Mathematical Excalibur

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Olympiad Corner

The 36th International Mathematical Olympiad wad held in Toronto, Canada on July, 1995. The following six problems were given to the contestants. (The country inside the parantheses are the problem proposers.) -Editors

First Day

Question 1. (Bulgaria)
Let A, B, C and D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at the points X and Y. The line XY meets BC at the point Z. Let P be a point on the line XY different from Z. The line CP intersects the circle with diameter AC at the points C and M, and the line BP intersects the circle with diameter BD at the points B and N. Prove that the lines AM, DN and XY are concurrent.

Question 2. (Russia)
Let a, b and c be positve real numbers such that abc=1. Prove that

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\geq\frac{3}{2}.$$

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Acknowledgment: Thanks to Debbie Leung for her help in typesetting.

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in TeX, MS Word and WordPerfect, are encouraged. The deadline for receiving material for the next issue is October 15, 1995. Send all correspondence to:

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Descartes' Rule of Signs

Andy Liu
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Let $P(x)$ be a polynomial of degree n with complex coefficients. The Fundamental Theorem of Algebra tells us that it has exactly n complex roots. We are interested in the number of real roots in the case where the coefficients are real. We may assume that the leading coefficient is 1 and the constant term is non-zero.

As an example, consider

$p(x) = x^6-6x^5+10x^4-2x^3-3x^2+4x-12.$

As it turns out, it has four real roots -1, 2 (with multiplicity 2) and 3, and two non-real roots i and $-i$.

In general, we may not be able to find the roots of $P(x)$. However, we can obtain some information about the number of positive roots from the number of sign-switches of $P(x)$. If we consider the sequence of the signs of the non-zero coefficients of $P(x)$ in order, a sign-switch is said to occur if a + is followed immediately by a − or vice versa.

For $p(x)$ above, the sequence is + − + − − + −. Hence the number of sign switches is 5.

The first part of Descartes' Rule of Signs states that the number of positive roots of $P(x)$ has the same parity as the number of sign-switches of $P(x)$. Clearly, the latter is even if and only if the constant term of $P(x)$ is positive (because the sign sequence begins and ends with +). What we have to prove is that the same goes for the number of positive roots of $P(x)$.

From the Fundamental Theorem of Algebra, $P(x)$ is a product of linear factors and irreducible quadratic factors. Now the constant term of a quadratic factor with a negative discriminant must be positive. The constant term of a linear factor is positive if and only if it corresponds to a negative root. It follows that the sign of the constant term of $P(x)$ is positive if and only if the number of positive roots of $P(x)$ is even.

Since the number of sign-switches of $p(x)$ is 5, we can tell that it has an odd number of positive roots without trying to find them.

The second part of Descartes' Rule of Signs states that the number of positive roots of $P(x)$ is less than or equal to the number of sign-switches of $P(x)$. We shall build up $P(x)$ as follows. Start with the product of all irreducible quadratic factors and all linear factors corresponding to negative roots. What we have to prove is that the number of sign-switches increases every time we introduce a linear factor corresponding to a positive root.

For any polynomial $Q(x)$ with real coefficients, leading coefficient 1 and a non-zero constant term, we group consecutive terms of the same signs together to express $Q(x)$ as an alternating sum of polynomials of positive coefficients. Then the sign-switches occur precisely between summands. We claim that when we multiply $Q(x)$ by $x-t$ for some positive number t, the original sign-switches are preserved, while at least one additional sign-switch occurs.

Consider each summand in turn. The leading coefficient is positive. This does not change after multiplication by x. However, we may have to combine it with $-t$ times the last term of the preceding summand. Since there is a sign-switch

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Descartes' Rule of Signs

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between the two summands, the term with which it is to be combined is also positive. This justifies the first claim. The second claim follows since the constant terms of $Q(x)$ and $(x-t)Q(x)$ have opposite signs. This completes the proof of Descartes' Rule of Signs.

Let us illustrate the proof of the second part with

$$p(x) = (x^2+1)(x+1)(x-2)^2(x-3).$$

We first let

$$q(x) = (x^2+1)(x+1) = x^3+x^2+x+1.$$

Since the number of sign-switches is 0, there is only one summand. We have

$$\begin{aligned} q_1(x) &= (x-2)\,q(x) \\ &= (x-2)\,(x^3+x^2+x+1) \\ &= x^4-x^3-x^2-x-2 \\ &= x^4-(x^3+x^2+x+2). \end{aligned}$$

$$\begin{aligned} q_2(x) &= (x-2)\,q_1(x) \\ &= (x-2)x^4 - (x-2)(x^3+x^2+x+2) \\ &= (x^5-2x^4) - (x^4- x^3- x^2-4) \\ &= x^5 - 3x^4 + x^3 + x^2 + 4. \end{aligned}$$

Note that we have combined the terms $-x^4$ and $-2x^4$ which have the same sign. Finally,

$$\begin{aligned} p(x) &= (x-3)\,q_2(x) \\ &= (x-3)x^5- (x-3)(3x^4) + (x-3)(x^3+x^2+4) \\ &= (x^6-3x^5)-(3x^5-9x^4)+(x^4-2x^3-3x^2+4x-12) \\ &= x^6 - 6x^5 +10x^4 - 2x^3 - 3x^2 + 4x - 12. \end{aligned}$$

We point out that using the same argument, we can prove that the number of negative roots of $P(x)$ is not greater than the number of sign-switches in $P(-x)$, and differs from it by an even number. For example, the number of sign-switches in $p(-x) = x^6+6x^5+10x^4+2x^3-3x^2-4x-12$ is 1, and we can conclude that $p(x)$ has exactly one negative root.

希臘幾何學的發展

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「幾何」一詞，拉丁文是 **geometria**，其中"geo-"代表「地」（與 geography, geology 中"geo-"的意思一致），而"metria"則與今天英文的"metric"相關，代表「量度」。兩部份合起來，就是「量地的學問」------原來古埃及的尼羅河每年都泛濫一次，摧毀河畔的農地，洪水過後，政府爲重新劃定的農地量度面積，以決定每戶所須繳付的賦稅；幾何學就在這情況下應運而生。後來，這些知識輾轉傳入希臘，逐漸發展成一套完整的學說。

希臘數學史可分爲三個時期：第一段從愛奧尼亞學派到柏拉圖學派爲止；第二段是亞歷山大前期，從歐幾里得到羅馬攻陷希臘爲止；最後則是亞歷山大後期。

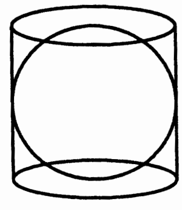
第一段時期的希臘共有六個主要學派，其中以奉行素食的畢達哥拉斯(**Pythagoras**)學派最負盛名，他們的「畢氏定理」(**Pythagorean Theorem**)是理科必備的工具。這時期，很多數學家都有重要的發現，雖然沒有建立一套完整的學說，但爲日後阿基米得等人的數學理論建立了一個良好的基礎。

第二段時期中最偉大的數學家可算是歐幾里得(**Euclid**)。歐幾里得深諳柏拉圖幾何的精髓，經過嚴謹的演繹和推論，寫成了<<原本>>(**Elements**)一書。<<原本>>爲幾何建立了一套完整的理論，在**1637**年笛卡爾(**Descartes**)引入「坐標幾何」前，它佔幾何學的領導地位。它也是用公理法建立起演繹數學體系的最早典範------所謂「公理法」，就是從一些大家都公認可以接受，毋須加以証明的「公理」出發，通過合乎邏輯的推論而得出被驗証的結果。這可說是人類從直觀事物邁向抽象思維的重要一步！

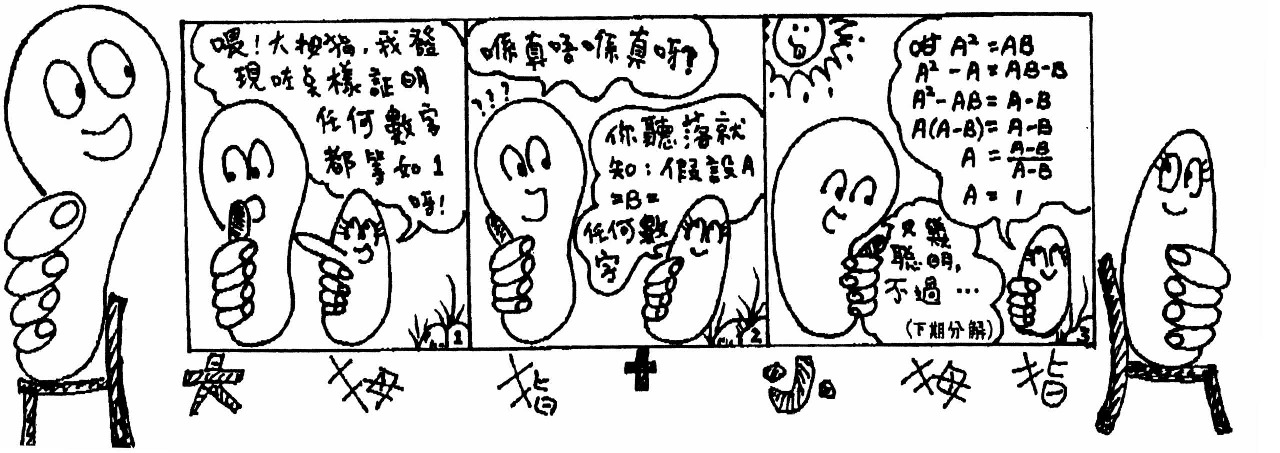
亞歷山大前期的另一位偉大的數學家是阿基米德。在他所著的<<圓的量度>>中，阿基米德利用外切與內接九十六邊形求得圓周率π的兩個近似值：

$$3\tfrac{10}{71} < \pi < 3\tfrac{1}{7}。$$

假設圓的半徑爲 **1**，則圓周剛好爲**2π**，此數值必須大於內接六邊形的周界，而六邊形的周界爲 **6**。故此求得 π 的下限爲 **3**。同理，利用圓形的外切六邊形，可求得 π 的上限爲 **3.4641**。假若我們把六邊形換作十二邊形、廿四邊形，…，則（內接或外切）多邊形的周界會越來越接近圓，而相應的上限及下限也會趨近 π 的眞實數値。阿基米得利用圓形的外切與內接九十六邊形，求得 $3\tfrac{10}{71} < \pi < 3\tfrac{1}{7}$。阿基米得的另一項建樹是體積的計算，例如圓球體積是它的外接圓柱體積的三份二。阿基米德還發現圓球的表面積恰巧也是外接圓柱表面積的三份二，他非常欣賞這定理，吩咐親人把這個圖形刻在他的墓碑上（見圖）。



最後一段是亞歷山大後期。這個時期的數學家，以歐幾里得的<<原本>>爲根據，作出了不少增潤修補的工作，而我們現在所學的平面幾何，亦在這時候逐漸形成。讀到這裏，同學們該明白到這門學問是眾多數學家集體智慧的結晶及長期辛勞的成果。



Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, address and school affiliation. Please send submissions to *Dr. Tsz-Mei Ko, Dept of EEE, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is October 15, 1995.

Problem 16. Let a, b, c, p be real numbers, with a, b, c not all equal, such that $a+\frac{1}{b}=b+\frac{1}{c}=c+\frac{1}{a}=p$. Determine all possible values of p and prove that $abc+p=0$. (Source: 1983 Dutch Mathematical Olympiad.)

Problem 17. Find all sets of positive integers x, y and z such that $x\le y\le z$ and $x^y+y^z=z^x$.

Problem 18. For real numbers a, b, c, define

$$f(a,b,c)=a+b-|a-b|-\big|a+b+|a-b|-2c\big|.$$

Show that $f(a,b,c)>0$ if and only if $f(b,c,a)>0$ if and only if $f(c,a,b)>0$.

Problem 19. Suppose A is a point inside a given circle and is different from the center. Consider all chords (excluding the diameter) passing through A. What is the locus of the intersection of the tangent lines at the endpoints of these chords?

Problem 20. For $n>1$, let $2n$ chess pieces be placed on any $2n$ squares of an $n\times n$ chessboard. Show that there are 4 pieces among them that formed the vertices of a parallelogram. (Note that if $2n-1$ pieces are placed on the squares of the first column and the first row, then there is no parallelogram. So $2n$ is the best possible.)

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Solutions

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Problem 11. Simplify

$$\sum_{n=1}^{1995}\tan n\tan(n+1).$$

(There is an answer with two terms involving tan1, tan1996 and integers.)

Solution: Independent solutions by **Iris CHAN Chau Ping** (St. Catherine's School for Girls, Kwun Tong), **CHAN Chi Kin** (Pak Kau English School), **CHAN Sze Tai, Angie** (Ming Kei College), **CHAN Wing Sum** (HKUST), **CHOW Chak On** (HKUST), **CHUI Yuk Man** (Queen Elizabeth School), **LEUNG Ka Fai** (Ju Ching Chu Sceondary School (Yuen Long)), **LIU Wai Kwong** (Pui Tak Canossian College), **Alex MOK Chi Chiu** (Homantin Government Secondary School), **TAM Tak Wing** (Delia Memorial School (Yuet Wah)) and **WOO Chin Yeung** (St. Peter's Secondary School).

From $\tan 1=\tan[(n+1)-n]=(\tan(n+1)-\tan n)/(1+\tan n\tan(n+1))$, we get

$$\sum_{n=1}^{1995}\tan(n)\tan(n+1)=\sum_{n=1}^{1995}\left(\frac{\tan(n+1)-\tan(n)}{\tan 1}-1\right)$$

$$=\frac{\tan 1996-\tan 1}{\tan 1}-1995=\frac{\tan 1996}{\tan 1}-1996.$$

Comments: This problem illustrates the *telescoping method* of summing a series, i.e., by some means, write a_n as $b_{n+1}-b_n$, then summing a_n will result in many cancellations yielding a simple answer.

Problem 12. Show that for any integer $n>12$, there is a right triangle whose sides are integers and whose area is between n and $2n$. (Source: 1993 Korean Mathematical Olympiad.)

Solution: **WONG Chun Keung**, St. Paul's Co-ed College.

Consider triangle A with sides $3d$, $4d$, $5d$, which has area $6d^2$. So for n in the interval $(3d^2+1,6d^2-1)$, triangle A has an area between n and $2n$. For $d\ge 3$, $6d^2-1-[3(d+1)^2+1]=3(d-1)^2-8>0$. So the intervals $(3d^2+1,6d^2-1)$ with $d=3,4,5,\ldots$ cover all positive integers n greater than or equal to 28. For $d=2$, triangle A has area 24, which takes care of the cases $n=13,14,\ldots,23$. Finally, the cases $n=24,25,26,27$ are taken care of by the triangle with sides 5, 12, 13, which has area 30.

Other commended solvers: **CHAN Wing Sum** (HKUST) and **LIU Wai Kwong** (Pui Tak Canossian College).

Problem 13. Suppose x_k, y_k $(k=1,2,\ldots,1995)$ are positive and $x_1+x_2+\cdots+x_{1995}=y_1+y_2+\cdots+y_{1995}=1$. Prove that

$$\sum_{k=1}^{1995}\frac{x_ky_k}{x_k+y_k}\le\frac{1}{2}.$$

Solution: Independent solution by **CHAN Chi Kin** (Pak Kau English School), **CHAN Wing Sum** (HKUST), **KWOK Wing Yin** (St. Clare's Girls' School) and **LEUNG Ka Fai** (Ju Ching Chu Secondary School (Yuen Long)).

Since $x_ky_k/(x_k+y_k)\le(x_k+y_k)/4$ (is equivalent to $(x_k-y_k)^2\ge 0$ by simple algebra), we get

$$\sum_{k=1}^{1995}\frac{x_ky_k}{x_k+y_k}\le\sum_{k=1}^{1995}\frac{x_k+y_k}{4}=\frac{1}{2}.$$

Other commended solvers: **Iris CHAN Chau Ping** (St. Catherine's School for Girls, Kwun Tong), **CHEUNG Lap Kin** (Hon Wah Middle School), **CHOW Chak On** (HKUST), **LIU Wai Kwong** (Pui Tak Canossian College), **Alex MOK Chi Chiu** (Homantin Government Secondary School), **TAM Tak Wing** (Delia Memorial School (Yuet Wah)), **WONG Chun Keung** (St. Paul's Co-ed College) and **WOO Chin Yeung** (St. Peter's Secondary School).

Problem 14. If ΔABC, $\Delta A'B'C'$ are (directly) similar to each other and $\Delta AA'A''$, $\Delta BB'B''$, $\Delta CC'C''$ are also (directly) similar to each other, then show that $\Delta A''B''C''$, ΔABC are (directly) similar to each other.

Solution: Independent solution by **CHAN Wing Sum** (HKUST) and **LIU Wai Kwong** (Pui Tak Canossian College).

We will use capital letters for points and small letters for the corresponding complex numbers. Since $\Delta AA'A''$, $\Delta BB'B''$, $\Delta CC'C''$ are (directly) similar to each other,

$$\frac{a''-a}{a'-a}=\frac{b''-b}{b'-b}=\frac{c''-c}{c'-c}=r.$$

Then $a''=ra'+(1-r)a$, $b''=rb'+(1-r)b$, $c''=rc'+(1-r)c$. Since ΔABC, $\Delta A'B'C'$ are (directly) similar to each other,

$$\frac{b-a}{c-a}=\frac{b'-a'}{c'-a'}.$$

Then

$$\frac{b''-a''}{c''-a''}=\frac{r(b'-a')+(1-r)(b-a)}{r(c'-a')+(1-r)(c-a)}=\frac{b-a}{c-a},$$

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Problem Corner

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which is equilvalent to $\Delta A''B''C''$ (directly) simlar to ΔABC.

Problem 15. Is there an infinite sequence $a_0, a_1, a_2, \cdots$ of non-zero real numbers such that for $n = 1, 2, 3, \cdots$, the polynomial

$$P_n(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

has exactly n distinct real roots? (Source: 1990 Putnam Exam.)

Solution: Yes. Take $a_0 = 1$, $a_1 = -1$ and proceed by induction. Suppose $a_0, \cdots, a_n$ have been chosen so that $P_n(x)$ has n distinct real roots and $P_n(x) \to \infty$ or $-\infty$ as $x \to \infty$ *depending upon whether* a_n *is positive or negative*. Suppose the roots of $P_n(x)$ is in the interval $(-T,T)$. Let $a_{n+1} = (-1)^{n+1}/M$, where M is chosen to be very large so that T^{n+1}/M is very small. Then $P_{n+1}(x) = P_n(x) + (-x)^{n+1}/M$ is very close to $P_n(x)$ on $[-T,T]$ because $|P_{n+1}(x) - P_n(x)| \le T^{n+1}/M$ for every x on $[-T,T]$. So, $P_{n+1}(x)$ has a sign change very close to every root of $P_n(x)$ and has the same sign as $P_n(x)$ at T. Since $P_n(x)$ and $P_{n+1}(x)$ take on different sign when $x \to \infty$, there must be another sign change beyond T. So $P_{n+1}(x)$ must have n+1 real roots.

Comments: **Liu Wai Kwong** sent in a more detail solution showing that the numbers can even be chosen to have the same sign.

Other commended solvers: **LIU Wai Kwong** (Pui Tak Canossian College).

Olympiad Corner

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Question 3. (Czech Republic)
Determine all integers $n > 3$ for which there exist n points $A_1, A_2, \cdots, A_n$ in the plane, and real numbers $r_1, r_2, \cdots, r_n$ satisfying the following two conditions:
(i) no three of the points $A_1, A_2, \cdots, A_n$ lie on a line;
(ii) for each triple i, j, k $(1 \le i < j < k \le n)$ the triangle $A_iA_jA_k$ has area equal to $r_i + r_j + r_k$.

Second Day

Question 4. (Poland)
Find the maximum value of x_0 for which there exists a sequence of positive real numbers $x_0, x_1, \cdots, x_{1995}$ satisfying the two conditions:
(i) $x_0 = x_{1995}$;
(ii) $x_{i+1} + \dfrac{2}{x_{i-1}} = 2x_i + \dfrac{1}{x_i}$ for each i = 1, 2, $\cdots$, 1995.

Question 5. (New Zealand)
Let $ABCDEF$ be a convex hexagon with

$$AB = BC = CD,$$
$$DE = EF = FA,$$

and

$$\angle BCD = \angle EFA = 60^\circ.$$

Let G and H be two points in the interior of the hexagon such that:

$$\angle AGB = \angle DHE = 120^\circ.$$

Prove that

$$AG + GB + GH + DH + HE \ge CF.$$

Question 6. (Poland)
Let p be an odd prime number. Find the number of subsets A of the set $\{1, 2, \cdots, 2p\}$ such that
(i) A has exactly p elements, and
(ii) the sum of all the elements in A is divisible by p.

IMO-95, Toronto, Canada

Kin Y. Li

On July 16, the Hong Kong team started their journey to Toronto, Canada for the thirty-sixth International Mathematical Olympiad. The flight took about 18 hours with one stop at Anchorage, Alaska. Shortly after arrival, the team was interviewed by local Chinese media. The Canadian host certainly publicized the event very well. During the entire period, the team stayed at the beautiful York University campus. The quarters provided were very comfortable; each person had his own room!

Opening ceremony came two days later and the examination followed. Team leaders and deputy leaders began markings and coordination soon afterward, while the students were given tours to Toronto's top attractions, such as Skydome, Ontario Science Center, Downtown Toronto, CN (Canadian National) Tower, Canada's Wonderland and of course, Niagara Falls. Meanwhile the scores were quickly decided. This year the team brought home two silvers, three bronzes and one honorable mention. (One silver was actually one mark short of a gold!) In the closing ceremony, the winners received their medals. Also, for entertainment, there were impressive performances, which included an awesome laser show. Throughout the events, there were many opportunities for students from different countries to get to know each other. Enjoying every moment of the whole trip, the team finally came home reluctantly on the evening of July 25. Everybody had fond memories and developed new friendships.

Photo at left: The 1995 Hong Kong Math Olympiad Team taken at the Kai Tak Airport before departure. From left to right are: LI Kin-Yin (leader), MOK Tze Tao, HO Wing Yip, POON Wai Hoi, CHEUNG Kwok Koon, YU Chun Ling, WONG Him Ting, and KWOK, Ka Keung (deputy leader).