Mathematical Excalibur

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Olympiad Corner

Below were the Day 1 problems of the Croatian Mathematical Olympiad which took place on May 5, 2018.

Problem A1. Let a, b and c be positive real numbers such that $a+b+c=2$. Prove that

$$\frac{(a-1)^2}{b}+\frac{(b-1)^2}{c}+\frac{(c-1)^2}{a} \geq \frac{1}{4}\left(\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a}\right).$$

Problem C1. Let n be a positive integer. A *good word* is a sequence of $3n$ letters, in which each of the letters A, B and C appears exactly n times. Prove that for every good word X there exists a good word Y such that Y cannot be obtained from X by swapping neighbouring letters fewer than $3n^2/2$ times.

Problem G1. Let k be a circle centered at O. Let $\overline{AB}$ be a chord of that circle and M its midpoint. Tangent on k at points A and B intersect at T. The line ℓ goes through T, intersects the shorter arc AB at the point C and the longer arc AB at the point D, so that $|BC|=|BM|$.

(continued on page 4)

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***February 15, 2019***.

For individual subscription for the next five issues for the 17-18 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Austrian Math Problems

Kin Y. Li

In this article, we would like to look at some of the Austrian Math Olympiad problems. This competition is going into its 50^{th} year. For the young math students, the Austrian math problems are treasures that are everlasting, especially the problems appeared in the recent decades. Below are some examples that we hope you will enjoy.

Example 1. (*Beginners Competition: June 7^{th}, 2001*) Prove that the number n^n-1 is divisible by 24 for all odd positive integer values of n.

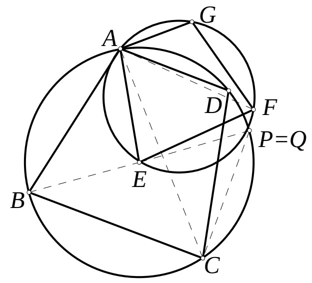
Solution. Since n is an odd positive integer, we can write $n=2k+1$ with $k=0,1,2,\ldots$. Substituting yields

$$n^n-1=n(n^{n-1}-1)=n(n^{2k}-1).$$

Since $1^2\equiv3^2\equiv5^2\equiv7^2\equiv1 \pmod 8$, we see that $n^{2k}\equiv1 \pmod 8$ certainly holds, and $n^{2k}-1$ is therefore divisible by 8.

If n is divisible by 3, we see that $n(n^{2k}-1)$ is certainly divisible by $3\cdot8=24$ as required. If n is not divisible by 3, we note that $1^2\equiv2^2\equiv1 \pmod 3$, and $n^{2k}\equiv1 \pmod 3$ holds, so that $n^{2k}-1$ is not only divisible by 8, but also by 3. It follows that $n^{2k}-1$ is therefore divisble by $3\cdot8=24$, and therefore so is $n(n^{2k}-1)$ as required.

Example 2 (*National Competition: June 6^{th}, 2002*) Let $ABCD$ and $AEFG$ be similar inscribed quadrilaterals, whose vertices are labeled counter-clockwise. Let P be the second common point of the circumcircles of the quadrilaterals beside A. Show that P must lie on the line connecting B and E.



Solution. Rotation and stretching with center A, $\angle BAC$ and factor $AB{:}AC$ maps B onto C and E onto F. This mapping therefore transforms the line $BE=BQ$ onto the line $FC=FQ$, whereby we let Q denote the point of intersection of lines BE and FC. Since this mapping rotates by $\angle BAC$, this is also the angle between the lines BQ and FQ, and since this is equal to $\angle BAC$ (or its supplement), Q must lie on the circumcircle of ΔABC, which is also the circumcircle of $ABCD$. By analogous reasoning, it must also lie on the circumcircle of $AEFG$, and we see that $P=Q$ must hold, which proves that P must lie on the line BE, as required.

Example 3 (*National Competition: May 26^{th}, 2004*). Prove without the use of calculus:

a) If a, b, c and d are real numbers, then

$$a^6+b^6+c^6+d^6-6abcd \geq -2$$

holds. When does equality hold?

b) For which positive integers k does there exist an inequality of the form

$$a^k+b^k+c^k+d^k-kabcd \geq M_k$$

that holds for all real values of a, b, c and d? Determine the largest possible values of M_k and determine when equality holds.

Solution. a) The given inequality can be proved by applying the AM-GM inequality as

$$\frac{a^6+b^6+c^6+d^6+1^6+1^6}{6} \geq |abcd| \geq abcd.$$

Equality holds for $|a|=|b|=|c|=|d|=1$, more precisely when (a,b,c,d) equals one of

$(1,1,1,1)$, $(1,1,-1,-1)$, $(1,-1,1,-1)$, $(-1,1,1,-1)$, $(1,-1,-1,1)$, $(-1,1,-1,1)$, $(-1,-1,1,1)$ or $(-1,-1,-1,-1)$.

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b) First of all, we note that no such number M_k can possibly exist if k is odd, since a choice of negative values for a, b, c and d with sufficiently large absolute value yields negative values with arbitrary large absolute value for the expression $a^k+b^k+c^k+d^k-kabcd$.

Similarly, no such number exists for k=2, since a choice of $a=b=c=d=r$ yields $a^2+b^2+c^2+d^2-2abcd = 4r^2-2r^4$, for which a choice of sufficiently large values of r again yields negative values with arbitrarily large absolute value.

This leaves even values of k with $k\geq 4$ to consider. In this case, choosing $a=b=c=d=1$ yield $a^k+b^k+c^k+d^k-kabcd = 4-k$, and as in a), we can apply AM-GM inequality to get

$$\frac{a^k+b^k+c^k+d^k+(k-4)1^k}{k} \geq |abcd| \geq abcd$$

with equality for the same values of (a,b,c,d) as in a).

Example 4 (*National Competition: June 6th, 2007*) We are given a convex n-gon with a triangulation, i.e. a division into triangles by non-intersecting diagonals. Prove that the n corners of the n-gon can each be labeled by the digits of 2007 such that any quadrilateral composed of two triangles in the triangulation with a common side has corners labeled by digits with the sum 9.

Solution. We shall prove this by induction on n. If n=4, we label the vertices 2, 0, 0, 7 and the claim holds. (Note that this is the only possible combination of digits summing to 9, since $4\cdot 2<9$ and $2\cdot 7>9$ hold. Also note that the three corners of any triangle must be labeled with three of the digits 2, 0, 0, 7.)

We now assume that the claim holds as stated for any convex n-gon, and consider a convex $(n+1)$-gon. Any triangulation of such an $(n+1)$-gon certainly contains at least one triangle (in fact, at least two), two of whose sides are consecutive sides of the $(n+1)$-gon with common vertex V. The n-gon obtained by removing this one triangle from the triangulation with the implied triangulation in the remaining n-gon as given can certainly be labeled as required.

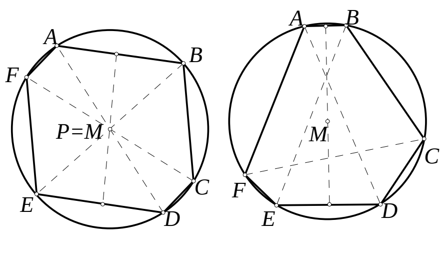
We now note that the triangle with vertex V only has a side in common with one other triangle of the triangulation, the corners of which are already labeled with three of the four required digits. Labelling V with the fourth digit results in a labeling of the $(n+1)$-gon with the required property.

Example 5 (*National Competition: June 3rd, 2010*) A diagonal in a hexagon is considered a *long* diagonal if it divides the hexagon into two quadrilaterals. Any two long diagonals divide the hexagon into two triangles and two quadrilaterals.

We are given a convex hexagon with the property that the division into pieces by any two long diagonals always yields two isosceles triangles with sides of the hexagon as bases. Show that such a hexagon must have a circumcircle.

Solution. Since any two opposing isosceles triangles (such as ABP and DEP) have a common angle at their vertices, they must be similar, and their bases therefore parallel. The angle bisector in their common vertex is therefore also the common altitude.

If all three diagonals of the hexagon intersect at M, this point is also a common point of all angle bisectors. It must therefore be the same distance from A to B, as it lies on the bisector of AB, but the same holds for B and C, C and D, and so on. This point is therefore equidistant from all corners of the hexagon, and is therefore the mid-point of the circumcircle of the hexagon.



If the diagonals of the hexagon do not have a common point, they form a triangle. The angle bisectors have a common point, namely the incenter of this triangle, which we again call M. The same holds for this point M as in the previous situation, and we once again have established the existence of a circumcircle of the hexagon, as claimed.

Example 6 (*National Competition: May 1st, 2015*) A *police emergency number* is a positive integer that ends with the digits 133 in decimal representation. Prove that every police emergency number has a prime factor larger than 7.

(*In Austria, 133 is the emergency number of the police*.)

Solution. Let $n=1000k+133$ be a police emergency number and assume that all its prime divisors are at most 7. It is clear from the last digit that n is odd and that n is not divisible by 5, so $1000k+133 = 3^a7^b$ for suitable integers $a,b\geq 0$. Thus, $3^a7^b\equiv 133 \pmod{1000}$.

This also implies $3^a7^b\equiv 133\equiv 5 \pmod 8$. We know that 3^a is congruent to 1 or 3 modulo 8 and 7^b is congruent to 1 or 7 modulo 8. In order for the product 3^a7^b to be congruent to 5 modulo 8, 3^a must therefore be congruent to 3 and 7^b must be congruent to 7. Therefore, we can conclude that a and b are both odd.

We also have $3^a7^b\equiv 133\equiv 3 \pmod 5$. As a and b are odd, 3^a and 7^b are each congruent to 3 or 2 modulo 5. Neither 3^2, nor $3\cdot 2$ is congruent to 3 modulo 5, a contradiction.

Example 7 (*National Competition: April 30th, 2016*) Consider 2016 points arranged on a circle. We are allowed to jump ahead by 2 or 3 points in clockwise direction. What is the minimum number of jumps required to visit all points and return to the starting point?

Solution. Clearly it takes at least 2016 jumps to visit all points. It is impossible to use only jumps of length 2 or only jumps of length 3 because this would confine us to a single residue class modulo 2 or 3 respectively.

If the problem could be solved with 2016 jumps, the total distance covered by these jumps would be strictly between $2\cdot 2016$ and $3\cdot 2016$ which makes a return to the original point impossible. Therefore, at least 2017 jumped are required.

This is indeed possible, for example with the following sequence of points on the circle

0,3,6,…,2013,2015, 2,5,…,2012,2014, 1,4,…, 2011, 2013,0.

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***February 15, 2019.***

Problem 526. Let $a_1=b_1=c_1=1$, $a_2=b_2=c_2=3$ and for $n\ge3$, $a_n=4a_{n-1}-a_{n-2}$,

$$b_n=\frac{b_{n-1}^2+2}{b_{n-2}},\ c_n=2c_{n-1}+\sqrt{3c_{n-1}^2-2}.$$

Prove that $a_n=b_n=c_n$ for all $n=1,2,3,\ldots$.

Problem 527. Let points O and H be the circumcenter and orthocenter of acute ΔABC. Let D be the midpoint of side BC. Let E be the point on the angle bisector of $\angle BAC$ such that $AE\perp HE$. Let F be the point such that $AEHF$ is a rectangle. Prove that points D, E, F are collinear.

Problem 528. Determine all positive integers m satisfying the condition that there exists a unique positive integer n such that there exists a rectangle which can be decomposed into n congruent squares and can also be decomposed into $n+m$ congruent squares.

Problem 529. Determine all ordered triples (x,y,n) of positive integers satisfying the equation $x^n+2^{n+1}=y^{n+1}$ with x is odd and the greatest common divisor of x and $n+1$ is 1.

Problem 530. A square can be decomposed into 4 rectangles with 12 edges. If square $ABCD$ is decomposed into 2005 convex polygons with degrees of A, B, C, D at least 2 and degrees of all other vertices at least 3, then determine the maximum number of edges in the decomposition.

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Solutions

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Problem 521. Given 20 points in space so that no three of them are collinear, prove that the number of planes determined by these points is not equal to 1111.

Solution. **CHUI Tsz Fung** (Ma Tau Chung Government Primary School), **Eren KIZILDAG** (MIT), **LEUNG Hei Chun** and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Assume the number of planes is 1111. The 20 points would define $(20\cdot19\cdot18)/3!$ = 1140 planes so that 1140−1111=29 triplets of points lie in the planes already determined by other triplets. If one of the planes contain 7 or more points, then there are $(7\cdot6\cdot5)/3!$ = 35 triplets of points in this plane and the number of triplets is greater than the number of planes by at least 35−1=34. So the greatest possible number of planes is 1140−34=1105. Clearly, this cannot happen if there are 1111 planes.

So each plane can contain at most 6 of the points. Let *a, b, c* be the number of planes containing 4, 5, 6 points respectively. When counting triplets, in cases k=4,5,6, we consider each plane containing k points $k(k-1)(k-2)/3!$ = 4, 10, 20 times, which are 3, 9, 19 times too many, respectively. So the number of planes satisfies $1140-3a-9b-19c = 1111$. Hence $3a+9b+19c=29$. However, there are no nonnegative integers a,b,c satisfying $3a+9b+19c=29$. So we arrive at a contradiction.

Other commended solvers: **ZHANG Yupei** (HKUST).

Problem 522. Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that for all real x and y,

$$(x-2)\,f(y)+f\,(y+2f(x)) = f\,(x+y\,f(x)).$$

Solution. **CHUI Tsz Fung** (Ma Tau Chung Government Primary School), **Eren KIZILDAG** (MIT), **Akash Singha ROY** (West Bengal, India), **Ioannis D. SFIKAS** (Athens, Greece), **George SHEN** and **Toshihiro SHIMIZU** (Kawasaki, Japan).

We will refer to the given equation as (*). In case $f(0)=0$, setting x=0 in (*), we get $f(y)=0$ for all y. In case $f(0)\neq0$, setting y=0, (*) becomes $(x-2)f(0)+f(2f(x)) = f(x)$ for all real x. If $f(x)=f(x')$, then $x=x'$ and so f is injective.

Next, putting x=2 into (*), we get $f(y+2f(2)) = f(2+yf(2))$ for all real y. Since f is injective, we get $y+2f(2) = 2+yf(2)$ for all real y. Setting y=0, we get $f(2)=1$. Since f is injective, $f(3)\neq1$. Setting x=3 and $y=3/(1-f(3))$ (which is $y=3+yf(3)$) into (*), we get $f(y+2f(3))=0$. So f has a root at $r=y+f(3)$. Next, setting $y=r$ in (*), we get $f(r+2f(x))=f(x+rf(x))$ for all real x. Since f is injective, we get $r+2f(x) = x+rf(x)$ for all real x.

Now due to $f(2)=1\neq0$, $r\neq2$. So $f(x)=(x-r)/(2-r)$. Finally, substituting $f(x)$ by $(x-r)/(2-r)$ we get r=1 so that $f(x)=x-1$. As a result, it is easy to check (*) has the two solutions $f(x)$=0 and $f(x)=x-1$.

Other commended solvers: **Alex Kin Chit O** (G.T. (Ellen Yeung) College).

Problem 523. Find all positive integers n for which there exists a polynomial $P(x)$ with integer coefficients such that $P(d) = (n/d)^2$ for each positive divisor d of n.

Solution. **CHUI Tsz Fung** (Ma Tau Chung Government Primary School), **Eren KIZILDAG** (MIT), **LEUNG Hei Chun**, **Toshihiro SHIMIZU** (Kawasaki, Japan) and **ZHANG Yupei** (HKUST).

For n=1, let $P(x)=x$, then $P(1)=1$ satisfies the condition. If n is a prime, then its only positive divisors are 1 and n and the conditions on P is $P(1)=n^2$ and $P(n)=1$. We can satisfy this with $P(x)=n^2+(n+1)(1-x)$.

Next we consider $n=km$ is not prime with $k,m>1$. We have conditions $P(1)=n^2$, $P(k)=m^2$, $P(m)=k^2$ and $P(n)=1$. For arbitrary integers a, b, by factoring, we see $P(a)-P(b)$ is divisible by $a-b$. So $n-k=k(m-1)$ divides $P(n)-P(k)$ = $1-m^2$ = $(1-m)(1+m)$. This leads to k divides $m+1$. Similarly, $n-m$ divides $P(n)-P(m)$ and so $m(k-1)$ divides $(1-k)(1+k)$ and m divides $k+1$. Hence, km divides $(k+1)(m+1)$ and it also divides $(k+1)(m+1)-km = k+m+1$. We must have $km\le k+m+1$, which implies that $km-k-m+1\le2$ or $(k-1)(m-1)\le2$. We may assume $k\le m$. Then the only possible case is k=2 and m=3 so that n=6.

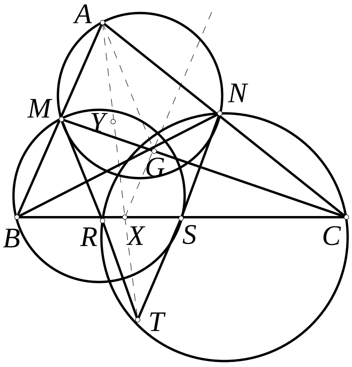
For n=6, we will find a polynomial P such that $P(1)=36$, $P(2)=9$, $P(3)=4$ and $P(6)=1$. We can apply the Lagrange interpolation formula to get $P(x)$ = $1-(x-6)(1+(x-3)(2x-5))$, which can be easily checked to satisfy $P(1)=36$, $P(2)=9$, $P(3)=4$ and $P(6)=1$.

Other commended solvers: **Akash Singha ROY** (West Bengal, India).

Problem 524. (proposed by *Andrew WU*, St. Albans School, Mc Lean, VA, USA) In ΔABC with centroid G, M

and N are the midpoints of AB and AC, and the tangents from M and N to the circumcircle of ΔAMN meet BC at R and S, respectively. Point X lies on side BC satisfying $\angle CAG = \angle BAX$. Show that GX is the radical axis of the circumcircles of ΔBMS and ΔCNR.

Solution. By Proposer.



Observe that BN is the radical axis of the circumcircles of ΔANM and ΔCNR. To prove this, we will show $BM{\cdot}BA=BR{\cdot}BC$ or equivalently that $AMRC$ is a cyclic quadrilateral. By the tangency condition, we have $\angle AMR = 180°-\angle ANM = 180°-\angle ACR$, so $AMRC$ is cyclic, as desired. Similarly, we have CM is the radical axis of the circumcircles of ΔANM and ΔBMS. Thus, by the radical center theorem, BN, CM and the radical axis of the circumcircles of ΔBMS and ΔCNR concur. This implies the centroid G lies on the radical axis.

Next, by properties of symmedians, we get lines MR, AX, NS concur at some point T. Suppose lines AX and MN meet at Y. Then by similar triangles, we have $RX/XS=MY/YN=BX/XC$ due to the facts that $\Delta TRS\sim\Delta TMN$ and $\Delta AMN\sim\Delta ABC$.

Thus, it follows that $XR{\cdot}XC=XS{\cdot}XB$. So X has equal power with respect to the circumcircles of ΔBMS and ΔCNR. Then line GX is the radical axis of ΔBMS and ΔCNR.

Other commended solvers: **CHUI Tsz Fung** (Ma Tau Chung Government Primary School), **Andrea FANCHINI** (Cantù, Italy), **LEUNG Hei Chun** and **Toshihiro SHIMIZU** (Kawasaki, Japan) and **ZHANG Yupei** (HKUST).

Problem 525. Find all positive integer n such that $n(n+2)(n+4)$ has at most 15 positive divisors.

Solution. **CHUI Tsz Fung** (Ma Tau Chung Government Primary School), **Ioan Viorel CODREANU** (Satulung, Maramures, Romania), **Eren KIZILDAG** (MIT), **LEUNG Hei Chun**, **Ioannis D. SFIKAS** (Athens, Greece), **Toshihiro SHIMIZU** (Kawasaki, Japan) and **ZHANG Yupei** (HKUST).

Let $a_n=n(n+2)(n+4)$ and let b_n be the number of positive divisors of a_n. The values of b_1 to b_{10} are 4, 10, 8, 14, 12, 24, 12, 28, 12, 40. Next, we recall if a positive integer m has prime factorization $p_1^{e_1}\cdots p_j^{e_j}$, then m has $(e_1+1)\cdots(e_j+1)$ positive divisors. If m divides a positive integer M, then M has at least as many divisors as m.

Let $n\ge 11$. If n is even, say $n=2k$, then $a_n=2^3k(k+1)(k+2)$. At least one of the numbers k, $k+1$, $k+2$ is divisible by 2 and exactly one of them is divisible by 3. Since $k\ge 6$, the numbers k, $k+1$, $k+2$ cannot all be powers of 2 or 3. So $k(k+1)(k+2)$ has a prime divisor p not equal to 2 or 3. Hence, $2^4 3p$ divides a_n and this implies that a_n has at least $5{\cdot}2{\cdot}2 = 20$ positive divisors.

Let $n\ge 11$ be odd. Then the numbers n and $n+2$ are relativity prime, as are $n+2$ and $n+4$ and also n and $n+4$. One of these three numbers is divisible by 3. This number has at least one other prime divisor p or else is a power of 3. In the latter case it is divisible by 3^3 since $n\ge 11$. Let q and r be prime divisors of the other two numbers. In the first case the number a_n is divisible by $3pqr$. The number n, $n+2$, $n+4$ are relatively prime, so 3, p, q, r are relatively prime. This implies that a_n has at least $2{\cdot}2{\cdot}2{\cdot}2 = 16$ divisors. In the second case a_n is divisible by 3^3qr. The primes 3, q, r are again distinct. So a_n has at least $4{\cdot}2{\cdot}2=16$ divisors.

The number a_n has at most 15 positive divisors only for $n=1, 2, 3, 4, 5, 7, 9$.

Other commended solvers: **Christos ALVANOS** (Mandoulides, Thessaloniki, Greece), **Alex Kin Chit O** (G.T. (Ellen Yeung) College) and **Akash Singha ROY** (West Bengal, India).

Olympiad Corner

(*Continued from page 1*)

Problem G1. (*cont.*) Prove that the circumcenter of the triangle ADM is the reflection of O across the line AD.

Problem N1. Determine all pairs (m,n) of positive integers such that

$$2^m = 7n^2+1.$$

Austrian Math Problems

(*Continued from page 2*)

<u>Example 8</u> (*National Competition: April 30th, 2017*) Anna and Berta play a game in which they take turns in removing marbles from a table. Anna takes the first turn. When at the beginning of a turn there are $n\ge 1$ marbles on the table, then the player whose turn it is removes k marbles, where $k\ge 1$ either is an even number with $k\le n/2$ or an odd number with $n/2\le k\le n$. A player wins the game if she removes the last marble from the table. Find the smallest $N\ge 100{,}000$ such that Berta can enforce a victory if there are exactly N marbles on the table in the beginning.

<u>Solution.</u> We claim that the losing situations are those with exactly $n=2^a-2$ marbles left on the table for all integers $a\ge 2$. All other situation are winning situations.

For $n=1$, the player wins by taking the single remaining marble. For $n=2$, the only possible move is to take $k=1$ marbles and the opponent wins in the next move. For $n\ge 3$, (1) if n is odd, the player takes all n marbles and wins; (2) if n is even, but not of the form 2^a-2, then n lies between two other numbers of that form, so there is a unique b with $2^b-2<n<2^{b+1}-2$. From $n\ge 3$, we get $b\ge 2$. So all 3 parts of the inequalities are even and so $2^b\le n\le 2^{b+1}-4$. By the induction hypothesis, we know 2^b-2 is a losing situation. Taking $k = n-(2^b-2) \le n/2$ marbles, we leave it to the opponent; (3) if n is even of the form 2^a-2, the player cannot leave a losing situation with 2^b-2 marbles to the opponent (where $b<a$ holds due to at least 1 marble must be removed and $b\ge 2$ holds as after a legal move starting from an even n, at least 1 marble remains). The player would then remove $k=2^a-2^b$ marbles. As $b\ge 2$, k is even and greater than $n/2$ due to $k\ge 2^{a-1}>2^{a-1}-1= n/2$, which is impossible. This means Berta can enforce a victory if and only if N is of the form 2^a-2. The smallest number $N\ge 100{,}000$ of this form is $N = 2^{17}-2 = 131{,}070$.