Mathematical Excalibur

Volume 2, Number 1 January-February, 1996

Olympiad Corner

Fourth Mathematical Olympiad of Taiwan:

First Day
Taipei, April 13, 1995

Problem 1. Let $P(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + a_nx^n$ be a polynomial with complex coefficients. Suppose the roots of $P(x)$ are $\alpha_1, \alpha_2, \cdots, \alpha_n$ with $|\alpha_1| > 1$, $|\alpha_2| > 1, \cdots, |\alpha_j| > 1$, and $|\alpha_{j+1}| \le 1, \cdots, |\alpha_n| \le 1$. Prove:

$$\prod_{i=1}^{j}|\alpha_i| \le \frac{\sqrt{|a_0|^2 + |a_1|^2 + \cdots + |a_n|^2}}{|a_n|}.$$

Problem 2. Given a sequence of integers $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$. One constructs a second sequence $|x_2 - x_1|$, $|x_3 - x_2|$, $|x_4 - x_3|$, $|x_5 - x_4|$, $|x_6 - x_5|$, $|x_7 - x_6|$, $|x_8 - x_7|$, $|x_1 - x_8|$. Such a process is called a single operation. Find all the 8-terms integral sequences having the following property: after finitely many single operations it becomes an integral sequence with all terms equal.

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Solution by Linear Combination

Kin-Yin Li

In mathematics, often we are interested in finding a solution to equations. Consider the following two problems:

Problem 1. Given real numbers $m_1, m_2, \ldots, m_n$ (all distinct) and $a_1, a_2, \ldots, a_n$, find a polynomial $v(x)$ such that $v(m_1) = a_1$, $v(m_2) = a_2, \ldots, v(m_n) = a_n$.

Problem 2. Given positive integers $m_1, m_2, \ldots, m_n$ (pairwise relatively prime) and integers $a_1, a_2, \ldots, a_n$, find an integer v such that $v \equiv a_1 \pmod{m_1}$, $v \equiv a_2 \pmod{m_2}, \ldots, v \equiv a_n \pmod{m_n}$.

Problem 1 comes up first in algebra and analysis (later in engineering and statistics). It is an *interpolation* problem, where we try to fit the values a_i at m_i (i.e., to find a polynomial whose graph passes through the points (m_1,a_1), (m_2,a_2), $\ldots$, (m_n,a_n)). Problem 2 comes up in number theory. It is a *congruence* problem, where we try to count objects by inspecting the remainders (i.e., to find a number which has the same remainder as a_i upon division by m_i).

There is a technique that can be applied to both problems. The idea is to solve first the special cases, where exactly one of the a_i's is 1 and all others 0. For problem 1, this is easily solved by defining (for $i = 1, 2, \ldots, n$) the polynomial $P_i(x)$ to be $(x-m_1)(x-m_2)\cdots(x-m_n)$ with the factor $(x-m_i)$ *omitted*, i.e.,

$$P_i(x) = \prod_{\substack{j=1 \\ j\ne i}}^{n}(x - m_j),$$

and $v_i(x) = P_i(x)/P_i(m_i)$. Then $v_i(m_i) = 1$ and $v_i(m_k) = 0$ for $k \ne i$ because $P_i(m_k) = 0$ (for $k \ne i$).

For problem 2, this is solved similarly by first defining (for $i = 1, 2, \ldots, n$) the integer P_i to be $m_1m_2\cdots m_n$ with the factor m_i *omitted*. Consider $P_i, 2P_i, \ldots, m_iP_i$. Upon division by m_i, no two of these will have the same remainder because the difference of any two of them is not divisible by m_i. So one of these, say c_iP_i, has remainder 1. Let $v_i = c_iP_i$, then $v_i \equiv 1 \pmod{m_i}$ and $v_i \equiv 0 \pmod{m_k}$ for $k \ne i$ because $P_i \equiv 0 \pmod{m_k}$.

Finally to solve problem 1 or 2 in general, we use the special case solutions $v_1, v_2, \ldots, v_n$ to form $v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$. It is now easy to check that the expression v solves both problems 1 and 2.

For problem 1,

$$v(x) = a_1\frac{P_1(x)}{P_1(m_1)} + \cdots + a_n\frac{P_n(x)}{P_n(m_n)}$$

is called *Lagrange's interpolation formula*. For problem 2, although the c_i's may be tedious to find, we know a solution $v = a_1c_1P_1 + \cdots + a_nc_nP_n$ exists. This is the assertion of the *Chinese remainder theorem*. Note also that if we add to v any multiple of $(x-m_1)(x-m_2)\cdots(x-m_n)$ in problem 1 or any multiple of $m_1m_2\cdots m_n$ in problem 2, we get other solutions.

The expression of v, involving a sum of multiples of $v_1, v_2, \ldots, v_n$, is so common in similar problems that it is now come to be called a *linear combination* of $v_1, v_2, \ldots, v_n$. In passing, note that the a_i's are numbers. However, the v_i's are polynomials in problem 1 and numbers in problem 2. Like vectors expressed in coordinates, the v_i's are objects that may take on different values at different positions. So functions corresponding to solutions of equations are often viewed as vectors (with infinitely many coordinates). Concepts like these are the foundation of *Linear*

(*continued on page 2*)

Solution by Linear Combination:
(*continued from page 1*)

Algebra, which studies the properties of solutions of these kind of problems in an abstract manner.

Example 1. If $f(x)$ is a polynomial of degree at most n and $f(k) = (n+1-k)/(k+1)$ for $k = 0, 1, \ldots, n$, find $f(n+1)$.

Solution 1. Applying Lagrange's interpolation formula, we define $P_k(x) = x(x-1)\cdots(x-n)$ with the factor $(x-k)$ *omitted.* Then $P_k(n+1) = (n+1)!/(n+1-k)$, $P_k(k) = (-1)^{n-k}k!(n-k)!$ and

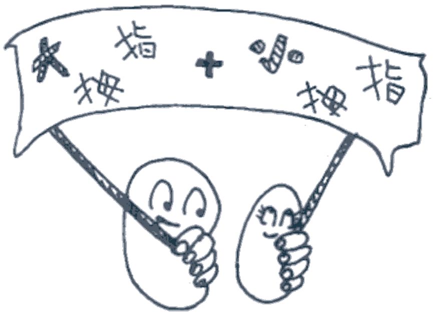
$$f(n+1) = (-1)^{n+1}\sum_{k=0}^{n}(-1)^{k+1}\frac{(n+1)!}{(k+1)!(n-k)!} = (-1)^n$$

where we used the binomial expansion of $(1-1)^{n+1}$ in the last step.

Solution 2. The polynomial $g(x) = (x+1)f(x) - (n+1-x)$ has degree at most $n+1$. We are given that $g(0) = g(1) = \cdots = g(n) = 0$. So $g(x) = Cx(x-1)\cdots(x-n)$. To find C, we set $x = -1$ and get $g(-1) = -(n+2) = C(-1)^{n+1}(n+1)!$. Therefore, $C = (-1)^n(n+2)/(n+1)!$ and $g(n+1) = (n+2)f(n+1) = (-1)^n(n+2)$, which implies $f(n+1) = (-1)^n$.

Example 2. Prove that for each positive integer n there exist n consecutive positive integers, none of which is an integral power of a prime number. (*Source:* 1989 IMO.)

Solution. Let $p_1, p_2, \ldots, p_{2n}$ be $2n$ distinct prime numbers and consider the congruence problem $v \equiv -1 \pmod{p_1p_2}$, $v \equiv -2 \pmod{p_3p_4}$, …, $v \equiv -n \pmod{p_{2n-1}p_{2n}}$. Since $p_1p_2, p_3p_4, \ldots, p_{2n-1}p_{2n}$ are pairwise relatively prime, by the Chinese remainder theorem, there is a positive integer solution v. Then each of the n consecutive numbers $v+1, v+2, \ldots, v+n$ is divisible by more than one prime number. So each is not a power of a prime number.



中國剩餘定理

郭宇權

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在中國南北朝時代著成的數學經典《孫子算經》中，有一道千古名題，名爲【物不知數】問題是

問： 今有物不知其數，三三數之賸二，五五數之賸三，七七數之賸二，問物幾何？

答曰： 二十三

這問題頗有猜謎的趣味，在中國民間頗廣流傳，在西方數學史上被稱爲【中國剩餘定理】。如果用現代數學符號來表示【孫子問題】，我們由已知條件

$$N \equiv 2 \pmod 3;$$
$$N \equiv 3 \pmod 5;$$
$$N \equiv 2 \pmod 7)$$

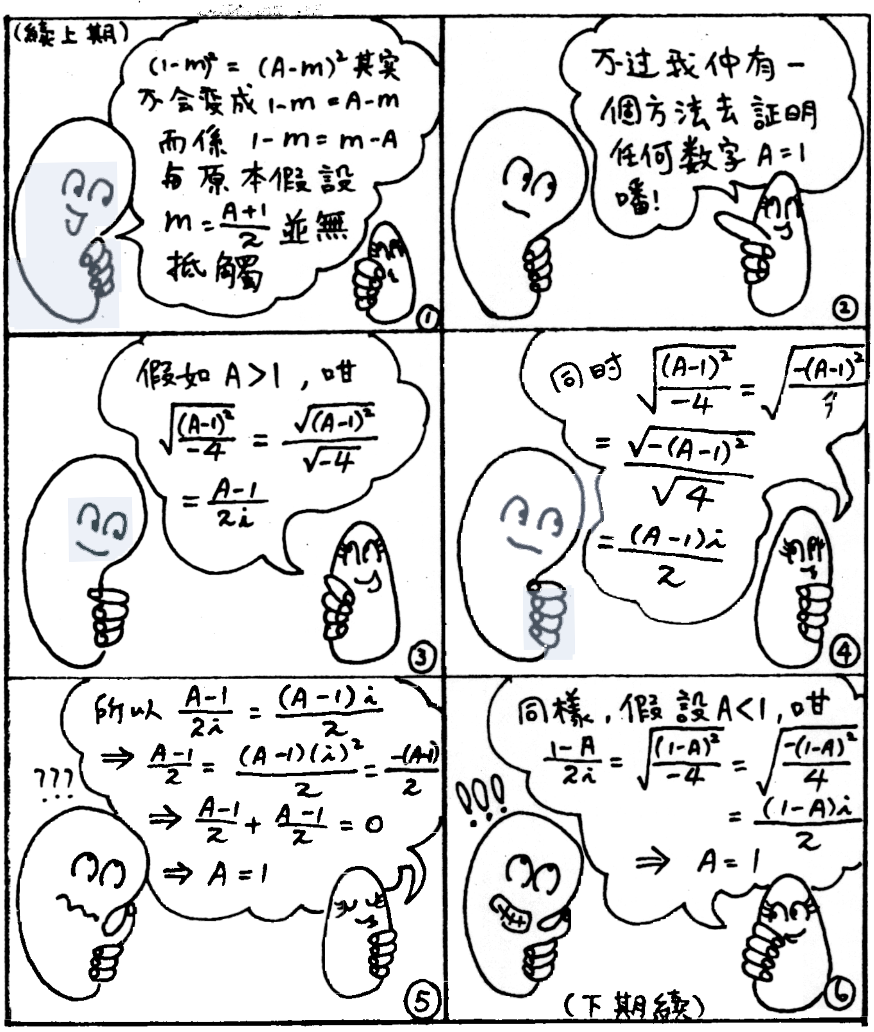
求最小的正整數N。這是一個一次同餘式組的問題，問題的求解法被編成【孫子歌】，是一首五絕詩

三人同行七十稀
五樹梅花廿一枝
七子團圓正半月
減百零五便得知

用現代算式表示是

$$N = 70\times2+21\times3+15\times2-105\times2 = 23$$

求解的手段是先找7與5的公倍數，用它除3餘1，這數是70；因此，70×2除3餘2。類似地，21是7與3的公倍數，用它除5餘1，所以21×3除5餘3；接著15×2除盡3與5，除7餘2。這三數的總和，滿足了所需條件；最後，減去3與5與7的公倍數，使N變爲滿足所有條件的最小正整數。這就得出所需答案。

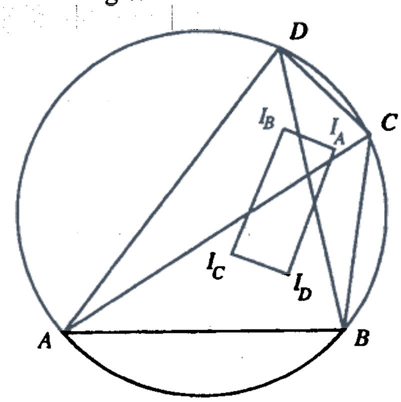


Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, address, school affiliation and grade level. Please send submissions to *Dr. Kin-Yin Li, Dept of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is February 28, 1996.

Problem 26. Show that the solutions of the equation $\cos \pi x = \frac{1}{3}$ are all irrational numbers. (*Source:* 1974 Putnam Exam.)

Problem 27. Let *ABCD* be a cyclic quadrilateral and let I_A, I_B, I_C, I_D be the incenters of ΔBCD, ΔACD, ΔABD, ΔABC, respectively. Show that $I_AI_BI_CI_D$ is a rectangle.



Problem 28. The positive integers are separated into two subsets with no common elements. Show that one of these two subsets must contain a three term arithmetic progression.

Problem 29. Suppose $P(x)$ is a nonconstant polynomial with integer coefficients and all coefficients are greater than or equal to -1. If $P(2) = 0$, show that $P(1) \neq 0$.

Problem 30. For positive integer $n > 1$, define $f(n)$ to be 1 plus the sum of all prime numbers dividing n multiplied by their exponents, *e.g.*, $f(40) = f(2^3 \times 5^1) = 1 + (2\times 3 + 5\times 1) = 12$. Show that if $n > 6$, the sequence $n, f(n), f(f(n)), f(f(f(n))), \ldots$ must eventually be repeating 8, 7, 8, 7, 8, 7, ...

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Solutions

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Problem 21. Show that if a polynomial $P(x)$ satisfies

$$P(2x^2 - 1) = \frac{P(x)^2}{2}.$$

it must be constant.

Solution 1: Independent solution by **LIU Wai Kwong** (Pui Tak Canossian College) and **YUNG Fai** (CUHK).

Construct a sequence $u_1 = 1$, $u_2 = -1$ and $u_n = \sqrt{\frac{u_{n-1}+1}{2}}$ for $n \geq 3$. We have $u_n < u_{n+1} < 1$ for $n \geq 2$ and $P(u_n) = (P(u_{n+1})^2/2) - 1$ for $n \geq 1$. Note that $P(u_n) \neq 0$ for $n \geq 1$ (otherwise $P(u_n) = 0$ would imply $P(u_{n-1})$, $P(u_{n-2})$, $\cdots$, $P(u_1)$ are rational, but $P(1) = 1 \pm \sqrt{3}$.) Differentiating the functional equation for P, we get $4xP'(2x^2-1) = P(x)P'(x)$. Since $P(1) \neq 4$, we get $P'(u_1) = P'(1) = 0$. This implies $0 = P'(u_2) = P'(u_3) = \cdots$. Therefore, $P'(x)$ is the zero polynomial and so $P(x)$ is constant.

Comments: This problem was from the 1991 USSR Math Winter Camp. Below we will provide a solution without calculus.

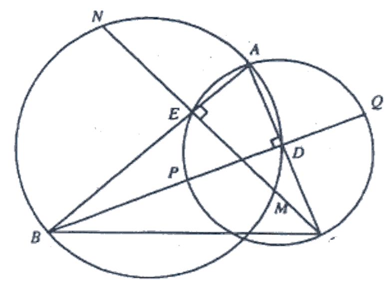
Solution 2: Suppose $P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ is such a polynomial with degree $n \geq 1$. Then

$$a_0(2x^2-1)^n + a_1(2x^2-1)^{n-1} + \cdots + a_n = \frac{(a_0x^n + a_1x^{n-1} + \cdots + a_n)^2}{2} - 1.$$

Comparing the coefficients of x^{2n}, we find $a_0 2^n = a_0^2/2$, so $a_0 = 2^{n+1}$. Suppose a_0, a_1, ..., a_k are known to be rational. Comparing the coefficients of x^{2n-k-1}, the left side yields a rational number involving a_0, ..., a_k, but the right side yields a number of the form a_0a_{k+1} plus a rational number involving a_0, ..., a_k. So a_{k+1} is also rational. Hence a_0, a_1, ..., a_n are all rational. Then $P(1) = a_0 + a_1 + \cdots + a_n$ is rational. However, $P(1) = (P(1)^2/2) - 1$ forces $P(1) = 1 \pm \sqrt{3}$, a contradiction. Therefore $P(x)$ must be constant.

Other commended solver: **William CHEUNG Pok Man** (S.T.F.A. Leung Kau Kui College).

Problem 22. An acute-angled triangle *ABC* is given in the plane. The circle with diameter *AB* intersects altitude *CE* and its extension at points *M* and *N*, and the circle with diameter *AC* intersects altitude *BD* and its extension at *P* and *Q*. Prove that the points *M, N, P, Q* lie on a common circle. (*Source*: 1990 USA Mathematical Olympiad).



Solution: William CHEUNG Pok Man (S.T.F.A. Leung Kau Kui College).

If *M, N, P, Q* are concyclic, then *A* must be the center because it is the intersection of the perpendicular bisectors of *PQ* and *MN*. So it suffices to show *AP* = *AM*.

Considering the similar triangles *ADP* and *APC*, we get *AD*/*AP* = *AP*/*AC*, i.e., $AP^2 = AD\times AC$. Similarly, $AM^2 = AE\times AB$. Since $\angle BEC = \angle BDC$, points *B, C, D, E* are concyclic. Therefore, $AD\times AC = AE\times AB$ and so *AP* = *AM*.

Other commended solvers: **HO Wing Yip** (Clementi Secondary School), **LIU Wai Kwong** (Pui Tak Canossian College), **Edmond MOK Tze Tao** (Queen's College), **WONG Him Ting** (HKU) and **YU Chun Ling** (Ying Wa College).

Problem 23. Determine all sequences $\{a_1, a_2, \ldots\}$ such that $a_1 = 1$ and $|a_n - a_m| \leq 2mn/(m^2+n^2)$ for all positive integers *m* and *n*. (*Source*: Past IMO problem proposed by Finland).

Solution: Independent solution by **CHAN Wing Sum** (HKUST), **LIU Wai Kwong** (Pui Tak Canossian College) and **YUNG Fai** (CUHK).

For fixed *m*,

$$\lim_{n\to\infty} |a_n - a_m| \leq \lim_{n\to\infty} \frac{2mn}{m^2+n^2} = 0$$

(*continued on page 4*)

Problem Corner

(*continued from page 3*)

So for all m,

$$a_m = \lim_{n\to\infty} a_n,$$

It follows that all terms are equal (to $a_1 = 1$.)

Problem 24. In a party, n boys and n girls are paired. It is observed that in each pair, the difference in height is less than 10 cm. Show that the difference in height of the k-th tallest boy and the k-th tallest girl is also less than 10 cm for $k = 1, 2, \ldots, n$.

Solution: Independent solution by **HO Wing Yip** (Clementi Secondary School) **and YU Chun Ling** (Ying Wa College).

Let $b_1 \ge b_2 \ge \cdots \ge b_n$ be the heights of the boys and $g_1 \ge g_2 \ge \cdots \ge g_n$ be those of the girls. Suppose for some k, $|b_k - g_k| \ge 10$. In the case $b_k - g_k \ge 10$, we have $b_i - g_j \ge 10$ for $1 \le i \le k$ and $k \le j \le n$. Consider the boys of height b_i ($1 \le i \le k$) and the girls of height g_j ($k \le j \le n$). By the pigeonhole principle, two of these n+1 must be paired originally. However, $b_i - g_j \ge 10$ contradicts the hypothesis. (The case $g_k - b_k \ge 10$ is handled similarly.) So $|b_k - g_k| < 10$ for all k.

Comments: This was a problem from the 1984 Tournament of the Towns, a competition started in 1980 at Moscow and Kiev and is now participated by students in dozens of cities in different continents.

Other commended solvers: **CHAN Wing Sum** (HKUST), **William CHEUNG Pok Man** (S.T.F.A. Leung Kau Kui College, **KU Yuk Lun** (HKUST), **LIU Wai Kwong** (Pui Tak Canossian College) and **WONG Him Ting** (HKU).

Problem 25. Are there any positive integers n such that the first four digits from the left side of n! (in base 10 representation) is 1995?

Solution 1: LIU Wai Kwong (Pui Tak Canossian College).

Let $[x]$ be the greatest integer not exceeding x and $\{x\} = x - [x]$. Also, let $a_j = 1 + j\times10^{-8}$, $b_0 = \log 10^8!$ and $b_j = \log 10^8! + (\log a_1 + \cdots + \log a_j)$ for $j > 0$. (For this solution, log means $\log_{10}$.) Observe that

(i) $0 < \log a_k \le \log a_{30000} < \log\dfrac{1996}{1995}$ for $k = 1, 2, \ldots, 30000$;

(ii) $\sum_{j=1}^{30000}\log a_j > 15000(\log a_1 + \log a_{30000}) > 1.$

Note the distance between $\{\log 1995\}$ and $\{\log 1996\}$ is $\log(1996/1995)$. Now $b_0, b_1, \ldots, b_{30000}$ is increasing and

$$b_{30000} - b_0 > 1 \quad \text{(by (ii))},$$

but

$$0 < b_{j+1} - b_j < \log\frac{1996}{1995} \quad \text{(by (i))}.$$

So there is a $k \le 30000$ such that

$$\{\log 1995\} < \{b_k\} < \{\log 1996\}$$

Now

$$\log 10^8! + \textstyle\sum_{j=1}^{k}\log a_j = \log(10^8 + k)! - 8k$$

implies

$$\{\log 1995\} < \{\log(10^8+k)!\} < \{\log 1996\}.$$

Adding $[\log 1995] = [\log 1996] = 3$, we have

$$\log 1995 < \log(10^8+k)! - m < \log 1996$$

for $m = [\log(10^8+k)!] - 3$. Therefore,

$$1995\times10^m < (10^8+k)! < 1996\times10^m.$$

Consequently, the number $(10^8+k)!$ begins with 1995.

Comments: With 1995 replaced by 1993, this problem appeared in the 1993 German Mathematical Olympiad. Below we will provide the (modified) official solution.

Solution 2: Let $m = 1000100000$. If $k < 99999$ and $(m+k)! = abcd\cdots$ (in base 10 representation), then $(m+k+1)! = abcd\cdots \times 10001\cdots = efgh\cdots$, where $efgh$ equals $abcd$ or the first four digits of $abcd$+1. So, the first four digits of each of $(m+1)!, (m+2)!, \ldots, (m+99999)!$ must be the same as or increase by 1 compared with the previous factorial. Also, because the fifth digit of $m+k$ ($k < 99999$) is 1, the fifth digit of $(m+k)!$ will be added to the first digit of $(m+k)!$ in computing $(m+k+1)!$. So, in any ten consecutive factorials among $(m+1)!$, $(m+2)!, \ldots, (m+99999)!$, there must be an increase by 1 in the first four digits. So the first four digits of $(m+1)!$, $(m+2)!, \ldots, (m+99999)!$ must take on all 9000 possible choices. In particular, one of these is 1995.

Olympiad Corner

(*continued from page 1*)

Problem 3. Suppose n persons meet in a meeting, every one among them is familiar with exactly 8 other participants of that meeting. Furthermore suppose that each pair of two participants who are familiar with each other have 4 acquaintances in common in that meeting, and each pair of two participants who are not familiar with each other have only 2 acquaintances in common. What are the possible values of n?

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Second Day
Taipei, April 15, 1995

Problem 4. Given n (where $n \ge 2$) distinct integers $m_1, m_2, \cdots, m_n$. Prove that there exist a polynomial $f(x)$ of degree n and with integral coefficients which satisfies the following conditions:

(i) $f(m_i) = -1$, for all $1 \le i \le n$.

(ii) $f(x)$ cannot be factorized into a product of two nonconstant polynomials with integral coefficients.

Problem 5. Let P be a point on the circumscribed circle of $\Delta A_1A_2A_3$. Let H be the orthocenter of $\Delta A_1A_2A_3$. Let B_1 (B_2, B_3 respectively) be the point of intersection of the perpendicular from P to A_2A_3 (A_3A_1, A_1A_2 respectively). It is known that the three points B_1, B_2, B_3 are colinear. Prove that the line $B_1B_2B_3$ passes through the midpoint of the line segment $\overline{PH}$.

Problem 6. Let a, b, c, d be integers such that $ad - bc = k > 0$, $(a,b) = 1$, and $(c,d) = 1$. Prove that there are exactly k ordered pairs of real numbers (x_1,x_2) satisfying $0 \le x_1, x_2 < 1$ and both $ax_1 + bx_2$ and $cx_1 + dx_2$ are integers.

Erratum: In the article 談談質數 in the last issue, 定理 4 (Chebychev Theorem) should be corrected as

$$\frac{n}{8\log n} \le \pi(n) \le \frac{12n}{\log n} \quad (\text{when } n \ge 2).$$