# Mathematical Excalibur

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## Olympiad Corner

*The following are five problems from the 24th USA Mathematical Olympiad held in April 27, 1995. The time limit for this competition was three and a half hours.* *-Editors*

**Problem 1.** Let $p$ be an odd prime. The sequence $(a_n)_{n\geq 0}$ is defined as follows: $a_0 = 0$, $a_1 = 1$, ..., $a_{p-2} = p-2$ and, for all $n \geq p-1$, $a_n$ is the least positive integer that does not form an arithmetic sequence of length $p$ with any of the preceding terms. Prove that, for all $n$, $a_n$ is the number obtained by writing $n$ in base $p-1$ and reading the result in base $p$.

**Problem 2.** A calculator is broken so that the only keys that still work are the sin, cos, tan, $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ buttons. The display initially shows 0. Given any positive rational number $q$, show that pressing some finite sequence of buttons will yield $q$. Assume that the calculator does real number calculations with infinite precision. All functions are in terms of radians.

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word are encouraged. The deadline for receiving material for the next issue is December 30, 1995.

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# 談談質數

王元教授

自然數是指1, 2, 3, ⋯ 之一。整數則是指 ⋯, −2, −1, 0, 1, 2, ⋯ 之一。自然數即正整數。二整數間可以定義和、差、乘運算，其結果仍爲整數，即"整數集合對加、減、乘運算是自封的"。

定理 1 (歐氏除法)：任二整數$a$及$b$ $(>0)$，必有整數$q$及$r$滿足

$$a = bq + r,\ \ 0 \leq r < b.$$

若在上式中$r = 0$，即$a = bq$，則稱$a$爲$b$之倍數，或$b$爲$a$之因數，記爲$b|a$。否則記爲$b\nmid a$。

自然數可以分成三類：

1：只有自然數1爲其因數；

$p$：恰有1與$p$爲其因數，這種數稱之爲質數。

$n$：除1與$n$之外，還有其他因數，這種數稱爲複合數。

凡能被2整除的整數稱爲偶數，否則稱爲奇數。

定理 2 (算術的基本定理 (Fundamental Theorem of Arithmetic))：非1之自然數皆可以唯一地表示爲質數之積。

由定理 2 可見在自然數中質數是基本的。最初之若干質數是由Eratosthenes 篩法得到的。例如要找出不超過50之質數，則先找出不超過 $\sqrt{50}$ 的質數，即 2, 3, 5, 7，再將 1, 2, ⋯, 50 排列如下：

1, **2**, **3**, 4, **5**, 6, **7**, 8, 9, 10,
**11**, 12, **13**, 14, 15, 16, **17**, 18, **19**, 20,
21, 22, **23**, 24, 25, 26, 27, 28, **29**, 30,
**31**, 32, 33, 34, 35, 36, **37**, 38, 39, 40,
**41**, 42, **43**, 44, 45, 46, **47**, 48, 49, 50,

去掉1；去掉2的倍數4, 6, ⋯, 50；在剩下的數中去掉3的倍數9, ⋯, 45及在剩下的數中去掉5與7的倍數(保留5與7)，最後剩下的都是質數：2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47.

質數表都是根據這一方法加以改進而造出來的。如Kulik曾編出不超過$10^8$的質數表。自從有了電腦之後，質數表就更大了。六十年代初，美國就在電腦中儲存了前$5\times10^8$個質數。但不管怎樣，我們至今仍然只知道有限多個質數，雖然有下面的著名定理：

定理 3 (歐幾里德 (Euclid))：質數有無窮多。

證：我們用反證法。若質數個數有限，則可以依次排列爲 $p_1, p_2, ..., p_s$。令$N = p_1p_2...p_s + 1$，若$N$爲質數，則$N > p_s$，矛盾。若$N$爲複合數，則$p_1, p_2, ..., p_s$ 皆非$N$之因數，否則就能整除1，此不可能，所以$N$有異於$p_1, p_2, ..., p_s$的質因數，矛盾。因此，質數有無窮多。定理證完。

目前所知道的大質數都是一些特殊形式的質數。形如

$$M_p = 2^p - 1,\ \ p爲質數$$

的質數叫梅森林質數(Mersenne Prime)。當$p$ = 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689, 9941, 11213, 19937, 21701, 23209, 44497, 86243, 132049, 216091時，$M_p$爲質數，最大者

$$2^{216091} - 1$$

共65050位，這是目前所知道的最大質數。但是否有無窮多Mersenne Prime？仍有待澄清。

還有費馬數 (Fermat Number)

$$F_n = 2^{2^n} + 1.$$

當$n = 0, 1, 2, 3, 4$時，$F_n$都是質數。Fermat曾猜想，當 $n = 0, 1, 2, \cdots$ 時，$F_n$都是質數。歐拉 (Euler) 證明了：

$$F_5 = 2^{2^5} + 1 = 641 \times 6700417.$$

從而否定了 Fermat 猜想。

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下面講兩個質數論方面的中心問題：

一、質數分佈問題。命$\pi(x)$表示不超過$x$的質數個數。例如 $\pi(50) = 15$, $\pi(100) = 25$等等。質數分佈論的中心問題之一是研究

(*continued on page 2*)

## 談談質數：

(*continued from page 1*)

$\pi(x)$的性質：我們如何來估計$\pi(x)$？定理3可以記為：

$$\pi(x)\to\infty.$$

定理 4 (車比雪夫 (Chebychev))：當$n\geq 2$時

$$\frac{n}{8\log n}\leq\pi(n)\leq\frac{12n}{\log n},$$

其中$\log\ n$表示$n$的自然對數 (natural logarithm)。

這一定理比定理3精密多了。在此就不證明了。高斯 (Gauss) 與蘭讓德 (Legendre) 曾猜想：

$$\pi(x)\sim\frac{x}{\log x}$$

即當$x\to\infty$時，$\pi(x)$與$\frac{x}{\log x}$之比趨於1。注意：Gauss猜想的形式與上式稍有不同。因此當$x$較大時，用$\frac{x}{\log x}$來估計$\pi(x)$時，應該是很精密的。例如：

| $x$ | $\pi(x)$ | $\frac{x}{\log x}$ |
|---:|---:|---:|
| 1,000 | 168 | 145 |
| 10,000 | 1229 | 1086 |
| 100,000 | 9592 | 8686 |
| 1,000,000 | 78498 | 72382 |
| 10,000,000 | 664579 | 620417 |

Gauss 與Legendre猜想是Hadamard與 de la Vallee Poussin 獨立證明的。人們稱它為質數定理。即

定理 5 (質數定理 (Prime Number Theorem))：$\pi(x)\sim\frac{x}{\log x}$.

質數的另一重要問題是兩個相鄰質數的間隔長度估計問題。有所謂的Bertrand假設：當$x\geq 1$時，在$x$與$2x$之間必有一個質數。這一假設也是Chebychev證明的，即

定理 6 (Chebychev)：當$x\geq 1$時，在$x$與$2x$之間必有一個質數。

定理6還可以有較大改進。作為定理3的進一步，我們可以考慮算術序列 (Arithmetic Progression)中的質數問題：若$l$, $q$為正整數，且適合 $(l,q)=1$ (即它們的最大公因數 $(l,q)$ 等於1) 問序列

$$l,\ l+q,\ l+2q,\ \cdots$$

中是否有無窮多個質數？當$l=q=1$時，

(*continued on page 4*)

# Systems of Distinct Representatives

*Kin-Yin Li*

Suppose in a school, there are some clubs. In the science club, the members are Bob and Cathy. In the dance club, the members are Bob, Mary, Joe and Emma. In the bridge club, the members are Joe, Emma, Paul and Cathy. In the debate club, the members are Bob and Cathy. Suppose a representative is to be elected from each club and no two clubs are allowed to have the same representative. Is this possible?

In the example, one possibility is to have Bob for science, Mary for dance, Joe for bridge and Cathy for debate. We say the collection Bob, Mary, Joe and Cathy is a *system of distinct representatives* (SDR) for the four clubs because each represents a different club.

If a new drama club is formed with only Bob and Cathy as members, then there is not any SDR for these five clubs because the science, debate and drama clubs together have only two members. So far, to decide whether there is a SDR for clubs or not is simple because there are not too many clubs. If the number of clubs increases, then the problem will become difficult. Naturally we would like to know if there is a method for knowing whether there exists any SDR for clubs or not. Also, we would like to know, when a SDR exists, how to find such a SDR.

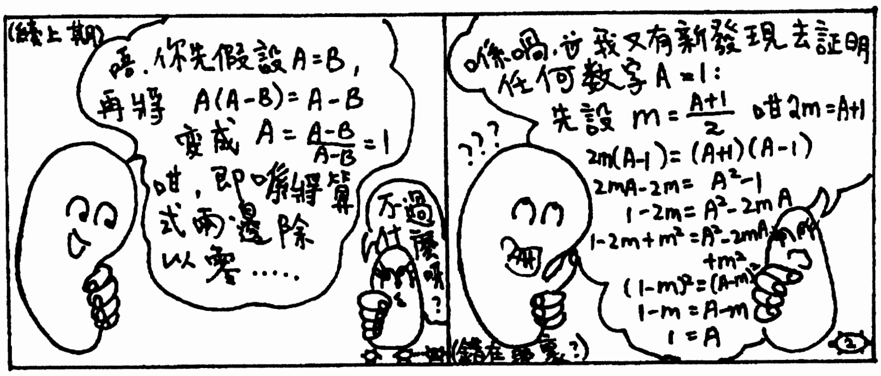
Suppose there are $n$ clubs. From the drama club situation above, we learned that if these $n$ clubs have a SDR, then every set of $m$ ($\leq n$) clubs together must have at least $m$ members. This gives us a necessary condition to check. In fact, there is a famous theorem, due to Philip Hall, that asserts the condition is also sufficient.

**Hall's Theorem.** *There exists a SDR for $n$ clubs if and only if every set of $m$ ($\leq n$) clubs together has at least $m$ members.*

Briefly, here is how to get a SDR inductively when the condition is met. If we are lucky that every set of $k$ ($< n$) clubs together has *more than* $k$ members, then pick a member as representative for a club and remove this member from the other $n-1$ clubs. The condition for the $n-1$ clubs will still be met. Inductively, we can find a SDR for these $n-1$ clubs.

If we are unlucky that there are $k$ ($< n$) clubs together having exactly $k$ members. Since $k < n$, inductively we can find a SDR for these $k$ clubs. Now remove these $k$ members from the other $n-k$ clubs. After removal, we can check that the condition for the remaining $n-k$ clubs will still be met. (This is because any $j$ of these remaining clubs together will contain the members of the $j+k$ clubs together, minus the $k$ removed members. That is, every set of $j$ ($\leq n-k$) remaining clubs has at least $(j+k)-k=j$ members.) So inductively we can find a SDR for the remaining $n-k$ clubs.

For another application of SDR, consider the situation of $n$ boys and $n$ girls in a party. Each boy knows some of the girls and vice versa. When is it possible to match each boy with a unique girl that he knows? This is simple if you understand Hall's theorem. For each boy, form a fan club consists of all the girls he knows. There is a matching if and only if there is a SDR for the $n$ fan clubs, i.e., every set of $m$ ($\leq n$) boys together must know at least $m$ girls.



## Problem Corner

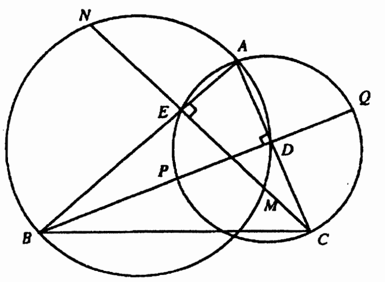
We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, address, school affiliation and grade level. Please send submissions to *Dr. Kin-Yin Li, Dept of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is December 30, 1995.

**Problem 21.** Show that if a polynomial $P(x)$ satisfies

$$P(2x^2-1)=\frac{P(x)^2}{2}-1,$$

it must be constant.

**Problem 22.** An acute-angled triangle $ABC$ is given in the plane. The circle with diameter $AB$ intersects altitude $CE$ and its extension at points $M$ and $N$, and the circle with diameter $AC$ intersects altitude $BD$ and its extension at $P$ and $Q$. Prove that the points $M, N, P, Q$ lie on a common circle. (*Source*: 1990 USA Mathematical Olympiad).




**Problem 23.** Determine all sequences $\{a_1, a_2, \ldots\}$ such that $a_1 = 1$ and $|a_n - a_m| \le 2mn/(m^2+n^2)$ for all positive integers $m$ and $n$. (*Source*: Past IMO problem proposed by Finland).

**Problem 24.** In a party, $n$ boys and $n$ girls are paired. It is observed that in each pair, the difference in height is less than 10 cm. Show that the difference in height of the $k$-th tallest boy and the $k$-th tallest girl is also less than 10 cm for $k = 1, 2, \ldots, n$.

**Problem 25.** Are there any positive integers $n$ such that the first four digits from the left side of $n!$ (in base 10 representation) is 1995?

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## Solutions

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**Problem 16.** Let $a$, $b$, $c$, $p$ be real numbers, with $a$, $b$, $c$ not all equal, such that $a+\frac{1}{b}=b+\frac{1}{c}=c+\frac{1}{a}=p$. Determine all possible values of $p$ and prove that $abc + p = 0$. (Source: 1983 Dutch Mathematical Olympiad.)

**Solution**: Official Solution.

Since $ca + 1 = ap$ and $bc + 1 = cp$, we get $ap^2 = cap + p = a(bc+1) + p = abc + a + p$. Hence $a(p^2-1) = abc + p$. Similarly, $b(p^2-1) = abc + p$ and $c(p^2-1) = abc + p$. Since $a$, $b$, $c$ are not all equal, $p = \pm1$ and then $abc + p = 0$. Both values of $p$ are possible by considering $(a,b,c) = (2,-1,1/2)$ and $(-2,1,-1/2)$.

*Comments*: Most solvers use repeated substitution to obtain the equation $(p^2-1)(a^2-ap+1) = 0$ (and similar equations for $b$ and $c$) and then show that $p = \pm1$. (Otherwise, $a^2 - ap + 1 = 0$ and the other two similar equations will lead to the contradiction $a = b = c$.) Solvers then use different approaches to find $abc$ for the two possible values of $p$ to prove $abc + p = 0$.

*Other commended solvers*: **CHAN Wing Sum** (HKUST), **William CHEUNG Pok Man** (S.T.F.A. Leung Kau Kui College), **Wallis LEUNG Ka-Wo** (HKUST) and **LIU Wai Kwong** (Pui Tak Canossian College).

**Problem 17.** Find all sets of positive integers $x$, $y$ and $z$ such that $x \le y \le z$ and $x^y + y^z = z^x$.

**Solution**: **William CHEUNG Pok Man** (S.T.F.A. Leung Kau Kui College).

Since $3^{1/3} > 4^{1/4} > 5^{1/5} > \ldots$, we have $y^z \ge z^y$ if $y \ge 3$. Hence the equation has no solution if $y \ge 3$. Since $1 \le x \le y$, the only possible values for $(x,y)$ are (1,1), (1,2) and (2,2). These lead to the equations $1 + 1 = z$, $1 + 2^z = z$ and $4 + 2^z = z^2$. The third equation has no solution since $2^z \ge z^2$ for $z \ge 4$ and (2,2,3) is not a solution to $x^y + y^z = z^x$. The second equation has no solution either since $2^z > z$. The first equation leads to the unique solution (1,1,2).

*Other commended solvers*: **HO Wing Yip** (Clementi Secondary School), **LIU Wai Kwong** (Pui Tak Canossian College) and **WONG Him Ting** (Salesian English School).

**Problem 18.** For real numbers $a$, $b$, $c$, define

$$f(a,b,c) = a+b-|a-b|-|a+b+|a-b|-2c|.$$

Show that $f(a,b,c) > 0$ if and only if $f(b,c,a) > 0$ if and only if $f(c,a,b) > 0$.

**Solution**: **William CHEUNG Pok Man** (S.T.F.A. Leung Kau Kui College).

We have $f(a,b,c) > 0$ if and only if $|a + b + |a-b| - 2c| < a + b - |a-b|$. Applying the fact that $|x| < y$ if and only if $x < y$ and $-x < y$ to the last inequality and simplifying, we see that $f(a,b,c) > 0$ if and only if $|a-b| < c$ and $c < a + b$. Applying the fact again to $|a-b| < c$ and transposing terms, we see that $f(a,b,c) > 0$ if and only if $a < b + c$ and $b < c + a$ and $c < a + b$. The assertion follows.

*Comments*: **LIU Wai Kwong** considers the six possible orderings $a \ge b \ge c$, $a \ge c \ge b$, etc. to show that $f(a,b,c) = f(b,c,a) = f(c,a,b) = 2(a + b + c - 2\max\{a,b,c\})$ and thus the assertion follows.

*Other commended solvers*: **Wallis LEUNG Ka-Wo** (HKUST) and **LIU Wai Kwong** (Pui Tak Canossian College).

**Problem 19.** Suppose $A$ is a point inside a given circle and is different from the center. Consider all chords (excluding the diameter) passing through $A$. What is the locus of the intersection of the tangent lines at the endpoints of these chords?

**Solution**: **WONG Him Ting** (Salesian English School).

Let $O$ be the center and $r$ be the radius. Let $A'$ be the point on $OA$ extended beyond $A$ such that $OA \times OA' = r^2$. suppose $BC$ is one such chord passing through $A$ and the tangents at $B$ and $C$ intersect at $D'$. By symmetry, $D'$ is on the line $OD$, where $D$ is the midpoint of $BC$. Since $\angle OBD' = 90°$, $OD \times OD' = OB^2$ $(= OA \times OA'.)$ So $\Delta OAD$ is similar to $\Delta OD'A'$. Since $\angle ODA = 90°$, $D'$ is on the line $L$ perpendicular to $OA$ at $A'$.

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## Problem Corner

(*continued from page 3*)

Conversely, for $D'$ on $L$, let the chord through $A$ perpendicular to $OD'$ intersect the circle at $B$ and $C$. Let $D$ be the intersection of the chord with $OD'$. Now $\Delta OAD$ and $\Delta OD'A'$ are similar right triangles. So $OD \times OD' = OA \times OA' = OB^2 = OC^2$, which implies $\angle OBD' = \angle OCD' = 90°$. Therefore, $D'$ is on the locus. This shows the locus is the line $L$.

*Other commended solvers*: **William CHEUNG Pok Man** (S.T.F.A. Leung Kau Kui College), **Wallis LEUNG Ka-Wo** (HKUST), **LIU Wai Kwong** (Pui Tak Canossian College) and **Bobby POON Wai Hoi** (St Paul's College).

**Problem 20.** For $n > 1$, let $2n$ chess pieces be placed on any $2n$ squares of an $n \times n$ chessboard. Show that there are 4 pieces among them that formed the vertices of a parallelogram. (Note that if $2n-1$ pieces are placed on the squares of the first column and the first row, then there is no parallelogram. So $2n$ is the best possible.)

**Solution: Edmond MOK Tze Tao** (Queen's College).

Let $m$ be the number of rows that have at least 2 pieces. (Then each of the remaining $n - m$ rows contains at most 1 piece.) For each of these $m$ rows, locate the leftmost square that contains a piece. Record the distances (i.e., number of squares) between this piece and the other pieces on the same row. The distances can only be 1, 2, …, $n$-1 because there are $n$ columns.

Since the number of pieces in these $m$ rows altogether is at least $2n - (n - m) = n + m$, there are at least $(n + m) - m = n$ distances recorded altogether for these $m$ rows. By the pigeonhole principle, at least two of these distances are the same. This implies there are at least two rows each containing 2 pieces that are of the same distance apart. These four pieces yield a parallelogram.

*Other commended solvers*: **William CHEUNG Pok Man** (S.T.F.A. Leung Kau Kui College), **HO Wing Yip** (Clementi Secondary School) and **WONG Him Ting** (Salesian English School).

## 談談質數：

(*continued from page 2*)

即得自然數集合，由定理3可知其中有無窮多質數。算術序列中包有無窮多質數這個問題是狄里希勒 (Dirichlet) 解決的，即

定理 7 (Dirichlet)：算術序列中有無窮多個質數。

我們還可以問：在算術序列中最小質數$P(l,q)$的上界估計？林尼克(Linnik)首先證明了：

定理 8 (Linnik)： $P(l,q) \le c_1 q^{c_2}$ ，其中$c_1, c_2$是兩個常數。

中國數學家潘承洞首先給出估計$c_2 = 5448$。現在最佳估計$c_2 = 5.5$是希斯-布朗(Heath-Brown)得到的。

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二、哥德巴赫猜想 (Goldbach Conjecture)。在Goldbach與Euler的通信中提出了這樣的猜想：

(i) 每一個偶數≥6都是兩個奇質數之和。
(ii)每一個奇數≥9都是三個奇質數之和。

猜想(ii)是猜想(i)的推論。事實上，如果(i)成立及$n$爲一個奇數≥9，則$n-3$爲偶數≥6，從而由(i)可知它是兩個質數$p_1$與$p_2$之和，即$n-3=p_1+p_2$，所以$n=3+p_1+p_2$，即(ii)成立。維諾格拉朵夫(Vinogradov)首先基本上證明了猜想(ii)，即

定理 9 (Vinogradov)：每個充分大的奇數都是三個奇質數之和。

關於猜想(i)，迄今仍然只有一些數值驗算，說明它可能是對的。例如有人在電腦上驗證過猜想(i)對於不超過$3\times10^8$的偶數都成立。首先是布倫(Brun)將Eratosthenes篩法加以改進，並證明了下面結果：

定理 10 (Brun)：每個大偶數都是兩個質因數不超過9的整數之和，簡記爲(9,9)。

Brun的結果與方法被很多數學家加以發展與改進。目前最好的結果是中國數學家陳景潤證明的，即

定理 11 (陳景潤)：每個大偶數都是一個質數及一個質因數不超過2的整數之和，簡記爲(1,2)。

類似地，還有所謂學生質數猜想：3, 5；5, 7；11, 13；…；10016957, 10016959；…；$10^9+7$, $10^9+9$；… 皆爲相差爲2的質數對，我們稱這樣一對質數爲學生質數對(twin primes)。已知小於100,000者有1,224對，小於1,000,000者有8,164對，現在所知道的最大學生質數對爲

$$260497545\times2^{6625}-1,\quad 260497545\times2^{6625}+1.$$

有一個著名猜想爲：學生質數對有無窮多？這是定理3的深化，迄今仍未能證明。這一猜想與Goldbach猜想(i)有深刻的內在聯繫。考慮不定方程式

$$ax+by=c.$$

其中$a,b,c$爲給予整數，試求這一方程式的質數解$x,y$問題？當$a=b=1$，$c$爲偶數≥6時，方程的可解性即Goldbach猜想(i)。當$a=1, b=-1, c=2$，方程式有無窮多解即相當於學生質數對猜想。用陳景潤證明(1,2)的方法可以證明：

定理 12 (陳景潤)：存在無窮多個質數$p$使$p+2$爲不超過2個質數之乘積。

## Olympiad Corner

(*continued from page 1*)

**Problem 3.** Given a nonisosceles, nonright triangle $ABC$, let $O$ denote the center of its circumscribed circle, and let $A_1$, $B_1$, and $C_1$ be the midpoints of sides $BC$, $CA$, and $AB$, respectively. Point $A_2$ is located on the ray $OA_1$ so that $\Delta OAA_1$ is similar to $\Delta OA_2A$. Points $B_2$ and $C_2$ on rays $OB_1$ and $OC_1$, respectively, are defined similarly. Prove that lines $AA_2$, $BB_2$, and $CC_2$ are concurrent, i.e., these three lines intersect at a point.

**Problem 4.** Suppose $q_0, q_1, q_2, \ldots$ is an infinite sequence of integers satisfying the following two conditions:

(i) $m-n$ divides $q_m - q_n$ for $m > n \ge 0$,
(ii) there is a polynomial $P$ such that $|q_n| < P(n)$ for all $n$.

Prove that there is a polynomial $Q$ such that $q_n = Q(n)$ for all $n$.

**Problem 5.** Suppose that in a certain society, each pair of persons can be classified as either *amicable* or *hostile*. We shall say that each member of an amicable pair is a *friend* of the other, and each member of a hostile pair is a *foe* of the other. Suppose that the society has $n$ persons and $q$ amicable pairs, and that for every set of three persons, at least one pair is hostile. Prove that there is at least one member of the society whose foes include $q(1-4q/n^2)$ or fewer amicable pairs.