhile, we encourage our readers to spend some spare time writing intriguing articles for the Mathematical Excalibur?

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For the 95-96 academic year, we plan to have five issues to be delivered on Sept, Nov, Jan, Mar and May. If you would like to receive your personal copy directly, send five stamped self-addressed envelopes to Dr. Tsz-Mei Ko, Hong Kong University of Science and Technology, Department of Electrical and Electronic Engineering, Clear Water Bay, Kowloon. Please write "Math Excalibur 95-96" at the lower left corner on all five envelopes.

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We have sent out the computer program FRACTINT to all interested readers. If you have requested but not yet received the software, contact Roger Ng.

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Are you interested in math or in winning a math olympiad gold medal? The Preliminary Selection Exam for the 1996 Hong Kong Math Olympiad Team will be held in Hong Kong Polytechnic University on May 27, 1995. You may ask your math teacher for further information if you are interested in participating in this exam. The 1996 IMO will be held in India.

Cryptarithms and Alphametics

Tsz-Mei Ko

A cryptarithm or alphametic is a puzzle to find the original digits in an encrypted equation which is made by substituting distinct letters for distinct digits in a simple arithmetic problem. Here is an example. Consider the alphametic

```
   AT
+   A
  TEE
```

in which each letter represents a distinct digit. The puzzle is to find the original digits each letter represents so that the result is arithmetically correct.

To solve this puzzle, we may reason as follows. Since T is the "carry" from the "tens" column, T must be equal to 1 and thus we get

```
   A1
+   A
  1EE
```

Now, on the tens column, since $A \neq E$, there must be a carry from the units column, i.e., $A+1 = 10+E$. Thus A=9 and E=0. Therefore, the solution should be

```
   91
+   9
  100
```

We may check our solution that it is arithmetically correct and each letter indeed represents a distinct digit (with A=9, E=0 and T=1). Also, from our reasoning, we see that the solution for this puzzle is unique.

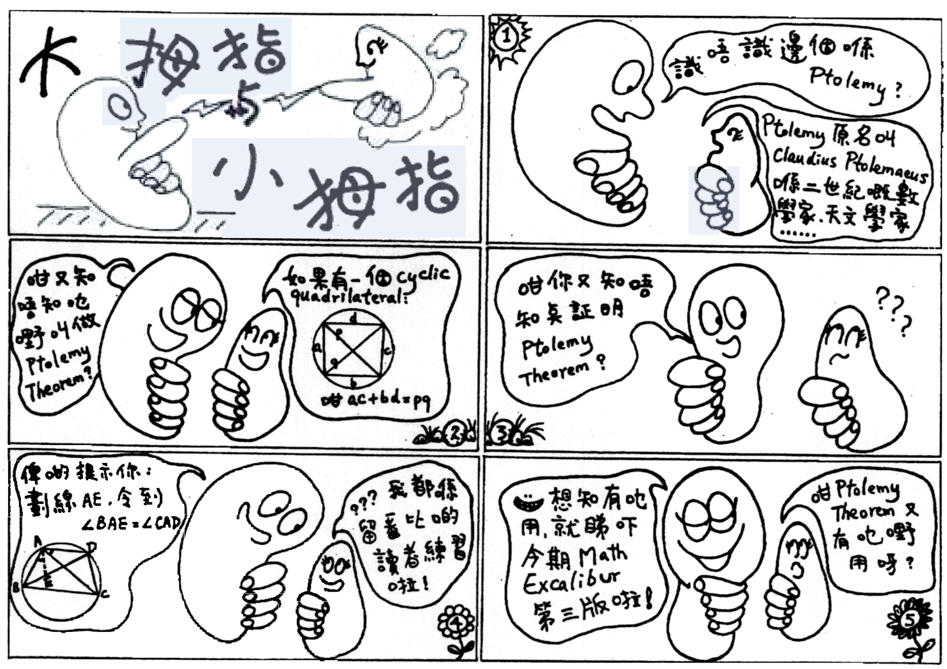
There are many amusing alphametics that make sense in English or some language. Here is one with a unique solution. Do you think you can solve it?

```
  FORTY
    TEN
+   TEN
  SIXTY
```

How about this cryptarithm in which the phrase "Qui Trouve Ceci" means "Who can solve this?" Each letter represents a distinct digit and each # represents any digit (not necessary to be distinct).

```
           CECI
QUI ) TROUVE
      ###
       ###
       ###
        ###
        ##E
         ###
         ###
           V
```

Mathematical Thumbnotes:



Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* Solutions to the following problems should be submitted by *June 10, 1995.*

Problem 11. Simplify

$$\sum_{n=1}^{1995} \tan(n)\tan(n+1).$$

(There is an answer with two terms involving tan 1, tan 1996 and integers.)

Problem 12. Show that for any integer $n > 12$, there is a right triangle whose sides are integers and whose area is between n and $2n$. (*Source:* 1993 Korean Mathematical Olympiad.)

Problem 13. Suppose x_k, y_k $(k = 1, 2, \cdots, 1995)$ are positive and $x_1 + x_2 + \cdots + x_{1995} = y_1 + y_2 + \cdots + y_{1995} = 1$. Prove that

$$\sum_{k=1}^{1995} \frac{x_k y_k}{x_k + y_k} \le \frac{1}{2}$$

Problem 14. Suppose ΔABC, $\Delta A'B'C'$ are (directly) similar to each other and $\Delta AA'A''$, $\Delta BB'B''$, $\Delta CC'C''$ are also (directly) similar to each other. Show that $\Delta A''B''C''$ is (directly) similar to ΔABC.

Problem 15. Is there an infinite sequence $a_0, a_1, a_2, \cdots$ of nonzero real numbers such that for $n = 1, 2, 3, \cdots$, the polynomial $P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ has exactly n distinct real roots? (*Source:* 1990 Putnam Exam.)

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Solutions

Problem 6. For quadratic polynomials $P(x) = ax^2 + bx + c$ with real coefficients satisfying $|P(x)| \le 1$ for $-1 \le x \le 1$, find the maximum possible values of b and give a polynomial attaining the maximal b coefficient.

Solution: Independent solution by **KWOK Wing Yin** (St. Clare's Girls' School), **Bobby POON Wai Hoi** (St. Paul's College), **SZE Hoi WING** (St. Paul's Co-ed College) and **WONG Chun Keung** (St. Paul's Co-ed College).

Since $b = (P(1) - P(-1))/2 \le 2/2 = 1$, the maximum possible values of b is at most 1. Now the polynomial $P(x) = x^2/2 + x - 1/2 = (x+1)^2/2 - 1$ satisfy the condition $|P(x)| \le 1$ for $-1 \le x \le 1$ because $0 \le x+1 \le 2$. So the maximum of b is 1.

Comments: With $-1 \le x \le 1$ replaced by $0 \le x \le 1$, the problem appeared in the 1968 Putnam Exam.

Other commended solvers: **CHAN Wing Sum** (HKUST), **CHEUNG Kwok Koon** (S.K.H. Bishop Mok Sau Tseng Secondary School), **W. H. FOK** (Homantin Government Secondary School), **Michael LAM Wing Young** (St. Paul's College), **LIN Kwong Shing** (University of Illinois) and **LIU Wai Kwong** (Pui Tak Canossian College).

Problem 7. If positive integers a, b, c satisfy $a^2 + b^2 = c^2$, show that there are at least three noncongruent right triangles with integer sides having hypotenuses all equal to c^3.

Solution: Independent solution by **LIN Kwong Shing** (University of Illinois) and **LIU Wai Kwong** (Pui Tak Canossian College).

Without loss of generality, assume $a \ge b$. The first triangle comes from $(c^3)^2 = (a^2+b^2)c^4 = (ac^2)^2 + (bc^2)^2$. The second triangle comes from $(c^3)^2 = (a^2 + b^2)^2c^2 = (a^4 - 2a^2b^2 + b^4 + 4a^2b^2)c^2 = [(a^2-b^2)c]^2 + [2abc]^2$. The third triangle comes from $(c^3)^2 = (a^2+b^2)^3 = (a^6 - 6a^4b^2 + 9a^2b^4) + (9a^4b^2 - 6a^2b^4 + b^6) = [a|a^2-3b^2|]^2 + [b(3a^2-b^2)]^2$.

For the first and second triangles, $2abc = ac^2$ or bc^2 implies $c = 2b$ or $2a$. Substitute $c = 2b$ or $2a$ into $a^2 + b^2 = c^2$ will lead to the contradiction $\sqrt{3} = a/b$ or b/a. So these two triangles cannot be congruent.

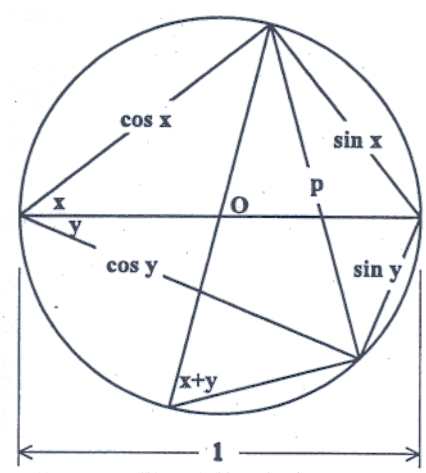
Similarly, for the first and third triangles, since $b(3a^2-b^2) = ac^2$ or bc^2 will lead to $\sqrt{2} = (a+b)/a$ or c/a by simple algebra, these two triangles cannot be congruent.

Finally, for the second and third triangles, $b(3a^2-b^2) = (a^2-b^2)c$ or $2abc$ will lead to $\sqrt{5} = (c-b)/b$ or $(c+a)/a$ (again by simple algebra). So these two triangles cannot be congruent.

Comments: Au Kwok Nin obtained the same triangles *systematically* by writing $c^6 = (c^3\cos n\theta)^2 + (c^3\sin n\theta)^2$ for $n = 1, 2, 3$ and expressed $\cos n\theta$, $\sin n\theta$ in terms of $\cos\theta = a/c$, $\sin\theta = b/c$. Cheung Kwok

(continued on page 4)

Proof Without Words



sin(x+y) = p = sin x cos y + cos x sin y

Problem Corner

(*continued from page 3*)

Koon observed that the greatest common divisors of the sides of the triangles were divisible by different powers of c, hence the triangles could not be congruent.

Other commended solvers: **AU Kwok Nin** (Tsung Tsin College), **CHAN Wing Sum** (HKUST), **CHEUNG Kwok Koon** (S.K.H. Bishop Mok Sau Tseng Secondary School) and **FUNG Tak Kwan & POON Wing Chi** (La Salle College).

Problem 8. (1963 Moscow Mathematical Olympiad) Let $a_1 = a_2 = 1$ and $a_n = (a_{n-1}^2 + 2)/a_{n-2}$ for $n = 3, 4, \cdots$. Show that a_n is an integer for $n = 3, 4, \cdots$.

Solution: Independent solution by **CHAN Chi Kin** (Pak Kau English School), **Michael LAM Wing Young** and **Bobby POON Wai Hoi** (St. Paul's College).

Since $a_1 = a_2 = 1$ and $a_n a_{n-2} = a_{n-1}^2 + 2$ for all integer $n \geq 3$, we have $a_n \neq 0$ and $a_n a_{n-2} - a_{n-1}^2 = 2 = a_{n+1}a_{n-1} - a_n^2$ for $n \geq 3$. We obtain $(a_{n+1}+a_{n-1})/a_n = (a_n+a_{n-2})/a_{n-1}$ by rearranging terms. Hence, the value of $(a_n+a_{n-2})/a_{n-1}$ is constant for $n \geq 3$. Since $(a_3+a_1)/a_2 = 4$, we have $(a_n+a_{n-2})/a_{n-1} = 4$, i.e., $a_n = 4a_{n-1} - a_{n-2}$ for $n \geq 3$. This shows that a_n is in fact an odd integer for all $n \geq 1$.

Comments: Most solvers observed that a_n depends on a_{n-1} and a_{n-2}, and thus guessed that a_n can be expressed as $ra_{n-1} + sa_{n-2}$ for some r, s. They went on to find $r = 4$ and $s = -1$ by setting $n = 3, 4$, then confirmed the guess by mathematical induction.

Other commended solvers: **CHAN Wing Sum** (HKUST), **CHEUNG Kwok Koon** (S.K.H. Bishop Mok Sau Tseng Secondary School), **HUI Yue Hon Bernard** (HKUST), **LIN Kwong Shing** (University of Illinois), **LIU Wai Kwong** (Pui Tak Canossian College) and **Alex MOK Chi Chiu** (Homantin Government Secondary School).

Problem 9. On sides AD and BC of a convex quadrilateral $ABCD$ with $AB < CD$, locate points F and E, respectively, such that $AF/FD = BE/EC = AB/CD$. Suppose EF when extended beyond F meets line BA at P and meets line CD at Q. Show that $\angle BPE = \angle CQE$.

Solution: Bobby POON Wai Hoi, St. Paul's College.

First construct parallelograms $ABGF$ and $CDFH$. Since BG, AD, CH are parallel, $\angle GBE = \angle HCE$. Also, $BG/CH = AF/DF = AB/CD = BE/CE$. So, ΔBGE is similar to ΔCHE. Then G, E, H must be collinear and $GE/HE = AB/CD = GF/HF$. Therefore, $\angle GFE = \angle HFE$ or $\angle BPE = \angle CQE$.

Other commended solvers: **CHEUNG Kwok Koon** (S.K.H. Bishop Mok Sau Tseng Secondary School), **W. H. FOK** (Homantin Government Secondary School), **Michael LAM Wing Young** (St. Paul's College) and **LIU Wai Kwong** (Pui Tak Canossian College).

Problem 10. Show that every integer $k>1$ has a multiple which is less than k^4 and can be written in base 10 with at most four different digits. [*Hint:* First consider numbers with digits 0 and 1 only.] (This was a problem proposed by Poland in a past IMO.)

Solution: Official IMO solution.

Choose n such that $2^{n-1} \leq k < 2^n$. Let S be the set of nonnegative integers less than 10^n that can be written with digits 0 or 1 only. Then S has 2^n elements and the largest number m in S is composed of n ones. Since $2^n > k$, by the pigeonhole principle, there are two numbers x, y in S which have the same remainder upon division by k, i.e., $x \equiv y \pmod{k}$. Then $|x-y|$ is a multiple of k and

$$|x-y| \leq m < 10^{n-1} \times 1.2 < 16^{n-1} \leq k^4.$$

Finally, considering the cases of subtracting a 0,1 digit by another 0,1 digit with possible carries, we see that $|x - y|$ can be written with digits 0, 1, 8, 9 only.

Similar Triangles ...

(*continued from page 1*)

Example 3. A triangle $A_1A_2A_3$ and a point P_0 are given in the plane. For $s \geq 4$, define $A_s = A_{s-3}$. For $k \geq 0$, define P_{k+1} to be the image of P_k under rotation with center at A_{k+1} through angle 120° clockwise. Prove that if $P_{1986} = P_0$, then $\Delta A_1A_2A_3$ is equilateral. (This was a problem on the 1986 IMO.)

Solution. We have $p_{k+1}-a_{k+1} = \omega(p_k-a_{k+1})$, where $\omega = \cos 120° - i\sin 120° = (-1-i\sqrt{3})/2$. Adding proper multiples of these equations (so as to cancel all p_k's), we consider

$$(p_{1986}-a_{1986})+\omega(p_{1985}-a_{1985})+\omega^2(p_{1984}-a_{1984}) + \cdots + \omega^{1985}(p_1-a_1)$$
$$= \omega(p_{1985} - a_{1986}) + \omega^2(p_{1984} - a_{1985}) + \omega^3(p_{1983} - a_{1984}) + \cdots + \omega^{1986}(p_0 - a_1).$$

Cancelling common terms on both sides, noting $\omega^{1986}p_0 = p_0 = p_{1986}$, then transposing all terms on the left side to the right, we get

$$0 = (1-\omega)(a_{1986}+\omega a_{1985}+\omega^2 a_{1984}+\cdots+\omega^{1985}a_1)$$
$$= 662(1-\omega)(a_3+\omega a_2+\omega^2 a_1)$$

by the definition of a_k and the fact $\omega^3 = 1$. Since $\omega \neq 1$, $\Delta A_1A_2A_3$ is equilateral.

Olympiad Corner

(*continued from page 1*)

Question 4. Let C be a circle with radius R and center O, and S a fixed point in the interior of C. Let AA' and BB' be perpendicular chords through S. Consider the rectangles $SAMB$, $SBN'A'$, $SA'M'B'$, and $SB'NA$. Find the set of all points M, N', M', and N when A moves around the whole circle.

Question 5. Find the minimum positive integer k such that there exists a function f from the set Z of all integers to $\{1, 2, \cdots, k\}$ with the property that $f(x) \neq f(y)$ whenever $|x-y| \in \{5, 7, 12\}$.

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Olympiad News:

Congratulations to CHEUNG Kwok Koon (F. 7, SKH Bishop Mok Sau Tseng Secondary School), HO Wing Yip (F. 6, Clementi Secondary School), MOK Tze Tao (F. 5, Queen's College), POON Wai Hoi Bobby (F. 6, St. Paul's College), WONG Him Ting (F. 7, Salesian English School) and YU Chun Ling (F. 6, Ying Wa College) for being selected as the 1995 Hong Kong Mathematical Olympiad Team Members. The selection was based on their outstanding performances in the Hong Kong Math Olympiad Training Program. They will represent Hong Kong to participate in the 36th International Mathematical Olympiad (IMO) to be held in Toronto, Canada this summer. Hong Kong was ranked 16 among 69 participating teams in 1994.