Mathematical Excalibur

Volume 21, Number 1 *October 2016 – March 2017*

Olympiad Corner

Below are the problems of the Final Round of the 65th Czech and Slovak Math Olympiad (April 4-5, 2016).

Problem 1. Let $p>3$ be a prime. Find the number of ordered sextuples (a,b,c,d,e,f) of positive integers, whose sum is $3p$, and all the fractions

$$\frac{a+b}{c+d},\frac{b+c}{d+e},\frac{c+d}{e+f},\frac{d+e}{f+a},\frac{e+f}{a+b}$$

are integers.

Problem 2. Let r and r_a be the radii of the inscribed circle and excircle opposite A of the triangle ABC. Show that if $r+r_a=|BC|$, then the triangle is right-angled.

Problem 3. Mathematics clubs are very popular in certain city. Any two of them have at least one common member. Prove that one can distribute rulers and compasses to the citizens in such a way that only one citizen get both (compass and ruler) and any club has to his disposal both, compass and ruler, from its members.

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Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***April 15, 2017***.

For individual subscription for the next five issues for the 17-18 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Miscellaneous Problems

Kin Y. Li

There are many Math Olympiad problems. Some are standard problems in algebra or in geometry or in number theory or in combinatorics, where there are some techniques for solving them. Then, there are problems that are not so standard, which cross two or more categories. In math problem books, they go under the category of miscellaneous problems. Some of these may arise due to curiosity. Then one may need to combine different facts to explain them. Below are some such problems we hope the readers will enjoy.

Example 1 (1995 USA Math Olympiad). A calculator is broken so that the only keys that still work are the sin, cos, tan, $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$ buttons. The display initially shows 0. Given any positive rational numbers q, show that pressing some finite sequence of buttons will yield q. Assume that the calculator does real number calculation with infinite precision. All functions are in terms of radians.

Solution. We will show that all numbers of the form $\sqrt{m/n}$, where m, n are positive integers, can be displayed by doing induction on $k=m+n$. (Since $r/s=\sqrt{r^2/s^2}$, these include all positive rational numbers.)

For $k=2$, pressing cos will display 1. Suppose the statement is true for integer less than k. Observe that if x is displayed, then letting $\theta=\tan^{-1}x$, we see

$$\cos^{-1}(\sin x)=\frac{\pi}{2}-\theta \quad and \quad \tan\left(\frac{\pi}{2}-\theta\right)=\frac{1}{x}.$$

So we can display $1/x=\tan(\cos^{-1}(\sin x))$. Therefore, to display $\sqrt{m/n}$ with $k=m+n$, we may assume $m<n$. By the induction step, since $(n-m)+m=n<k$, $\sqrt{(n-m)/m}$ can be displayed. Then using

$$\phi=\tan^{-1}\sqrt{(n-m)/m} \quad and \quad \cos\phi=\sqrt{m/n},$$

we can display $\sqrt{m/n}$. This completes the induction.

Example 2 (1986 Brazilian Math Olympiad). A ball moves endlessly on a circular billiard table. When it hits the edge it is reflected. Show that if it passes through a point on the table three times, then it passes through it infinitely many times.

Solution. Suppose AB and BC are two successive chords of the ball's path. By the reflection law, $\angle ABO = \angle OBC$. Now $\triangle OAB$ and $\triangle OBC$ are isosceles. So $\angle AOB = \angle BOC$. Hence, $AB=BC$. Then every chord of the path has the same length d.

We now claim that through any given point P inside the circle there are at most two chords with length d. Let AB and CD be a chord containing P, with $AP=a$ and $CP=b$. The power of P with respect to the circle is $PA\cdot PB=PC\cdot PD$, which is $a(d-a)=b(d-b)$. Hence, $a=b$ or $a+b=d$. This means that P always divides the chord containing it in two segments of fixed lengths a and $d-a$. Now if three chords passes through P, the circle with center P and radius a would cut the circle of the billiard table three times, a contradiction.

Thus if the path passes through P more than twice, then on two occasions it must be moving along the same chord AB. That implies $\angle AOB$ is a rational multiple of 2π and hence the path will traverse AB repeatedly.

Example 3. Is there a way to pack 250 $1\times1\times4$ bricks into a $10\times10\times10$ box?

Solution. Assign coordinate (x,y,z) to each of the cells, where $x,y,z=0,1,\ldots,9$. Let the cell (x,y,z) be given color $x+y+z$ (mod 4). Note each $1\times1\times4$ brick contain all 4 colors exactly once. If the packing is possible, then there are exactly 250 cells of each color. However, a direct counting shows there are 251 cells of color 1, a contradiction. So such packing is impossible.

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Example 4 (2013 Singapore Math Olympiad). Six musicians gathered at a chamber music festival. At each scheduled concert some of the musicians played while the others listened as members of the audience. What is the least number of such concerts which would need to be scheduled so that every two musicians each must play for the other in some concert?

Solution. Let the musicians be A,B,C, D,E,F. We first show that four concerts are sufficient. The four concerts with the performing musicians: $\{A,B,C\}$, $\{A,D,E\}$, $\{B,D,F\}$ and $\{C,E,F\}$ satisfy the requirement. We shall now prove that three concerts are not sufficient. Suppose there are only three concerts. Since everyone must perform at least once, there is a concert where two of the musicians, say A, B, played. But they must also played for each other. Thus we have A played and B listened in the second concert and vice versa in the third. Now C,D,E,F must all perform in the second and third concerts since these are the only times when A and B are in the audience. It is not possible for them to perform for each other in the first concert. Thus the minimum is 4.

Example 5 (1999 Brazilian Math Olympiad). Prove that there is at least one nonzero digit between the 1,000,000th and the 3,000,000th decimal digits of $\sqrt{2}$.

Solution. Let us suppose that all digits between the 1,000,000th and the 3,000,000th decimal digits of $\sqrt{2}$ are zeros. Then

$$\sqrt{2}=\frac{n}{10^{1,000,000}}+\varepsilon, \qquad (*)$$

where n is a positive integer and $\varepsilon > 0$ satisfy

$$n<2\cdot 10^{1,000,000} \text{ and } \quad \varepsilon<(10^{-3})^{10^{1,000,000}}.$$

By squaring (*), we can get

$$\begin{aligned} &2\cdot 10^{2,000,000}-n^2 \\ &=2n\varepsilon 10^{1,000,000}+\varepsilon^2 10^{2.000.000}. \end{aligned}$$

However, the left side is a positive integer and the right side is less than 1, which is a contradiction.

Example 6 (1995 Russian Math Olympiad). Is it possible to fill in the cells of a 9×9 table with positive integers ranging from 1 to 81 in such a way that the sum of the elements of every 3×3 square is the same?

Solution. Place 0,1,2,3,4,5,6,7,8 on the first, fourth and seventh rows. Place 3,4,5,6,7,8,0,1,2 on the second, fifth and eigth rows. Place 6,7,8,0,1,2,3,4,5 on the third, sixth and ninth rows. Then every 3×3 square contains 0 to 8. Consider this table and its 90° rotation. For each cell, fill it with the number $9a+b+1$, where a is the number in the cell originally and b is the number in the cell after the table is rotated by 90°. By inspection, 1 to 81 appears exactly once and every 3×3 square has sum 9×36+36+9=369.

Example 7. Can the positive integers be partitioned into infinitely many subsets such that each subset is obtained from any other subset by adding the same integer to each element of the other subset?

Solution. Yes. Let A be the set of all positive integers whose odd digit positions (from the right) are zeros. Let B be the set of all positive integers whose even digit positions (from the right) are zeros. Then A and B are infinite set and the set of all positive integers is the union of $a+B=\{a+b\colon\ b\in B\}$ as a range over the element of A. (For example, 12345 = 2040+10305 $\in$ 2040+B.)

Example 8 (2015 IMO Shortlisted Problem proposed by Estonia). In Lineland there are $n\geq 1$ towns, arranged along a road running from left to right. Each town has a left bulldozer (put to the left of the town and facing left) and a right bulldozer (put to the right of the town and facing right). The sizes of the $2n$ bulldozers are distinct. Every time when a right and left bulldozer confront each other, the larger bulldozer pushes the smaller one off the road. On the other hand, the bulldozers are quite unprotected at their rears; so if a bulldozers reaches the rear-end of another one, the first one pushes the second one off the road, regardless of their sizes.

Let A and B be two towns, with B being to the right of A. We say that town A can *sweep* town B *away* if the right bulldozer of A can move over to B pushing off all bulldozers it meets. Similarly, B can sweep A away if the left bulldozer of B can move to A pushing off all bulldozers of the towns on its way.

Prove that there is exactly one town which cannot be swept away by any other one.

Solution. Let T_1, T_2,…,T_n be the towns enumerated from left to right. Observe first that, if town T_a can sweep away town T_b, then T_a also can sweep away every town located between T_a and T_b.

We prove by induction on n. The case n=1 is trivial. For the induction step, we first observe that the left bulldozer in T_1 and the right bulldozer in T_n are completely useless, so we may forget them forever. Among the other $2n-2$ bulldozers, we choose the largest one. Without loss of generality, it is the right bulldozer of some town T_k with $k<n$.

Surely, with this right bulldozer T_k can sweep away all towns to the right of it. Moreover, none of these towns can sweep T_k away; so they also cannot sweep away any town to the left of T_k. Thus, if we remove the towns T_{k+1}, T_{k+2},…,T_n, none of the remaining towns would change its status of being (un)sweepable away by the others.

Applying the induction hypothesis to the remaining towns, we find a unique town among T_1,T_2,…,T_k which cannot be swept away. By the above reasons, it is also the unique such town in the initial situation. Thus the inductive step is established.

Example 9 (1991 Brazilian Math Olympiad). At a party every woman dances with at least one man, and no man dances with every woman. Show that there are men M and M' and women W and W' such that M dances with W, M' dances with W', but M does not dance with W', and M' does not dance with W.

Solution. Let M be one of the men who dance with the maximal number of women, W' one of the women he doesn't dance with, and M' one of the men W' dances with. If M' were to dance with every woman that M dances with, then the maximality of the number of women that M dances with would be contradicted, so there is a woman W that dances with M but not with M'.

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***April 15, 2017.***

Problem 496. Let a,b,c,d be real numbers such that $a+\sin b > c+\sin d$, $b+\sin a > d+\sin c$. Prove that $a+b>c+d$.

Problem 497. Let there be three line segments with lengths 1, 2, 3. Let the segment of length 3 be cut into $n\geq 2$ line segments. Prove that among these $n+2$ segments, there exist three of them that can be put to form a triangle where each side is one of the three segments.

Problem 498. Determine all integers $n>2$ with the property that there exists one of the numbers $1,2,\ldots,n+1$ such that after its removal, the n numbers left can be arranged as $a_1,a_2,\ldots,a_n$ with no two of $|a_1-a_2|$, $|a_2-a_3|$, …, $|a_{n-1}-a_n|$, $|a_n-a_1|$ being equal.

Problem 499. Let ABC be a triangle with circumcenter O and incenter I. Let Γ be the escribed circle of $\triangle ABC$ meeting side BC at L. Let line AB meet Γ at M and line AC meet Γ at N. If the midpoint of line segment MN lies on the circumcircle of $\triangle ABC$, then prove that points O, I, L are collinear.

Problem 500. Determine all positive integers n such that there exist $k\geq 2$ positive rational numbers such that the sum and the product of these k numbers are both equal to n.

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Solutions

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Problem 491. Is there a prime number p such that both p^3+2008 and p^3+2010 are prime numbers? Provide a proof.

Solution. **Adnan ALI** (Atomic Energy Central School 4, Mumbai, India), **Ioan Viorel CODREANU** (Secondary School Satulung, Maramures, Romania), **Prithwijit DE** (HBCSE, Mumbai, India), **EVGENIDIS Nikolaos** (M. N. Raptou High School, Palaiokastrou 10, Agia, Greece), **Karaganda** (Nazarbaev iIntellectual School, Nurligenov Temirlan - 9 grade student), **Koopa KOO**, **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S6), **Mark LAU**, **Toshihiro SHIMIZU** (Kawasaki, Japan), **Anderson TORRES**, **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

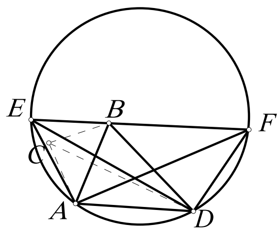
Let p be a prime. If $p\neq 7$, then $p^3\equiv -1$ or 1 (mod 7). Since $2008 \equiv -1$ (mod 7) and $2010\equiv 1$ (mod 7), so either p^3+2008 or p^3+2010 is divisible by 7, hence composite. If $p = 7$, then $p^3+2010 = 2353 = 13\times 181$ is composite. Therefore, there is no such prime.

Problem 492. In convex quadrilateral $ADBE$, there is a point C within $\triangle ABE$ such that

$$\angle EAD+\angle CAB=180^\circ=\angle EBD+\angle CBA.$$

Prove that $\angle ADE=\angle BDC$.

Solution. **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S6).



Let F be the second intersection of the circumcircle of $\triangle EAD$ and line EB. Then $\angle DBF=180^\circ-\angle EBD=\angle CBA$. Moreover,

$$\begin{aligned}\angle BDF &= 180^\circ-\angle AEB-\angle ADB\\ &= 180^\circ-(360^\circ-\angle EAD-\angle EBD)\\ &= 180^\circ-(\angle CAB+\angle CBA) =\angle BCA.\end{aligned}$$

These two relations give $\triangle BDF\sim\triangle BCA$. So $BD/BF=BC/BA$. Together with $\angle DBF =\angle CBA$, we have $\triangle BDC\sim\triangle BFA$. Then $\angle ADE=\angle AFE=\angle BFA=\angle BDC$.

Other commended solvers: **Toshihiro SHIMIZU** (Kawasaki, Japan), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Problem 493. For $n \geq 4$, prove that $x^n-x^{n-1}-x^{n-2}-\cdots-x-1$ cannot be factored into a product of two polynomials with rational coefficients, both with degree greater than 1.

Solution. **Prithwijit DE** (HBCSE, Mumbai, India) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Let $P_n(x) = x^n-x^{n-1}-x^{n-2}-\cdots-x-1$ and $Q_n(x)= (x-1)P_n(x)= x^{n+1}-2x^n+1$. The cases n = 2 or 3 follow directly from the rational root theorem. For $n\geq 4$, the Descartes' rule of signs shows there is a positive root r. It is easy to check $P_n(\sqrt{3}) < 0$. So $r>\sqrt{3}$.

If $P_n(s)=0$ with $|s|>1$, then $Q_n(s)=0$, which implies $|s|^n\,|s-2|=1$. We get $2\leq|s-2|+|s| = |s|^{-n}+|s|$. So $Q_n(|s|) \geq 0$. Since $Q_n(x)<0$ for $1<x<r$, we must have $|s|\geq r$. On the other hand, if $P_n(t)=0$ and $|t|<1$, then $1=|t-2||t|^n \leq 3|t|^n$. It follows that the absolute value of the product of all roots t of $P_n(x)$ with $|t|<1$ is at least 1/3. So r is the only root of $P_n(x)$ with absolute value greater than 1.

Assume $P_n(x)=f(x)g(x)$, where $f(x)$, $g(x)$ are monic polynomials with integer coefficients and $f(r)=0$. Then if $g(x)$ has positive degree, its roots would have absolute value less than 1 and so $|g(0)|<1$. This contradicts the constant term of $g(x)$, being $g(0)$, must be ± 1.

Other commended solvers: **Anderson TORRES**.

Problem 494. In a regular n-sided polygon, either 0 or 1 is written at each vertex. By using non-intersecting diagonals, Bob divides this polygon into triangles. Then he writes the sum of the numbers at the vertices of each of these triangles inside the triangle. Prove that Bob can choose the diagonals in such a way that the maximal and minimal numbers written in the triangles differ by at most 1.

Solution. **Adnan ALI** (Atomic Energy Central School 4, Mumbai, India) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

If all numbers written at the vertices of the polygon are equal, then the claim holds trivially. Hence assume that there are both zeros and ones among the numbers at the vertices. We prove by induction that, for every convex polygon, the partition into triangles can be chosen in such a way that Bob writes either 1 or 2 to each triangle.

If $n=3$, then this claim holds since the sum of the numbers at the vertices of a triangle can be neither 0 nor 3. If $n=4$, then draw the diagonal that connects the vertices where 0 and 1 are written, respectively, or, if such a diagonal does not exist, then an arbitrary diagonal. In both cases, only sums 1 and 2 can arise. If $n\geq 5$, then choose two consecutive vertices with different labels and a third vertex P that is not neighbor to either of them. Irrespective of whether the label of P is 0 or 1, we can draw the diagonal from it to one of the two consecutive vertices chosen before so that the labels of its endpoints are different. Now the polygon is divided into two convex polygons with smaller number of vertices so that both 0 and 1 occur among their vertex labels. By the induction hypothesis, both polygons can be partitioned into triangles with sum of labels of vertices either 1 or 2.

Other commended solvers: **William FUNG**.

Problem 495. The lengths of each side and diagonal of a convex polygon are rational. After all the diagonals are drawn, the interior of the polygon is partitioned into many smaller convex polygonal regions. Prove that the sides of each of these smaller convex polygons are rational numbers.

Solution. **Adnan ALI** (Atomic Energy Central School 4, Mumbai, India), **Toshihiro SHIMIZU** (Kawasaki, Japan) and **Anderson TORRES**.

We only need to show the quadrilateral case, since if this is showed, then the length of any segment of a diagonal connecting a vertex to an intersection point with other diagonal would be rational. Let $ABCD$ be a quadrilateral with all sides and diagonals have rational lengths. Let $\alpha=\angle ABD$ and $\beta=\angle DBC$. Let P be the intersection of AC and BD. Since

$$\cos\alpha=\frac{AB^2+BD^2-AD^2}{2AB\cdot BD},$$

$\cos\alpha$ is rational. Similarly, $\cos\beta$ and $\cos(\alpha+\beta)=\cos\angle ABC$ are rational. Then, since $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$, so $\sin\alpha\sin\beta$ is also rational. Also, $\sin^2\beta=1-\cos^2\beta$ is rational. Thus, $\sin\alpha/\sin\beta=\sin\alpha\sin\beta/\sin^2\beta$ is rational. Then, AP/PC = area(ABD)/area(DBC) = $(AB\cdot BD\sin\alpha)/(BD\cdot BC\sin\beta)$ is rational. Therefore, AP and PC are rational. Similarly, PB and PD are rational.

Other commended solvers: **Corneliu MĂNESCU-AVRAM** (Transportation High School, Ploieşti, Romania).

Olympiad Corner

(*Continued from page 1*)

Problem 4. For positive a, b, c, it holds $(a+c)(b^2+ac)=4a$. Find the maximal possible value of $b+c$ and find all triples (a,b,c), for which the value is attained.

Problem 5. There is $|BC|=1$ in a triangle ABC and there is a unique point D on BC such that $|DA|^2=|DB|\cdot|DC|$. Find all possible values of the perimeter of ABC.

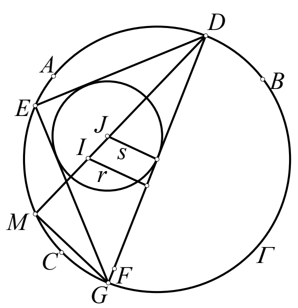
Problem 6. There is a figure of a prince on a field of a 6×6 square chessboard. The prince can in one move jump either horizontally or vertically. The lengths of the jumps are alternately either one or two fields, and the jump on the next field is the first one. Decide whether one can choose the initial field for the prince, so that the prince visits in an appropriate sequence of 35 jumps every field of the chessboard.

Miscellaneous Problems

(*Continued from page 2*)

<u>*Example 10.*</u> Two triangles have the same incircle. If a circle passes through five of the six vertices of the two triangles, then must it also pass the sixth vertex?

<u>*Solution.*</u> Let ABC and DEF be the triangles. Let A, B, C, D, E be on the same circle Γ, with radius R and center O. Suppose that F does not belong to Γ. Let $G\neq D$ be the intersection of DF with Γ. Let $\theta=\angle EDF=\angle EDG$. Let I and r be the common incenter and the inradius of $\triangle ABC$ and $\triangle DEF$. Let J and s be the incenter and the inradius of $\triangle DEG$.



We will prove that the incircle of $\triangle ABC$ and $\triangle DEG$ coincide. First, we prove that $I=J$ by showing $IM=JM$. It is well known that $IM=2R\sin(\theta/2)=EM$. From Euler's formula, $OI^2=R^2-2Rr$, which implies that the power of I with respect to Γ is $IM\cdot ID=2Rr$. Since $ID=r/\sin(\theta/2)$, we have $IM=2R\sin(\theta/2)=JM$. So $I=J$. This also proves $r=s$. Hence, the incircle of $\triangle ABC$ and $\triangle DEG$ are the same. Then $F=G$ follows.

<u>*Example 11 (1988 Brazilian Math Olympiad).*</u> A figure on a computer screen shows n points on a sphere, no four coplanar. Some pairs of points are joined by segments. Each segment is colored red or blue. For each point there is a key that switches the colors of all segments with that point as endpoint. For every three points there is a sequence of key presses that make all three segments between them red. Show that it is possible to make all the segments on the screen red. Find the smallest number of key presses that can turn all the segments red, starting from the worst case.

<u>*Solution.*</u> Consider three of the points. The parity of the number of blue segments of the triangle with these points as vertices doesn't change while switching the keys. Since it is possible to make all three segments red, the number of blue segments in each triangle is even.

Let P be one of the n points. Let A be the set of points connected to P by red points and B be the set of points connected to P by blue segments. Let $A_1, A_2\in A$. So PA_1 and PA_2 are both red and thus A_1A_2 is red. Now consider $B_1B_2\in B$. Then PB_1 and PB_2 are both blue and B_1B_2 is red. Finally consider $A\in A$ and $B\in B$. PA is red and PB is blue, so AB is blue. Put P in A. All this reasoning shows that segments in the same set are red and segments connecting points in different sets are blue.

Switching all points in set A will make all segments red. Indeed, all segments in A will change twice, one time from each of its edges, all segments connecting points from A and B will change once, turning from blue to red and segments in B won't change. This proves the first part.

For the second part, notice first that one needs to switch each point at most once. Let $|A|=k$ and $|B|=n-k$. If we switch a point from A and b points from B, we change at most $a(n-k)+bk$ blue segments. Suppose without loss of generality that $k\leq n-k$, hence $k\leq[n/2]$. Then $k(n-k)\leq a(n-k)+bk\leq a(n-k)+b(n-k)$, hence $k\leq a+b$. So the number of key presses is at most k and in the worst case, $[n/2]$. This number is needed to make all segments red if $|A|=[n/2]$.