Mathematical Excalibur

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Olympiad Corner

Below were the Day 2 problems of the Croatian Mathematical Olympiad which took place on May 6, 2018.

Problem A2. determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(xf(y))=(1-y)f(xy)+x^2y^2f(y)$$

holds for all real numbers x and y.

Problem C2. Let n be a positive integer. Points $A_1, A_2, \ldots, A_n$ are located on the inside of a circle, and points $B_1, B_2, \ldots, B_n$ are on the circle, so that the lines $A_1B_1, A_2B_2, \ldots, A_nB_n$ are mutually disjoint. A grasshopper can jump from point A_i to point A_j (for $i,j\in\{1,\ldots,n\}$, $i\neq j$) if and only if the lines A_iA_j does not go through any of the inner points of the lines $A_1B_1, A_2B_2, \ldots, A_nB_n$.

Problem G2. Let ABC be an acute-angled triangle such that $|AB| < |AC|$. Point D is the midpoint of the shorter arc BC of the circumcircle of the triangle ABC. Point I is the incenter of the triangle ABC, and point J is the reflection of I across the line BC.

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On-line: http://www.math.ust.hk/excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***May 25, 2019***.

For individual subscription for the next five issues for the 18-19 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Sum of Digits of Positive Integers

Pedro Pantoja, Natal/RN, Brazil

In this short article we will explore some types of problems in number theory about the sum of digits of a positive integer.

Throughout this article, $S(a)$ will denote the sum of the digits of a positive integer a. For example $S(12)=1+2=3$, $S(349)=3+4+9=16$. Let $c(n,m)$ denote the total number of carries, which arises when adding a and b, for example $c(100,4)=0$, $c(23,17)=1$, $c(88,99)=2$.

Proposition 1. For positive integer a, we have

i) $S(a)\le a$;
ii) $S(a)\equiv a \pmod 9$;
iii) if a is even, then $S(a+1)-S(a)=1$;
iv) $S(a+b)=S(a)+S(b)-9c(a,b)$, in particular, $S(a+b)\le S(a)+S(b)$;
v) $S(ab)\le\min\{aS(b),bS(a)\}$;
vi) $S(ab)\le S(a)S(b)$;
vii) $S(a)\le 9([\log a]+1)$.

Proof. i) and ii) are obvious.

iii) If a is even, then $S(a+1)-S(a)=1$. In fact, a and $a+1$ differ only in the unit digit, which for a will be 0, 2, 4, 6 or 8 and for $a+1$ will be, respectively, 1, 3, 5, 7 or 9.

iv) We proceed by induction on the maximal number of digits k of b and a. If both b and a are single digit numbers, then we have just two cases. If $b+a<10$, then we have nocarries and clearly $S(b+a)=b+a=S(b)+S(a)$. If on the other hand, $b+a=10+k\ge 10$, then

$$S(b+a) = 1+k = 1+(b+a-10) = S(b)+S(a)-9.$$

Assume that the claim holds for all pairs with at most k digits each. Let

$$b = b_1+n\cdot 10^{k+1} \text{ and } a = a_1+n\cdot 10^{k+1},$$

where b_1 and a_1 are at most k digit numbers. If there is no carry at the $k+1^{\text{st}}$ digit, then $c(b,a)=c(b_1,a_1)$ and thus

$$\begin{aligned}S(b+a) &= S(b_1+a_1) + m + n\\ &= S(b_1)+m+S(a_1)+n-9c(n_1,m_1)\\ &= S(b)+S(a)-9c(b,a).\end{aligned}$$

If there is a carry, then $c(n,a) = 1 + c(n_1,ma_1)$ and thus

$$\begin{aligned}S(b+a) &= S(b_1+a_1)+m+n-9\\ &= S(b_1)+m+S(a_1)+n-9(c(b_1,a_1)+1)\\ &= S(b)+S(a)-9c(b,a).\end{aligned}$$

This finishes the induction and we are done.

v) Because of symmetry, in order to prove v), it suffices to prove that $S(ab) \le aS(b)$. The last inequality follows by applying the subadditivity (iv) property repeatedly. Indeed, $S(2b)=S(b+b)\le S(b)+S(b) = 2S(b)$. After a steps we obtain

$$\begin{aligned}S(ab) &= S(b+\cdots+b)\\ &\le S(b)+\cdots+S(b) = aS(b).\end{aligned}$$

vi) and vii) Left as exercises for the reader.

For applications, we provide

Example 1: Find all positive integers with $n \le 1000$ such that $n = (S(n))^3$.

Solution: The perfect cube numbers smaller than 1000 are 1, 8, 27, 64, 125, 216, 343, 512, 729. From these numbers the only one that satifies the conditions of the problem is $n = 512$.

Example 2: (*MAIO-2012*) Evaluate

$$S(1) - S(2) + S(3) - S(4) + \cdots + S(2011) - S(2012).$$

Solution: The problem becomes trivial using Proposition 1, item iii). We have $S(3)-S(2)=1$, $S(5)-S(4) = 1$, …, $S(2011) - S(2010) = 1$ and $S(1) = 1$, $S(2012) = 5$. Therefore, $S(1) - S(2) + S(3) - S(4) + \cdots + S(2011) - S(2012) =1+1005-5 =1001$.

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Example 3: (*Nordic Contest 1996*) Show that there exists an integer divisible by 1996 such that the sum of its decimal digits is 1996.

Solution. We affirm that the number m = 199619961996…199639923992 satisfies the conditions of the statement. Note that $S(m)=25\cdot 78+2\cdot 23=1996$. On the other hand, m is divisible by 1996, since m equals

$$1996\cdot 100010001000\ldots 1000200002.$$

Example 4: Find $S(S(S(S(2018^{2018}))))$.

Solution: Using proposition 1, item vii) several times we have

$$S(2018^{2018})\le 9([2018\log 2018]+1) < 60030,$$

$$S(S(2018^{2018}))\le 9([\log 60030]+1) < 45,$$

$$S(S(S(2018^{2018})))\le 9([\log 45]+1) < 18.$$

On the other hand, $2018^{2018}\equiv 2^{2018} = (2^3)^{672}\cdot 2^2\equiv 4 \pmod 9$. Hence,

$$S(S(S(2018^{2018}))) = 4 \text{ or } 13.$$

So $S(S(S(S(2018^{2018})))) = 4$.

Example 5: Prove that $S(n)+S(n^2)+S(n^3)$ is a perfect square for infinitely many positive integers n that are not divisible by 10.

Solution: Let us prove that the numbers of the form $n=10^{m^2}-1$ satisfy the problem. The result follows immediately because there are infinitely many number of this form. Firstly, $S(n)=9m^2$ and

$$n^2 = 10^{2m^2} - 2\cdot 10^{m^2} + 1 = 99..9800..01$$

where there are m^2-1 9's and 0's. Then $S(n^2)=9m^2$. Similarly,

$$S(n^3)=99\ldots 9700\ldots 0299\ldots 9,$$

where there are m^2-1 9's and 0's and m^2 9's at the end. Then $S(n^3)=18m^2$. Finally, $S(n)+S(n^2)+S(n^3)=36m^2$.

Remark 1: The numbers of the previous problem are registered in On-Line Encyclopedia of Integer Sequences (OEIS) A153185. Some examples of such numbers: 9, 18, 45, 90, 171, 180, 207, 279, 297, 396, 414, 450, 459, ….

Remark 2: Notice that sometimes mathematical intuition deceives us. That is, the nine numbers 1, 11, 111, …, 111…1 satisfy $S(n^2) = (S(n))^2$. Unfortunately, the next number in this family is

$$1111111111^2 = 1234567900987654321.$$

So $S(1111111111) = 10$, but $S(1111111111^2) = 82$. The smallest positive integer such that $S(n) = 10$ and $S(n^2) = 100$ is $n = 1101111211$.

Example 6: We say that a superstitious number is equal to 13 times a sum of its digits. Find all superstitious numbers.

Solution: Obviously there is no superstitious number with one digit. If a two digit number ab is superstitious, then $10a+b=13(a+b)$, that is $3a+12b=0$, which is impossible.

If a three-digit number abc is superstitious, we would have $100a+10b+c=13(a+b+c)$. that is $29a=b+4c$. The maximum possible value for $b+4c$ is 45 (for $b=c=9$). So a must be 1 and the equation $29=b+4c$ has solutions $(b,c) = (1,7),(5,6)$, and $(9,5)$. The numbers 117, 156 and 195 are the only superstitious numbers with three digits.

If a four-digit number abcd is superstitious, it would result in $1000a+100b+10c+d = 13(a+b+c+d)$. As the number on the left is at least 1000 and the number on the right is at most $13\cdot 36=468$, there is no superstitious numbers of four digits. Finally, there is no superstitious number with more than four digits, since each added digit contributes at least 1,000 to the number on the left, while the one on the right contributes at most $13\cdot 9=117$. So the only superstitious numbers are 117, 156 and 195.

Example 7: (*Romanian Team Selection Test 2002*) Let $a, b > 0$. Prove that the sequence $S([an+b])$ contains a constant subsequence.

Solution. For any positive integer k, let n_k equals $[(10^k+a-b)/b]$. Then

$$10^k = a\left(\frac{10^k+a-b}{a}-1\right)+b$$

$$< an_k+b = a\left[\frac{10^k+a-b}{a}\right]+b$$

$$\le 10^k+b.$$

It follows that $10^k=[an_k+b]\le 10^k+b$.

If k is sufficiently large, that is $10^{k-1}>b$, it follows from above that S_{n_k} is one plus the sum of the digits of one of the numbers t in the set $\{0,1,\ldots,[b]\}$. Since k takes infinitely many values and the set of the numbers t is finite, it follows that for infinitely many k, the sum of digits of numbers $[an_k+b]$ is the same.

Example 8: (*2016 IMO Shortlisted Problem*) Find all polynomials $P(x)$ with integer coefficients such that for any positive integer $n\ge 2016$, the integer $P(n)$ is positive and

$$S(P(n)) = P(S(n)). \qquad (*)$$

Solution: Let

$$P(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0.$$

Clearly $a_d > 0$. There exists an integer $m>1$ such that $|a_i|<10^m$ for all $0\le i\le d$. Consider $n=9\cdot 10^k$ for a sufficiently large integer k in (*). If there exists an index $0\le i\le d-1$ such that $a_i<0$, then all digits of $P(n)$ in positions from 10^{ik+m+1} to $10^{(i+1)k-1}$ are all 9's Hence, we have $S(P(n))>9(k-m-1)$. On the other hand, $P(S(n))=P(9)$ is a fixed constant. Therefore, (*) cannot hold for large k. This shows $a_i>0$ and for all $0\le i\le d-1$. Hence, $P(n)$ is an integer formed by the nonnegative integers $a_d 9^d$, $a_{d-1}9^{d-1}$,…, a_0 by inserting some zeros in between.

This yields

$$S(P(n))=S(a_d 9^d)+S(a_{d-1}9^{d-1})+\cdots+S(a_0).$$

Combining with (*), we have

$$S(a_d 9^d)+S(a_{d-1}9^{d-1})+\cdots+S(a_0) = P(9) = a_d 9^d + a_{d-1}9^{d-1}+\cdots+a_0.$$

As $S(m)\le m$ for any positive integer m, with equality when $1\le m\le 9$, this forces each $a_i 9^i$ to be a positive integer between 1 and 9. In particular, this shows $a_i=0$ for $i>2$ and hence $d\le 1$. Also, we have $a_1\le 1$ and $a_0\le 9$. If $a_1=1$ and $1\le a_0\le 9$, we take $n=10^k+(10-a_0)$ for sufficiently large k in (*). This yields a contradiction. Since

$$S(P(n)) = S(10^k+10) = 2 \ne 11 = P(11-a_0) = P(S(n)).$$

The zero polynomial is also rejected since $P(n)$ is positive for large n. The remaining candidates are $P(x)=x$ or $P(x)=a_0$ where $1\le a_0\le 9$, all of which satisfy (*), and hence are the only solutions.

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***May 25, 2019.***

Problem 531. $BCED$ is a convex quadrilateral such that $\angle BDC = \angle CEB = 90°$ and BE intersects CD at A. Let F, G be the midpoints of sides DE, BC respectively. Let O be the circumcenter of ΔBAC. Prove that lines AO and FG are parallel.

Problem 532. Prove that there does not exist a function f:(0,+∞)→(0,+ ∞) such that for all $x,y>0$,

$$f^2(x) \geq f(x+y)(f(x)+y).$$

Problem 533. Let ℤ and ℕ be the sets of all integers and all positive integers respectively. Let $r, s \in \mathbb{N}$. Prove that there are exactly $(r+1)^{s+1}-r^{s+1}$ functions $g:[1,s]\cap\mathbb{N}\to[-r,r]\cap\mathbb{Z}$ such that for all $x,y\in[1,s]\cap\mathbb{N}$, we have $|g(x)-g(y)|\leq r$.

Problem 534. Prove that for any two positive integers m and n, there exists a positive integer k such that $2^k - m$ has at least n distinct prime divisors.

Problem 535. Determine all integers $n>4$ such that it is possible to color the vertices of a regular n-sided polygon using at most 6 colors such that any 5 consecutive vertices have distinct colors.

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Solutions

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Problem 526. Let $a_1=b_1=c_1=1$, $a_2=b_2=c_2=3$ and for $n\geq3$, $a_n=4a_{n-1}-a_{n-2}$,

$$b_n = \frac{b_{n-1}^2+2}{b_{n-2}}, c_n = 2c_{n-1}+\sqrt{3c_{n-1}^2-2}.$$

Prove that $a_n=b_n=c_n$ for all n =1,2,3,….

Solution. **Angel Gerardo Napa BERNUY** (PUCP University, Lima, Peru), **CHUI Tsz Fung** (Ma Tau Chung Government Primary School), **DBS Maths Solving Team** (Diocesan Boy's School), **Prithwijit DE** (HBCSE, Mumbai, India), **O Long Kin Oscar** (St. Joseph's College), **TAM Choi Nang Julian** (Yan Chai Hospital Law Chan Chor Si College), **Duy Quan TRAN** (University of Medicine and Pharmacy at Ho Chi Minh City, Vietnam) and **Bruce XU** (West Island School).

The cases n = 1,2 can easily be checked. For $n\geq3$, $b_nb_{n-2} = b_{n-1}^2+2$ implies $b_{n+1}b_{n-1} = b_n^2+2$. Subtracting these and factoring, we get $(b_{n+1}-b_{n-1})/b_n=(b_n-b_{n-2})/b_{n-1}$. Then

$$(b_n-b_{n-2})/b_{n-1}= (b_{n-1}-b_{n-3})/b_{n-2}$$
$$= \cdots = (b_3-b_1)/b_2 = 4.$$

Hence, $b_n = 4b_{n-1}- b_{n-2}$ for $n \geq 3$. Since $a_1=b_1$ and $a_2=b_2$, $a_n=b_n$ for all n = 1,2,3,…. Next, from

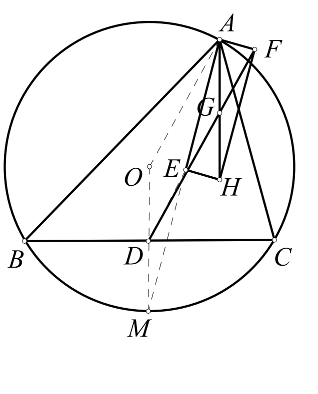
$$c_n = 2c_{n-1}+\sqrt{3c_{n-1}^2-2},$$

we can see c_n is strictly increasing and for $n\geq2$, $(c_n-2c_{n-1})^2 = 3c_{n-1}^2-2$. Then $c_n^2 - 4c_nc_{n-1} + c_{n-1}^2 = -2$ and $c_{n+1}^2-4c_{n+1}c_n + c_n^2 = -2$. Subtracting these and factoring, we get $(c_{n+1}-c_{n-1})(c_{n+1}-4c_n+c_{n-1}) = 0$. As $c_{n+1} > c_{n-1}$, we get $c_{n+1}=4c_n-c_{n-1}$ for $n\geq2$. So $a_n=b_n=c_n$ for all n = 1,2,3,….

Other commended solvers: **AISINGIUR To To**, **Alvin LUKE** (Portland, Oregon, USA), **Corneliu MĂNESCU-AVRAM** (Ploieşti, Romania), **Ioannis D. SFIKAS** (Athens, Greece), **Toshihiro SHIMIZU** (Kawasaki, Japan), **SO Tsz To** (S.K.H. Lam Woo Memorial Secondary School), **Nicuşor ZLOTA** ("Traian Vuia" Technical College, Focşani, Romania), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** (Buzău, Romania).

Problem 527. Let points O and H be the circumcenter and orthocenter of acute ΔABC. Let D be the midpoint of side BC. Let E be the point on the angle bisector of $\angle BAC$ such that $AE\perp HE$. Let F be the point such that $AEHF$ is a rectangle. Prove that points D, E, F are collinear.

Solution. **Alvin LUKE** (Portland, Oregon, USA).



Connect AO, OD and extend OD to meet the circumcircle of ΔABC at M. Then $OD\perp BC$ and M bisects arc BC. Also, A, E, M are collinear. Observe AE, AF are internal and external bisectors of $\angle BAC$. So $AE\perp AF$.

Since $HE\perp AE$ and $HF\perp AF$, so $AEHF$ is a rectangle. Hence, segments AH and EF bisect each other. Let AH and EF meet at G. Then AG=½AH= ½EF=EG.

Also, OA=OM and $OD \parallel AH$. So

$$\angle OAE=\angle OME=\angle EAG=\angle GEA.$$

So (*) $EG \parallel OA$.

Next, observe O and H are the circumcenter and the orthocenter of of ΔABC respectively. Since $OD\perp BC$, so OD=½AH=AG. Finally, connect DG. We see $AODG$ is a parallelogram. So (**) $DG \parallel OA$. Therefore, by (*) and (**), D, E, G, F are collinear.

Other commended solvers: **Angel Gerardo Napa BERNUY** (PUCP University, Lima, Peru), **CHUI Tsz Fung** (Ma Tau Chung Government Primary School), **DBS Maths Solving Team** (Diocesan Boy's School), **Prithwijit DE** (HBCSE, Mumbai, India), **Andrea FANCHINI** (Cantú, Italy), **Jon GLIMMS, Corneliu MĂNESCU-AVRAM** (Ploieşti, Romania), **Apostolos MANOLOUDIS**, **George SHEN, Toshihiro SHIMIZU** (Kawasaki, Japan), **Mihai STOENESCU** (Bischwiller, France), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** (Buzău, Romania).

Problem 528. Determine all positive integers m satisfying the condition that there exists a unique positive integer n such that there exists a rectangle which can be decomposed into n congruent squares and can also be decomposed into $n+m$ congruent squares.

Solution. **Angel Gerardo Napa BERNUY** (PUCP University, Lima, Peru), **CHUI Tsz Fung** (Ma Tau Chung Government Primary School) , and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Suppose rectangle $ABCD$ can be decomposed into $n+m$ unit squares and also into n squares with sides equal x. Let $x = a/b$ with $\gcd(a,b) = 1$. Then the area of rectangle $ABCD$ is $n+m$ as well as $n(a/b)^2$. Then from $n+m = n(a/b)^2$, we can solve for n to get

$$n=\frac{mb^2}{a^2-b^2}=\frac{mb^2}{(a-b)(a+b)}.$$

Since $\gcd(b,a+b) = \gcd(b,a-b) = \gcd(a,b) = 1$, so $(a-b)(a+b) \mid m$. Now $a+b$, $a-b$ are of the same parity. If m is the product of positive integers i, j, k with j, k odd and greater than 1, then $(a+b,a-b) = (j,k)$ or $(jk,1)$ leading to $n=i(j-k)^2/4$ or $i(jk-1)^2/4$, contradicting the uniqueness of n. So m can have at most one odd factor greater than 1, i.e. $m=2^c$ or $2^c p$ with p an odd prime.

In case $m=2^c$, for $c=1,2$, there is no n; for $c=3$, $m=8$ and $(a,b)=(2,4)$, $n=1$; for $c\geq 4$, $(a+b,a-b)=(4,2)$ or $(8,2)$ resulting in $n = 2^{c-3}$ or 2^{c-4} contradicting the uniqueness of n.

In case $m=2^c p$, for $c=0$, $m=p$ and $(a+b,a-b) = (p,1)$, $n = (p-1)^2/4$; for $c = 1$, $(a+b,a-b) = (p,1)$, $n = (p-1)^2/2$; for $c = 2$, $(a+b,a-b) = (p,1)$, $n = (p-1)^2$; for $c\geq 3$, $(a+b,a-b) = (4,2)$ or $(p,1)$ contradict the uniqueness of n.

So the only solutions are $m = 8, p, 2p, 4p$, where p is an odd prime.

Other commended solvers: **Victor LEUNG Chi Shing** and **Charles POON Tsz Chung**.

Problem 529. Determine all ordered triples (x,y,n) of positive integers satisfying the equation $x^n+2^{n+1} = y^{n+1}$ with x is odd and the greatest common divisor of x and $n+1$ is 1.

Solution. **Alvin LUKE** (Portland, Oregon, USA) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

When $n=1$, let $y=t$ be an integer at least 3 and $x=t^2-4$ are solutions. When $n\geq 2$,

$$x^n = y^{n+1}-2^{n+1}=(y-2)\sum_{k=0}^{n}2^k y^{n-k}.$$

For any prime factor p of y−2, from above, we see x must be a multiple of p. As x is odd, p is also odd. As $\gcd(x,n+1) = 1$, we see $\gcd(x,(n+1)2^n) = 1$. Then p is not a factor of $(n+1)2^n$. Now

$$S=\sum_{k=0}^{n}2^k y^{n-k}\equiv\sum_{k=0}^{n}2^n=(n+1)2^n \pmod{y-2}.$$

Hence, p is not a factor of S. So we have $\gcd(y-2,S) = 1$. So $S=T^n$ for some positive integer T. Since y is positive, y is at least 3.

When $n\geq 2$, we have

$$y^n < S=T^n < (y+2)^n. \qquad (*)$$

So $T = y+1$. However, when y is even, $S \equiv y^n \pmod 2$ is even, but then $S = (y+1)^n$ is odd by (*). Similarly, when y is odd, $S\equiv y^n \pmod 2$ is odd, but then $S=(y+1)^n$ is even by (*). Again this leads to a contradiction.

In conclusion, when integer n is at least 2, there are no solutions. So the only solution are $x=t^2-4$, $y=t$, $n=1$, where integer $t\geq 3$.

Other commended solvers: **Ioannis D. SFIKAS** (Athens, Greece).

Problem 530. A square can be decomposed into 4 rectangles with 12 edges. If square $ABCD$ is decomposed into 2005 convex polygons with degrees of A, B, C, D at least 2 and degrees of all other vertices at least 3, then determine the maximum number of edges in the decomposition.

Solution. **CHUI Tsz Fung** (Ma Tau Chung Government Primary School), **DBS Maths Solving Team** (Diocesan Boy's School) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Let v, e, f be the number of vertices, edges and faces used in decomposing the square respectively. By Euler's formula, we have $v-e+f = 1$ (omitting the exterior of the square).

Let $d(V)$ be the number of edges connected to V. Let V be a vertex on the square other than A,B,C,D. Then $d(V)\geq 3$, which is the same as $d(V)\leq 3d(V)-6$.

Now there are $v-4$ vertices not equal to A, B, C, D. The sum of the degrees of the $v-4$ vertices other than A, B, C, D is $2e-[d(A)+d(B)+d(C)+d(D)]$, which is at least $3(v-4)$. Since $d(A)$, $d(B)$, $d(C)$, $d(D)$ ≥ 2, we get

$$\begin{aligned}2e-8 &\geq 2e-[d(A)+d(B)+d(C)+d(D)]\\ &\geq 3(v-4) = 3v-12.\end{aligned}$$

Since $v-e+f = 1$, $3e = 3v+3f-3 \leq 2e+1+3f$, which simplies to $e \leq 3f+1$.

For equality case, we can decompose the unit square into rectangles of size 1 by 1/2005, which has $3\times 2005+1=6016$ edges.

Olympiad Corner

(Continued from page 1)

Problem G2. (*cont.*) Line DJ intersects the circumcircle of the triangle ABC at the point E which lies on the shorter arc AB. Prove that $|AI|=|IE|$ holds.

Problem N2. Let n be a positive integer. Prove that there exists a positive integer k such that

$$51^k - 17$$

is divisible by 2^n.

Sums of Digits …

(Continued from page 2)

Next, we will provide some exercises for the readers.

Problem 1: (*Mexico 2018*) Find all pairs of positive integers (a,b) with $a>b$ which simultaneously satisfy the following two conditions

$$a \mid b+S(a) \quad\text{and}\quad b \mid a+S(b).$$

Problem 2: (*Lusophon 2018*) Determine the smallest positive integer a such that there are infinitely many positive integer n for which you have $S(n)-S(n+a) = 2018$.

Problem 3: (*Cono Sur 2016*) Find all n such that $S(n)(S(n)-1) = n-1$.

Problem 4: (*Iberoamerican 2014*) Find the smallest positive integer k such that

$$\begin{aligned}S(k) = S(2k) &= S(3k) = \cdots\\ &= S(2013k) = S(2014k).\end{aligned}$$

Problem 5: (*OMCC 2010*) Find all solutions of the equation $n(S(n)-1) = 2010$.

Problem 6: (*Iberoamerican 2012*) Show that for all positive integers n there are n consecutive positive integers such that none is divisible by the sum of their respective digits.