Mathematical Excalibur

Volume 3, Number 2 March-May, 1997

Olympiad Corner

The Ninth Asian Pacific Mathematics Olympiad, March 1997:

Time Allowed: 4 hours.
Each question is worth 7 points.

Problem 1. Given

$$S=1+\frac{1}{1+\frac{1}{3}}+\frac{1}{1+\frac{1}{3}+\frac{1}{6}}+\cdots+\frac{1}{1+\frac{1}{3}+\frac{1}{6}+\cdots+\frac{1}{1993006}},$$

where the denominators contain partial sums of the sequence of reciprocals of triangular numbers. Prove that $S>1001$.

Problem 2. Find an integer n, with $100\le n\le 1997$, such that $\frac{2^n+2}{n}$ is also an integer.

Problem 3. Let ABC be a triangle inscribed in a circle and let

$$l_a=\frac{m_a}{M_a},\ l_b=\frac{m_b}{M_b},\ l_c=\frac{m_c}{M_c},$$

(*continued on page 4*)

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Acknowledgment: Thanks to Catherine NG, EEE Dept, HKUST for general assistance.

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word are encouraged. The deadline for receiving material for the next issue is July 10, 1997.

For individual subscription for the remaining issue for the 96-97 academic year, send us a stamped self-addressed envelope. Send all correspondence to:

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由圓周率到四年一閏

香港道教聯合會青松中學
梁子傑

有時，一個簡單的電腦程序，就可以令我們發現不少有趣的數學現象，以下便是一個好例子：

```
PROGRAM rational_real; {written in MS QuickPascal}
VAR
   numer, denom : LongInt;
   devi, min, real_no : Double;
BEGIN
   min := 10; real_no := pi;
   FOR denom := 1 TO 50000 DO
      BEGIN
         numer := round(real_no * denom);
         devi := abs(numer/denom - real_no);
         IF devi < min THEN
            BEGIN
               min := devi;
               writeln (numer, '/', denom)
            END
      END
END.
```

這個程序是用來尋找圓周率π的有理數近似值的。程序令分母(denom)由1開始，先計算出最接近的分子(numer)的數值，然後計算出這個有理數近似值(numer/denom)跟圓周率的偏差(devi)，如果偏差比以前的小，就將該有理數印出來。

不出兩秒鐘，電腦就會計算出結果：

3/1, 13/4, 16/5, 19/6, 22/7, 179/57, 201/64, 223/71, 245/78, 267/85, 289/92, 311/99, 333/106, 355/113, 52163/16604, 52518/16717, 52873/16830, 53228/16943, …

從以上的結果，我們不難發現這個現象：並不是每當分母增加時，所計算出來的有理數近似值就一定較準確，好似當分母介乎於8至56之間時，以7作爲分母的近似值就比它們準確了。

如果大家細心地觀察一會，相信亦會發現在這些分母之中，出現了一些「跳躍」的現象，好似由7跳至57、由113跳至16604等。再留心看看，每次跳躍之後，分母增加的幅度亦有關係：7之後的57、64、71等，就相隔7；113之後的16604、16717等，就相隔113。同時分子亦有相類似的關係。奇怪嗎？爲甚麼會有這個現象出現呢？

要解釋以上的現象，我們就需要認識一個很特別的表達數值的方法，它就是「連分數」(continued fraction)。所謂「連分數」就是利用一連串的倒數來表達一個數的數值，例如：

$$\frac{1057}{498}=2+\frac{61}{498}=2+\frac{1}{\frac{498}{61}}$$

$$=2+\frac{1}{8+\frac{10}{61}}=2+\frac{1}{8+\frac{1}{6+\frac{1}{10}}}.$$

我們並且用這個記號來表示以上的結果：$\frac{1057}{498}=[2;8,6,10]$。

到了今天，數學家經已發現了很多有關連分數的性質，其中一項就是「漸近分數」的現象。以上述數字爲例，如果我們逐次選取連分數中部份的數字，即 $[2]=2$，$[2;8]=2+\frac{1}{8}=\frac{17}{8}$，$[2;8,6]=2+\frac{1}{8+\frac{1}{6}}=\frac{104}{49}$，$[2;8,6,10]=\frac{1057}{498}$，所得到的一列分數，就叫做「部份連分數」。我們可以證明「部份連分數」是一列越來越漸近原本數值的分數，換句話講，在數列中每一個分數都可以表示爲原本數值的約數，而且後者的準確性會比前者佳；當然，後者分母的數值卻比前者的大，應用起來就不及前者方便了。

回到上面電腦程序的結果，我們發現如果將圓周率π表示成連分數的話，我們有 $\pi=[3;7,15,1,292,1,1,\ldots]$。（因爲圓周率是一個無理數，它的連分數表達式自然是無窮盡的。）寫出π的部份連分數，我們有 $[3]=3$，$[3;7]=\frac{22}{7}$，$[3;7,15]=\frac{333}{106}$，$[3;7,15,1]=\frac{355}{113}$，$[3;7,15,1,292]=\frac{103993}{33102}$，……等等。而這些部份連分數，不是和電腦程序計算出來的結果相同嗎？

不過，電腦程序計算出來的結果卻比「部份連分數」爲多，這是因爲部份連分數的現象祇是一個充分條件，而不是一個必要條件，所以「部份連分數」並不是一個完整的漸近分數的數列。雖然如此，一個完整的數列亦可以利用「連分數」來表達出來。留意 $\frac{13}{4}=[3;4]$，$\frac{16}{5}=[3;5]$，$\frac{19}{6}=[3;6]$，$\frac{22}{7}=[3;7]$；另外，$\frac{179}{57}=[3;7,8]$，$\frac{201}{64}=[3;7,9]$……等等；不難看出，每當分母增加至和「部份連分數」相同的數值時，下一個漸近分數

就將會由下一個「部份連分數」的一半開始。例如，$\frac{355}{113}=[3;7,15,1]$，下一個部份連分數是[3;7,15,1,292]，而292的一半等於146，故此$\frac{355}{113}$之後的漸近分數應等於$[3;7,15,1,146]=\frac{52163}{16604}$。

以上的現象同時解釋了，為何在數列中，分母數值的間隔會和「跳躍」前的分母數值相等的現象。大家祇要將[3;7,8]和[3;7,9]計算一次，就會明白為何分子的差距和分母的差距，剛好等於[3;7]的分子和分母了。

在未討論下一個問題前，值得指出的是，在電腦程序計算出的漸近分數中，有幾個數值是十分「著名的」。例如：$\frac{3}{1}$，這是人類對圓周率最早期的約數，中國古籍中就有「徑一周三」的記載。另外$\frac{22}{7}$更毋須多介紹了。$\frac{223}{71}$是古希臘數學家阿基米德提出的約數；而$\frac{355}{113}$就最先由中國南北朝時代的數學家祖沖之提出的。當知道下一個漸近分數的分母將會高達16604時，相信大家都會明白為何祖沖之計算出圓周率的七位小數約數之後，要經過多達一千年的時間，才有人能夠計到更佳的圓周率近似值！

其實漸近分數並不是一些數字的玩意，它有不少實際的應用價值，其中一項就是閏年的計算。跟據資料顯示，地球環繞太陽一周需要365日5小時48分46秒，化為分數就即是有$365\frac{10463}{43200}$日。其中的365日當然沒有問題，但剩下的$\frac{10463}{43200}$日又如何處理呢？完全不理，那麼祇要每過5年，就會有一整天的時間差距了。（又或者每43200年，就會相差了10463日。）因此我們在曆法上就訂立了閏年的制度：即每隔一些年份就在那年增加一日，藉此保持曆法上不會出現偏差。但是，到底要經過多少年才有一次閏年呢？我們不可能經過4萬多年後，才增加萬多日來補償。不過我們可以通過計算$\frac{10463}{43200}$的漸近分數來獲得答案。

首先我們將本文開始的電腦程序中的real_no句子改為：real_no := 10463/43200; 執行後，可得到結果：

0/1, 1/3, 1/4, 4/17, 5/21, 6/25, 7/29, 8/33, 23/95, 31/128, 101/417, 132/545, ...

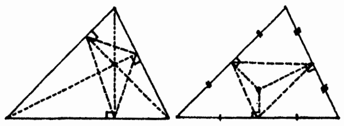
從結果中的$\frac{1}{4}$可知，我們應該每4年就要多加一日。按此比例，每100年就應有25個閏年。但由$\frac{23}{95}$和$\frac{31}{128}$可以知道，每100年其實祇需要24個閏年。所以如果我們祇跟著「四年一閏」的方法來編寫曆

(continued on page 4)

老師不教的幾何（三）

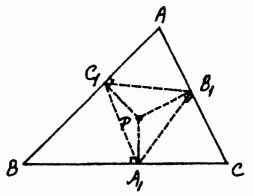
張百康

我們較早前蹓過的垂足三角形(orthic triangle)和中點三角形(medial triangle)有甚麼共同的性質呢？大家請重溫一下這兩個三角形（圖一）。



（圖一）

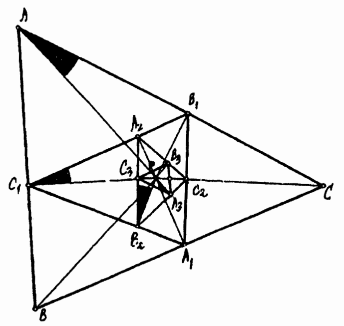
其實，它們可以看成是更一般情況的兩個特例：令P為已知三角形ABC內任意一點。從P作垂直線垂直於這三角形的三邊，連這三垂線的垂足可得另一三角形$A_1B_1C_1$，稱為三角形ABC相對於踏板點(pedal point) P的踏板三角形(pedal triangle)（圖二）。



（圖二）

分別以三角形ABC的垂心和外心作踏板點便可得垂足三角形和中點三角形作為ABC的踏板三角形。踏板三角形有甚麼有趣的性質呢？在1892年出版的一本幾何書中，編輯J. Neuberg提出及證明了踏板三角形的一個周期性現象：

以P作為三角形$A_1B_1C_1$的踏板點，可以得到$A_1B_1C_1$的踏板三角形$A_2B_2C_2$。繼續這作法以P為踏板點，可以得到另一個踏板三角形$A_3B_3C_3$，如此類推(圖三)。



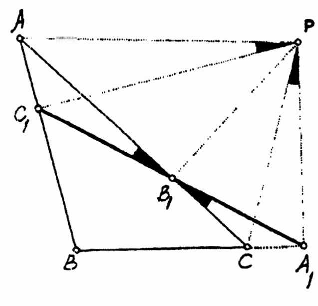
（圖三）

由於各有一對對角是直角，因此下列的四邊形都是外接四邊形：AC_1PB_1、$C_1B_2PA_2$和$B_2A_3PC_3$。因此$\angle B_1AP$、$\angle B_1C_1P$ $(=\angle A_2C_1P)$、$\angle A_2B_2P$ $(=\angle C_3B_2P)$和$\angle C_3A_3P$依次兩兩為等弧上的圓周角。所以$\angle B_1AP=\angle C_3A_3P$；同理，$\angle C_1AP=\angle B_3A_3P$。這兩結果告訴我們：$\angle BAC=\angle B_3A_3C_3$；同理，$\angle ACB=\angle A_3C_3B_3$和$\angle ABC=\angle A_3B_3C_3$。因此三角形$ABC$和$A_3B_3C_3$相似。顯而易見，踏板三角形有下列周期性：

$$\Delta ABC\sim\Delta A_3B_3C_3\sim\Delta A_6B_6C_6\sim\cdots,$$
$$\Delta A_1B_1C_1\sim\Delta A_4B_4C_4\sim\Delta A_7B_7C_7\sim\cdots$$
和 $\Delta A_2B_2C_2\sim\Delta A_5B_5C_5\sim\Delta A_8B_8C_8\sim\cdots$。

如果踏板點在三角形ABC外部時，這周期性還成立嗎？答案在一般情況下是肯定的，大家可參考上述證明加以修改便可。有沒有例外情況？如果大家懂得用互動幾何軟件如Cabri Geometry或Geometer's Sketchpad，這是一個有意義的探究活動。通過探究，大家應該發現：如果踏板點P在三角形ABC的外接圓上，則踏板三角形$A_1B_1C_1$退化為一直線，也沒有其他的踏板點使踏板三角形退化為直線。這直線稱為辛姆生線(Simson Line)。辛姆生(Robert Simson)是十七、八世紀的數學家，但後人在他的著作中找不到這性質的證明，反而是William Wallace在1797年發表了下列證明：

設A_1、B_1、C_1共線（圖四），則$\angle AB_1C_1$和$\angle A_1B_1C$為對頂角，所以相等。已知$\angle PA_1C=\angle PB_1C=90°$及$\angle PB_1A=\angle PC_1A=90°$，所以$P$、$A_1$、$C$、$B_1$四點共圓，且$P$、$B_1$、$C_1$、$A$也共圓。由此推出$\angle A_1PC=\angle A_1B_1C=\angle AB_1C_1=\angle APC_1$。並且，$\angle PA_1B=\angle PC_1B=90°$，所以$P$、$A_1$、$B$、$C_1$四點也共圓，因此，$\angle A_1PC_1$和$\angle C_1BA_1$互補。但$\angle A_1PC_1=\angle A_1PC+\angle CPC_1=\angle APC_1+\angle CPC_1=\angle APC$，所以$\angle APC$和$\angle ABC$ $(=\angle C_1BA_1)$也互補。故A、B、C、P四點共圓，即P點在三角形ABC的外接圓上。



（圖四）

(continued on page 4)

Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, address, school affiliation and grade level. Please send submissions to *Dr. Kin-Yin Li, Dept of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is July 10, 1997.

Problem 56. Find all prime numbers p such that $2^p + p^2$ is also prime.

Problem 57. Prove that for real numbers $x, y, z > 0$,

$$\frac{x^2}{x+y}+\frac{y^2}{y+z}+\frac{z^2}{z+x}\geq\frac{x+y+z}{2}.$$

Problem 58. Let ABC be an acute-angled triangle with $BC > CA$. Let O be its circumcenter, H its orthocenter, and F the foot of its altitude CH. Let the perpendicular to OF at F meet the side CA at P. Prove that $\angle FHP = \angle BAC$. (*Source*: unused problem in the 1996 IMO.)

Problem 59. Let n be a positive integer greater than 2. Find all real number solutions $(x_1, x_2, \cdots, x_n)$ to the equation

$$(1-x_1)^2 + (x_1-x_2)^2 + \cdots + (x_{n-1}-x_n)^2 + x_n^2 = \frac{1}{n+1}.$$

(Source: 1975 British Mathematical Olympiad)

Problem 60. Find (without calculus) a fifth degree polynomial $p(x)$ such that $p(x) + 1$ is divisible by $(x - 1)^3$ and $p(x) - 1$ is divisible by $(x + 1)^3$.

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Solutions

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Problem 51. Is there a positive integer n such that $\sqrt{n-1}+\sqrt{n+1}$ is a rational number?

Solution: Gary NG Ka Wing (STFA Leung Kau Kui College, Form 4).

Assume there is a positive integer n such that

$$\sqrt{n-1}+\sqrt{n+1}=r$$

is rational. Squaring and simplifying, we get

$$\sqrt{n^2-1}=\frac{r^2-2n}{2}$$

is also rational. However, for $n > 1$, if $\sqrt{n^2-1} = a/b$ for some positive integers a, b having no common factor greater than 1, then $a^2 = b^2(n^2-1)$, which implies b also divides a. So b must be 1. Now for $n > 1$,

$$n^2 > n^2 - 1 = a^2 > (n - 1)^2$$

is impossible. So $n = 1$, but then

$$\sqrt{n-1}+\sqrt{n+1}=\sqrt{2}$$

is irrational. Therefore, no such n exists.

Other commended solvers: **CHAN Ming Chiu** (La Salle College, Form 6), **CHAN Wing Sum** (HKUST), **William CHEUNG Pok Man** (S.T.F.A. Leung Kau Kui College, Form 6), **CHOI Wing Shan Winnie** (St. Stephen's Girls' College, Form 6), **LEUNG Shun Ming** (La Salle College, Form 4), **LIU Wai Kwong** (Pui Tak Canossian College), **TSE Wing Ho** (Ho Fung College, Form 5), **Sam YUEN Man Long** (STFA Leung Kau Kui College, Form 4) and **YUNG Fai** (CUHK).

Problem 52. Let a, b, c be distinct real numbers such that $a^3 = 3(b^2+c^2) - 25$, $b^3 = 3(c^2+a^2) - 25$, $c^3 = 3(a^2+b^2) - 25$. Find the value of abc.

Solution: CHEUNG Pok Man (S.T.F.A. Leung Kau Kui College, Form 6), **YEUNG Yi Pok** (Pui Shing Catholic Secondary School, Form 7) and **YUNG Fai** (CUHK).

Let a, b, c be roots of

$$x^3 - px^2 + qx - r = 0.$$

Then $p = a + b + c$, $q = ab + bc + ca$ and $r = abc$. Since $a^2 + b^2 + c^2 = p^2 - 2q$, so

$$a^3 = 3(b^2 + c^2) - 25 = 3(p^2-2q-a^2) - 25.$$

This is equivalent to $a^3 + 3a^2 + (25 + 6q - 3p^2) = 0$. Then a is a root of $x^3 + 3x^2 + (25 + 6q - 3p^2) = 0$. Similarly, b and c are roots of this equation. Comparing coefficients of the two equations, we get $p = -3$, $q = 0$ and $abc = r = -(25 + 6q - 3p^2) = 2$.

Other commended solvers: **LIU Wai Kwong** (Pui Tak Canossian College), **TSE Wing Ho** (Ho Fung College, Form 5) and **Sam YUEN Man Long** (STFA Leung Kau Kui College, Form 4) ∘

Problem 53. For ΔABC, define A' on BC so that $AB + BA' = AC + CA'$ and similarly define B' on CA and C' on AB. Show that AA', BB', CC' are concurrent. (The point of concurrency is called the *Nagel point* of ΔABC.)

Solution: CHEUNG Pok Man (S.T.F.A. Leung Kau Kui College, Form 6), **LIU Wai Kwong** (Pui Tak Canossian College) and **YEUNG Yi Pok** (Pui Shing Catholic Secondary School, Form 7)

Let $a = BC$, $b = CA$, $c = AB$ and $s = (AB + BC + CA)/2$. Since $AB + BA' = s = AC + CA'$, we have $BA' = s - c$ and $CA' = s - b$. Similarly, $CB' = s - a$, $AB' = s - c$, $AC' = s - b$ and $BC' = s - a$. Then

$$(CA'/BA')(AB'/CB')(BC'/AC') = 1.$$

So by the converse of Ceva's theorem, AA', BB', CC' are concurrent.

Other commended solvers: **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 4) and **Sam YUEN Man Long** (STFA Leung Kau Kui College, Form 4).

Problem 54. Let R be the set of real numbers. Find all functions $f : R \to R$ such that

$$f(f(x+y)) = f(x+y) + f(x)f(y) - xy$$

for all $x, y \in R$. (Source: 1995 Byelorussian Mathematical Olympiad (Final Round))

Solution: YUNG Fai (CUHK).

Putting $y = 0$, we get

$$f(f(x)) = [1 + f(0)]f(x).$$

Replacing x by $x + y$, we get

$$[1+f(0)]f(x+y) = f(f(x+y)) = f(x+y) + f(x)f(y) - xy,$$

which simplifies to

$$f(0)f(x+y) = f(x)f(y) - xy.$$

(*continued on page 4*)

Problem Corner

(*continued from page 3*)

Putting $y = 1$, we get

$$f(0)f(x+1) = f(x)f(1) - x.$$

Putting $y = -1$ and replacing x by $x+1$, we get

$$f(0)f(x) = f(x+1)f(-1) + x + 1.$$

Eliminating $f(x+1)$ in the last two equations, we get

$$[f^2(0) - f(1)f(-1)]f(x) = [f(0) - f(-1)]x + f(0).$$

If $f^2(0) - f(1)f(-1) \neq 0$, then $f(x)$ is linear. If $f^2(0) - f(1)f(-1) = 0$, then putting $x = 0$ in the last equation, we get $f(0) = 0$. In this case, the displayed equation above implies $f(x)f(y) = xy$. Then $f(x)f(1) = x$ for all $x \in R$. So $f(1) \neq 0$ and $f(x)$ is linear.

Finally, substituting $f(x) = ax + b$ into the original equation, since $f(x)$ cannot be constant, we find $a = 1$ and $b = 0$, i.e., $f(x) = x$ for all $x \in R$.

Other commended solvers: **CHAN Wing Sum** (HKUST) and **William CHEUNG Pok Man** (S.T.F.A. Leung Kau Kui College, Form 6).

Problem 55. In the beginning, 65 beetles are placed at different squares of a 9×9 square board. In each move, every beetle creeps to a horizontal or vertical adjacent square. If no beetle makes either two horizontal moves or two vertical moves in succession, show that after some moves, there will be at least two beetles in the same square. (Source: 1995 Byelorussian Mathematical Olympiad (Final Round))

Solution: William CHEUNG Pok Man (S.T.F.A. Leung Kau Kui College, Form 6) and **YUNG Fai** (CUHK).

Assign an ordered pair (a,b) to each square with a, $b = 1, 2, \ldots, 9$. Divide the 81 squares into 3 types. Type A consists of squares with both a and b odd, type B consists of squares with both a and b even and type C consists of the remaining squares. The numbers of squares of the types A, B and C are 25, 16 and 40, respectively.

Assume no collision occurs. After two successive moves, beetles in type A squares will be in type B squares. So the number of beetles in type A squares are at most 16 at any time. Then there are at most 32 beetles in type A or type B squares at any time. Also, after one move, beetles in type C squares will go to type A or type B squares. So there are at most 32 beetles in type C squares at any time. Hence there are at most 64 beetles on the board, a contradiction.

Other commended solvers: **Sam YUEN Man Long** (STFA Leung Kau Kui College, Form 4.

Olympiad Corner

(*continued from page 1*)

where m_a, m_b, m_c are the lengths of the angle bisectors (internal to the triangle) and M_a, M_b, M_c are the lengths of the angle bisectors extended until they meet the circle. Prove that

$$\frac{l_a}{\sin^2 A} + \frac{l_b}{\sin^2 B} + \frac{l_c}{\sin^2 C} \geq 3,$$

and that equality holds iff ABC is equilateral.

Problem 4. Triangle $A_1A_2A_3$ has a right angle at A_3. A sequence of points is now defined by the following iterative process, where n is a positive integer. From A_n ($n \geq 3$), a perpendicular line is drawn to meet $A_{n-2}A_{n-1}$ at A_{n+1}.

(a) Prove that if this process were continued indefinitely, then one and only one point P is interior to every triangle $A_{n-2}A_{n-1}A_n$, $n \geq 3$.

(b) Let A_1 and A_3 be fixed points. By considering all possible locations of A_2 on the plane, find the locus of P.

Problem 5. Suppose that n persons A_1, A_2, …, A_n ($n \geq 3$) are seated in circle and that A_i has a_i objects such that

$$a_1 + a_2 + \cdots + a_n = nN$$

where N is a positive integer. In order that each person has the same number of objects, each person A_i is to give or to receive a certain number of objects to or from its two neighbours A_{i-1} and A_{i+1}, where A_{n+1} means A_1 and A_0 means A_n. How should this distribution be performed so that the total numbers of objects transferred is minimum?

由圓周率到四年一閏

(*continued from page 1*)

法，一百年後就會多了一日。因此在今天我們使用的曆法之中，年份能夠被4整除的，例如1996年，就定爲閏年，但如果年份能夠被100整除的話，例如1900年，就不是閏年了。

再算一算，就知道每400年就有96個閏年，416年就應有96+4=100個閏年。不過，這結果又不乎合 $\frac{101}{417}$ 這個條件！故此，曆法上又需要在每400年中增加一日，就好似2000年，因爲這數字能被400整除，這年又變回一年閏年了！

公元2000年快到了，大家渴望見一見這400年才有一次的閏年嗎？

老師不教的幾何（三）

(*continued from page 1*)

把上述推理逆轉過來，恰巧也成立，因此只有外接圓上的點能使踏板三角形退化爲辛姆生線。

踏板三角形的周期性可否推廣到n邊形呢？大家不妨先用四邊形來試試。B. M. Stewart在1940年證明了：n邊形的第n個踏板n邊形相似於原n邊形（刊於 American Mathematical Monthly 第七卷第462-466頁）。

台灣師範大學附屬中學初中二年級的孫君儀同學最近以踏板多邊形作爲研究課題，獲得1997年台灣科學展覽第三名。她借助 Geometer's Sketchpad發現了一些有趣性質並加以證明，大家不妨試試探討，甚至再推廣。這些性質包括：

- 對於凹n邊形和自交n邊形，第n個踏板n邊形是否和原n邊形相似？
- 踏板點在n邊形外部，類似性質是否存在？有甚麼條件會使踏板n邊形不存在？
- 第n個踏板n邊形和原n邊形的面積比是多少？
- 垂足改爲夾$x°$角時，類似性質是否存在？
- 踏板點在何處可使第三垂足三角形的面積最大？