Mathematical Excalibur

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Olympiad Corner

Below were the problems of the Balkan Mathematical Olympiad which took place in Belgrade, Serbia on May 9, 2018.

Time allowed was 270 minutes. Each problem was worth 10 points

Problem 1. A quadrilateral $ABCD$ is inscribed in a circle k, where $AB > CD$ and AB is not parallel to CD. Point M is the intersection of the diagonals AC and BD and the perpendicular from M to AB intersects the segment AB at the point E. If EM bisects the angle CED, prove that AB is a diameter of the circle k. (*Bulgaria*)

Problem 2. Let q be a positive rational number. Two ants are initially at the same point X in the plane. In the n-th minute (n=1,2,…) each of them chooses whether to walk due north, east, south or west and then walks the distance of q^n metres. After a whole number of minutes, they are at the same point in the plane (not necessarily X), but have not taken exactly the same route within that time. Determine all possible values of q. (*United Kingdom*)

(continued on page 4)

Acknowledgment: Thanks to Sindy Ting, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***December 1, 2018***.

For individual subscription for the next five issues for the 17-18 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Miscellaneous Inequalities

Kin Y. Li

There are many kinds of inequality problems in mathematical Olympiad competitions. Some of these can be solved by applying certain powerful inequalities such as rearrangement or majorization or Muirhead's inequalities. Some can be solved by techniques like tangent line methods using a bit of differential calculus.

In this article, we will be looking at some inequality problems that are not solved by these kinds of powerful tools and techniques.

Example 1. (*1983 IMO Shortlisted Problem proposed by Finland*) Let p and q be integers with q>0. Show that there exists an interval I of length $1/q$ and a polynomial P with integral coefficients such that

$$\left|P(x)-\frac{p}{q}\right|<\frac{1}{q^2}$$

for all $x\in I$.

Solution. Pick $P(x) = p((qx-1)^{2n+1}+1)/q$ and $I = [1/(2q),\ 3/(2q)]$. Then all the coefficients of P are integers and

$$\left|P(x)-\frac{p}{q}\right|=\left|\frac{p}{q}(qx-1)^{2n+1}\right|\le\left|\frac{p}{q}\right|\frac{1}{2^{2n+1}}$$

for all $x\in I$. Choose n large so that $2^{2n+1} > |pq|$. Then we are done.

Example 2 (*1994 IMO*) Let m and n be positive integers. The set $A=\{a_1, a_2, \ldots, a_m\}$ is a subset of $1,2,\ldots,n$. Whenever $a_i+a_j\le n$, $1\le i\le j\le m$, a_i+a_j also belong to A. Prove that

$$\frac{a_1+a_2+\cdots+a_m}{m}\ge\frac{n+1}{2}.$$

Solution. We may assume that $a_1>a_2>\cdots>a_m$. We claim that for $i=1,2,\ldots,m$,

$$a_i+a_{m+1-i}\ge n+1. \qquad (*)$$

If not, then $a_i+a_{m+1-i}, \ldots, a_i+a_{m-1}, a_i+a_m$ are i different elements of A greater than a_i, which is impossible. By adding the cases $i=1,2,\ldots,m$ of (*), we get

$$2(a_1+\cdots+a_m)\ge m(n+1).$$

The result follows.

Example 3 (2001 IMO Shortlisted Problem proposed by Bulgaria). Find all positive integers $a_1, a_2, \ldots, a_n$ such that

$$\frac{99}{100}=\frac{a_0}{a_1}+\frac{a_1}{a_2}+\cdots+\frac{a_{n-1}}{a_n},$$

where $a_0=1$ and $(a_{k+1}-1)a_{k-1}\ge a_k^2(a_k-1)$ for $k=1,2,\ldots,n-1$.

Solution. Let $a_1, a_2, \ldots, a_n$ satisfy the conditions of the problem. Then $a_k>a_{k-1}$ and hence $a_k\ge 2$ for $k=1,2,\ldots,n$. The inequality $(a_{k+1}-1)a_{k-1}\ge a_k^2(a_k-1)$ can be rewritten as

$$\frac{a_{k-1}}{a_k}+\frac{a_k}{a_{k+1}-1}\le\frac{a_{k-1}}{a_k-1}.$$

Adding these inequalities for $k=i+1,\ldots,n-1$ and using $a_{n-1}/a_n< a_{n-1}/(a_n-1)$, we obtain

$$\frac{a_i}{a_{i+1}}+\cdots+\frac{a_{n-1}}{a_n}<\frac{a_i}{a_{i+1}-1}.$$

Then

$$\frac{a_i}{a_{i+1}}\le\frac{99}{100}-\frac{a_0}{a_1}-\cdots-\frac{a_{i-1}}{a_i}<\frac{a_i}{a_{i+1}-1} \qquad (*)$$

for $i=1,2,\cdots,n-1$. Now given $a_0,a_1,\ldots,a_i$, there is at most one possibility for a_{i+1}. By (*), this yields $a_1=2$, $a_2=5$, $a_3=56$, $a_4=78400$. These values satisfy the condition of the problem. So this is a unique solution.

Example 4 (1999 Polish Math Olympiad). Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be integers. Prove that

$$\sum_{1\le i<j\le n}(|a_i-a_j|+|b_i-b_j|)\le\sum_{1\le i<j\le n}|a_i-b_j|.$$

(continued on page 2)

Solution. For integer x, let $f_{\{a,b\}}(x)=1$ if either $a\le x<b$ or $b\le x<a$ and $f_{\{a,b\}}(x)=0$ otherwise. Observe that when a,b are integers, $|a-b|$ equals the sum of $f_{\{a,b\}}(x)$ over all integers x. Now fix an integer x and suppose $a_\le$ is the number of values of i for which $a_i\le x$.

Define $a_>$, $b_\le$, $b_<$ analogously. We have

$$\begin{aligned}&(a_\le-b_\le)+(a_>-b_>)\\&=(a_\le+a_<)-(b_\le+b_>)\\&=n-n=0,\end{aligned}$$

which implies $(a_\le-b_\le)(a_>-b_>)\le 0$. Thus

$$a_\le a_> + b_\le b_> \le a_\le b_> + a_> b_\le.$$

Now

$$a_\le a_> = \sum_{1\le i<j\le n} f_{\{a_i,b_j\}}(x).$$

because both sides count the same set of pairs and the other terms reduce similarly, yielding

$$\sum_{1\le i<j\le n} f_{\{a_i,a_j\}}(x)+f_{\{b_i,b_j\}}(x)\le \sum_{1\le i<j\le n} f_{\{a_i,b_j\}}(x).$$

Because x was an arbitrary integer, this last inequality holds for all integers x. Summing over all integers x and using our first observation, we get the desired inequality. Equality holds if and only if the above inequality is an inequality for all x, which is true precisely when the a_i equal the b_i in some order.

Example 5 (2007 Chinese Math Olympiad). Let a,b,c be complex numbers. Let $|a+b|=m$, $|a-b|=n$ and $mn\ne 0$. Prove that

$$\max\{|ac+b|,|a+bc|\}\ge \frac{mn}{\sqrt{m^2+n^2}}.$$

Solution. Since

$$\begin{aligned}&\max\{|ac+b|,|a+bc|\}\\ \ge\ &\frac{|b|\cdot|ac+b|+|a|\cdot|a+bc|}{|b|+|a|}\\ \ge\ &\frac{|b(ac+b)-a(a+bc)|}{|a|+|b|}=\frac{|b^2-a^2|}{|a|+|b|}\\ \ge\ &\frac{|b+a|\cdot|b-a|}{\sqrt{2(|a|^2+|b|^2)}}\end{aligned}$$

and $m^2+n^2=|a-b|^2+|a+b|^2=2(|a|^2+|b|^2)$, so

$$\max\{|ac+b|,|a+bc|\}\ge \frac{mn}{\sqrt{m^2+n^2}}.$$

Example 6 (1999 Balkan Math Olympiad). Let $x_0,x_1,x_2,\ldots$ be a non-decreasing sequence of nonnegative integers such that for every $k\ge 0$, the number of terms of the sequence which are less than or equal to k is finite; let this number be y_k. Prove that for all positive integers m and n,

$$\sum_{i=0}^n x_i+\sum_{j=0}^m y_j\ge (n+1)(m+1).$$

Solution. Under the given construction, $y_s\le t$ if and only if $x_t>s$. Thus the sequences $x_0,x_1,x_2,\ldots$ and $y_0,y_1,y_2,\ldots$ are dual, meaning that applying the given algorithm to $y_0,y_1,y_2,\ldots$ will restore the original $x_0,x_1,x_2,\ldots$.

To find $x_0+x_1+\cdots+x_n$, observe that among the numbers $x_0,x_1,\cdots,x_n$, there are exactly y_0 terms equal to 0, y_1-y_0 terms equal to 1, … and $y_{x_{n-1}}-y_{x_{n-2}}$ terms equal to x_{n-1}, while the remaining $n+1-x_{n-1}$ terms equal to x_n. Hence, $x_0+x_1+\cdots+x_n$ equals

$$\begin{aligned}&\sum_{i=1}^{x_n-1} i(y_i-y_{i-1})+x_n(n+1-y_{x_n-1})\\&=-y_0-y_1-\cdots-y_{x_{n-1}}+(n+1)x_n.\end{aligned}$$

First suppose that $x_n-1\ge m$. Write $x_n-1=m+k$ for $k\ge 0$. Because $x_n>m+k$, from our initial observations we have $y_{m+k}\le n$. Then

$$n+1\ge y_{m+k}\ge y_{m+k-1}\ge\cdots\ge y_m.$$

So

$$\begin{aligned}\sum_{i=0}^n x_i+\sum_{j=0}^m y_j &= (n+1)x_n-\left(\sum_{j=0}^{x_{n-1}} y_j-\sum_{i=0}^m y_i\right)\\ &= (n+1)x_n-\sum_{i=m+1}^{m+k} y_i\\ &\ge (n+1)(m+k+1)-k(n+1)\\ &= (n+1)(m+1).\end{aligned}$$

Next suppose that $x_n-1<m$. Then $x_n\le m$ implies $y_m>n$, which implies $y_m-1\ge n$. Because $x_0,x_1,x_2,\ldots$ and $y_0,y_1,y_2,\ldots$ are dual, we may apply the same argument with the roles of the two sequences reversed. This completes the proof.

Example 7 (2007 Chinese Girls' Math Olympiad). Let m,n be integers, $m>n\ge 2$, $S=\{1,2,\ldots,m\}$ and $T=\{a_1,a_2,\ldots,a_n\}$ be a subset of S. Suppose every two elements of T are not both the divisors of any element of S. Prove that

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}<\frac{m+n}{m}.$$

Solution. For $i=1,2,\ldots,n$, let k_i be the integer such that $k_i\le m/a_i<k_i+1$. Let $T_i=\{ka_i : k=1,\ldots,k_i\}$. Then $|T_i|=k_i$ Since every two elements of T are not both the divisors of any element of S, so if $i\ne i'$, then $T_i\cap T_{i'}$ is empty. Hence,

$$\sum_{i=1}^n k_i=\sum_{i=1}^n |T_i|=|T|\le|S|=m.$$

Since $m/a_i<k_i+1$, we have

$$m\sum_{i=1}^n 1/a_i\le\sum_{i=1}^n(k_i+1)\le m+n.$$

Dividing by m, we get the desired conclusion.

Example 8 (1987 IMO Shortlisted Problem proposed by Netherland). Given five real numbers u_0, u_1, u_2, u_3, u_4, prove that it is always possible to find five real numbers v_0, v_1, v_2, v_3, v_4 that satisfy the following conditions:

(i) $u_i-v_i\in\mathbb{N}$.
(ii) $\Sigma_{0\le i<j\le 4}(v_i-v_j)^2<4$.

Solution. Observe that

$$\begin{aligned}\sum_{0\le i<j\le 4}(v_i-v_j)^2 &= \sum_{0\le i<j\le 4}[(v_i-v)-(v_j-v)]^2\\ &= 5\sum_{i=0}^4(v_i-v)^2-\left(\sum_{i=0}^4(v_i-v)\right)^2\\ &\le 5\sum_{i=0}^4(v_i-v)^2.\end{aligned}$$

Let us take v_i's satisfying the last line with $v_0\le v_1\le v_2\le v_3\le v_4\le 1+v_0$. Define $v_5=1+v_0$. We see that one of the differences $v_{i+1}-v_i$, $i=0,\ldots,4$, is at most 1/5. Let $v=(v_{i+1}+v_i)/2$. Then place the other three v_i's in $[v-1/2,v+1/2]$. Now we have $|v-v_i|\le 1/10$, $|v-v_{i+1}|\le 1/10$ and $|v-v_k|\le 1/2$ for any k other than i and $i+1$. Finally, we have

$$\sum_{0\le i<j\le 4}(v_i-v_j)^2\le 5(2(1/10)^2+3(1/2)^2)<4.$$

Example 9 (2000 Romanian Math Olympiad). Let $n\ge 1$ be an odd positive integer and x_1, x_2, …, x_n be real numbers such that $|x_{k+1}-x_k|\le 1$ for $k=1,2,\ldots,n-1$. Show that

$$\sum_{k=1}^n|x_k|-\left|\sum_{k=1}^n x_k\right|\le\frac{n^2-1}{4}.$$

Solution. Let P, N be the sets of positive, negative numbers among x_1, x_2, …, x_n respectively. Without loss of generality, assume that there are more k such that x_k is negative than there are k such that x_k is positive. Let $(a_1,\ldots,a_n)$ be a permutation of $(x_1,\ldots,x_n)$ such that $a_1,\ldots,a_n$ is a nondecreasing sequence. By construction, $|P|\le(n-1)/2$.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for sending solutions is ***December 1, 2018.***

Problem 521. Given 20 points in space so that no three of them are collinear, prove that the number of planes determined by these points is not equal to 1111.

Problem 522. Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that for all real x and y,

$$(x-2)f(y)+f(y+2f(x))=f(x+yf(x)).$$

Problem 523. Find all positive integers n for which there exists a polynomial $P(x)$ with integer coefficients such that $P(d)=(n/d)^2$ for each positive divisor d of n.

Problem 524. (proposed by *Andrew WU*, St. Albans School, Mc Lean, VA, USA) In ΔABC with centroid G, M and N are the midpoints of AB and AC, and the tangents from M and N to the circumcircle of ΔAMN meet BC at R and S, respectively. Point X lies on side BC satisfying $\angle CAG=\angle BAX$. Show that GX is the radical axis of the circumcircles of ΔBMS and ΔCNR.

Problem 525. Find all positive integer n such that $n(n+2)(n+4)$ has at most 15 positive divisors.

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Solutions

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Problem 516. Determine all triples (p,m,n) of positive integers such that p is prime and $2^mp^2+1=n^5$ holds.

Solution. **CHUI Tsz Fung** (Ma Tau Chung Government Primary School) and **ZHANG Yupei** (HKUST).

Let $q=n^4+n^3+n^2+n+1$. Then $2^mp^2=(n-1)q$ and $\gcd(n-1,q)=\gcd(n-1,5)=1$ or 5. Now $q>1$ is odd and so p is an odd prime. Let $p=2k+1$. Then $\gcd(2^m,p^2)=1$. So $n-1=2^m$, $q=p^2$. Then $n=2^m+1$. So $n^4+n^3+n^2+n=p^2-1$ can be expressed as

$$(2^{2m}+2^{m+1}+2)(2^{2m}+3\cdot 2^m+2)=4k(k+1).$$

If $m\ge 2$, then the left side is 4 (mod 8) and the right side is 0 (mod 8). Hence, $m=1$. Then $p=11$ and $n=3$. So $(p,m,n)=(11,1,3)$ only.

Other commended solvers: **Ioan Viorel CODREANU** (Satulung, Maramures, Romania), **Akash Singha ROY** (West Bengal, India), **Ioannis D. SFIKAS** (Athens, Greece), **Toshihiro SHIMIZU** (Kawasaki, Japan) and **Nicuşor ZLOTA** ("Traian Vuia" Technical College, Focşani, Romania).

Problem 517. For all positive x and y, prove that

$$x^2y^2(x^2+y^2-2)\ge(xy-1)(x+y).$$

Solution. **CHUI Tsz Fung** (Ma Tau Chung Government Primary School).

Let $k=xy$. We have

$$2\sqrt{k}-\frac{2k+2}{2\sqrt{k}}-\frac{k-1}{k^2}$$
$$=\frac{(\sqrt{k}-1)^2(k\sqrt{k}+2k+2\sqrt{k}+1)}{k^2}\ge 0$$

Since $2\sqrt{k}=2\sqrt{xy}\le x+y$, so

$$\frac{x^2+y^2-2}{x+y}=x+y-\frac{2xy+2}{x+y}$$

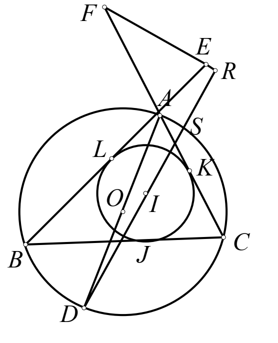
$$\ge 2\sqrt{k}-\frac{2k+2}{2\sqrt{k}}\ge\frac{k-1}{k^2}=\frac{xy-1}{x^2y^2}.$$

Then $x^2y^2(x^2+y^2-2)\ge(xy-1)(x+y)$.

Other commended solvers: **LEUNG Hei Chun**, **Paolo PERFETTI** (Math Dept, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy), **Ioannis D. SFIKAS** (Athens, Greece), **Nicuşor ZLOTA** ("Traian Vuia" Technical College, Focşani, Romania).

Problem 518. Let I be the incenter and AD be a diameter of the circumcircle of ΔABC. Let point E be on the ray BA and point F be on the ray CA. If the lengths of BE and CF are both equal to the semiperimeter of ΔABC, then prove that lines EF and DI are perpendicular.

Solution. **ZHANG Yupei** (HKUST).



Let circle ABC intersect line DI at S. Let K, J, L be the feet of the perpendiculars from I to sides AC, CB, BA of ΔABC respectively. Since AD is a diameter of the circumcircle of ΔABC, we get $\angle ASD=\angle AKI=\angle ALI=90°$. So A,S,K,I,L are concyclic.

Next, $\angle BLS=180°-\angle ALS=180°-\angle AKS=\angle CKS$ and $\angle LBS=\angle KCS$. So ΔBLS, ΔCKS are similar. Since $BE=CF$, $AF/AE=BL/CK=SB/SC$. We get $\angle EAF=\angle CAB=\angle CSB$. So $\Delta EAF\cong\Delta CSB$. Then $\angle SBC=\angle SAC=\angle EFA$. We get $EF\parallel AS$. Then $DI\perp EF$.

Other commended solvers: **Andrea FANCHINI** (Cantù, Italy), **William KAHN** (Sidney, Australia), **Akash Singha ROY** (West Bengal, India), **Ioannis D. SFIKAS** (Athens, Greece), and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Problem 519. Let A and B be subsets of the positive integers with 10 and 9 elements respectively. Suppose for every $x,y,u,v\in A$ satisfying $x+y=u+v$, we have $\{x,y\}=\{u,v\}$. Prove that the set $A+B=\{a+b: a\in A, b\in B\}$ has at least 50 elements.

Solution. **CHUI Tsz Fung** (Ma Tau Chung Government Primary School).

If a_1, $a_2\in A$ and b_1, $b_2\in B$ such that $a_1+b_1=a_2+b_2$, then $a_1-a_2=b_2-b_1$ (with $a_1\ne a_2$ and $b_1\ne b_2$). Assume the equation $x+b_1=y+b_2$ has two distinct solutions $(x,y)=(a_3, a_4)$ and (a_5, a_6) such that a_3, a_4, a_5, $a_6\in A$. Then we have $a_3-a_4=b_2-b_1=a_5-a_6$, which implies $a_3+a_6=a_4+a_5$. By the condition of A, we have $\{a_3, a_6\}=\{a_4, a_5\}$. Then we have 2 cases.

Case 1: $a_3=a_4$ and $a_5=a_6$. From $a_3+b_1=a_4+b_2$, we get $b_1=b_2$. Then $|a_3-a_4|+|b_1-b_2|=0$, contradiction.

Case 2: $a_3=a_5$ and $a_4=a_6$. Then $(a_3, a_4)=(a_5, a_6)$, contradiction.

So $x+b_1=y+b_2$ has at most one solution. Since there are 36 choices of $b_1\ne b_2\in B$, so there must be 36 solutions of (a_1, a_2, b_1, b_2) such that $a_1\ne a_2\in A$, $b_1\ne b_2\in B$ and $a_1+b_1=a_2+b_2$.

However, we have a_1+b_1, $a_2+b_2\in A+B$. Since $A+B$ has 90 not necessary distinct elements, so $A+B$ has at least 54 distinct elements. In particular, $A+B$ has at least 50 distinct elements.

Other commended solvers: **William KAHN** (Sidney, Australia), **Akash Singha ROY** (West Bengal, India), **George SHEN, Toshihiro SHIMIZU** (Kawasaki, Japan) and **ZHANG Yupei** (HKUST).

Problem 520. Let P be the set of all polynomials $f(x)=ax^2+bx$, where a,b are nonnegative integers less than 2010^{18}. Find the number of polynomials f in P for which there is a polynomial g in P such that $g(f(k))\equiv k \pmod{2010^{18}}$ for all integers k.

Solution. **William KAHN** (Sidney, Australia) and **George SHEN**.

We will show that there exists $Q(x) = cx^2+dx$ for $P(x) = ax^2+bx$ if and only if $2^8 1005^9|a$ and $\gcd(2010,b)=1$. Then it follows that the answer is $2\cdot 2010^9\cdot 2010^{18}(1-1/2)(1-1/3)(1-1/5)(1-1/67) = 2^5 3\cdot 11\cdot 2010^{26}$.

Assume that $Q(P(n))\equiv n \pmod{2010^{18}}$ for all n. Then $n\to P(n)$ is one-to-one $\pmod{2010^{18}}$ and using the Chinese remainder theorem we deduce that $n\to P(n)$ is one-to-one $\pmod{p^{18}}$ for p in $\{2,3,5,67\}$.

Let $p\in\{2,3,5,67\}$. If $p|b$, then $P(p^{17})\equiv P(0) \pmod{p^{18}}$ gives a contradiction. Hence, $p\nmid b$. If $p\nmid a$, then $P(-a^{-1}b)\equiv P(0) \pmod{p^{18}}$ gives a contradiction. So $p\mid a$. Hence $2010\mid a$ and $\gcd(2010,b) = 1$. In particular, $(b(a^2-b^2))^{-1} \pmod{2010^{18}}$ exists. Since

$$\begin{aligned}&Q(P(1)) \equiv 1 \pmod{2010^{18}}\\ \Rightarrow\ & c(a+b)^2+d(a+b)\equiv 1 \pmod{2010^{18}}\\ \Rightarrow\ & 2b(a^2-b^2)c \equiv 2a \pmod{2010^{18}}\end{aligned}$$

and

$$\begin{aligned}&Q(P(-1)) \equiv -1 \pmod{2010^{18}}\\ \Rightarrow\ & c(a-b)^2+d(a-b) \equiv -1 \pmod{2010^{18}}\\ \Rightarrow\ & 2b(a^2-b^2)d\equiv -2(a^2+b^2) \pmod{2010^{18}}\end{aligned}$$

we have

$c\equiv(b(a^2-b^2))^{-1}a+2010^{18}e \pmod{2010^{18}}$ and $d\equiv -(b(a^2-b^2))^{-1}(a^2+b^2) + 2010^{18}e \pmod{2010^{18}}$, where $e = 0$ or ½.

Therefore,

$$\begin{aligned}&Q(P(x))-x\\ \equiv\ & -(b(a^2-b^2))^{-1}a^2x(x-1)(x+1)(ax+b)\\ &+2010^{18}ex(x-1)\\ \equiv\ & -(b(a^2-b^2))^{-1}a^2x(x-1)(x+1)(ax+2b)\\ &\pmod{2010^{18}}.\end{aligned}$$

Now if $x=2$, we get $2010^{18}\mid 2^2 3a^2$, hence $2^8 1005^9\mid a$.

Conversely, if $2^8 1005^9\mid a$ and $\gcd(2010,b)=1$, then we can define c and d as above. Since $2\mid n(n-1)$ and $2\mid an+2b$ for all n, $Q(P(n))\equiv n \pmod{2010^{18}}$ follows.

Other commended solvers: **Toshihiro SHIMIZU** (Kawasaki, Japan) and **ZHANG Yupei** (HKUST).

Olympiad Corner

(*Continued from page 1*)

Problem 3. Alice and Bob play the following game: They start with two non-empty piles of coins. Taking turns, with Alice playing first, each player choose a pile with an even number of coins and moves half of the coins of this pile to the other piles. The game ends if a player cannot move, in which case the other player wins. (*Cyprus*)

Problem 4. Find all primes p and q such that $3p^{q-1}+1$ divides 11^p+17^p. (*Bulgaria*)

Miscellaneous Inequalities

(*Continued from page 2*)

Suppose that $1\le i\le n-1$. In the sequence $x_1,\ldots,x_n$, there must be two adjacent terms x_k and x_{k+1} which are separated by the interval (a_i, a_{i+1}), i.e. such that either $x_k\le a_i\le a_{i+1}\le x_{k+1}$ or $x_{k+1}\le a_i\le a_{i+1}\le x_k$. So $a_{i+1}-a_i\le|x_k-x_{k+1}|\le 1$. That is $a_1,\ldots,a_n$ is a nondecreasing sequence of terms, such that any two adjacent terms differ by at most 1.

Let σ_P denote the sum of the numbers in P. We claim that $\sigma_P\le(n^2-1)/8$. This is certainly true if P is empty.

If P is nonempty, then the elements of P are $a_i\le a_{i+1}\le\cdots\le a_n$ for some $2\le i\le n$. Because $a_{i-1}\le 0$ by assumption and $a_i\le a_{i-1}+1$ from the previous paragraph, we have $a_i\le 1$.

Similarly, $a_{i+1}\le a_i+1\le 2$ and so on up to $a_n\le|P|$. Hence, $\sigma_P\le 1+2+\cdots+|P|$. From $|P|\le(n-1)/2$, we get $\sigma_P\le(n^2-1)/8$, as claimed.

Let σ_N denote the sum of the numbers in N. The left-hand side of the required inequality then equals

$$\begin{aligned}&|\sigma_P-\sigma_N|-|-\sigma_P-\sigma_N|\\ \le\ & |2\sigma_P|\\ \le\ & 2\left(\frac{n^2-1}{8}\right)=\frac{n^2-1}{4}\end{aligned}$$

as needed.

<u>*Example 10 (2000 Asia Pacific Math Olympiad).*</u> Let n, k be positive integers with $n > k$. Prove that

$$\frac{1}{n+1}\cdot\frac{n^n}{k^k(n-k)^{n-k}}<\frac{n!}{k!(n-k)!}<\frac{n^n}{k^k(n-k)^{n-k}}.$$

<u>*Solution.*</u> By the binomial theorem, we have $n^n=(k+(n-k))^n=a_0+\cdots+a_n$, where for $i=0,1,\ldots,n$,

$$a_i=\binom{n}{i}k^i(n-k)^{n-i}>0.$$

We claim that

$$\frac{n^n}{n+1}<a_i<n^n.$$

The right inequality holds because $n^n = a_0+\cdots+a_n>a_i$. To prove the left inequality, it suffices to prove that a_i is larger than $a_0,\ldots, a_{i-1}, a_{i+1}, \ldots, a_n$ because then

$$n^n=\sum_{m=0}^{n}a_m<\sum_{m=0}^{n}a_i=(n+1)a_i.$$

Next, we will show a_i is increasing for $i\le k$ and decreasing for $i\ge k$. Observe that

$$\binom{n}{i}=\frac{i+1}{n-i}\binom{n}{i+1}.$$

Hence

$$\frac{a_i}{a_{i+1}}=\frac{\binom{n}{i}k^i(n-k)^{n-i}}{\binom{n}{i+1}k^{i+1}(n-k)^{n-i-1}}=\frac{n-k}{n-i}\cdot\frac{i+1}{k}.$$

This expression is less than 1 when $i<k$ and it is greater than 1 when $i\ge k$. In other words, $a_0<\cdots<a_k$ and $a_k>\cdots>a_n$ as desired.