Mathematical Excalibur

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Olympiad Corner

The 38th International Mathematical Olympiad, Mar del Plata, Argentina:

First day (July 24, 1997)
Each problem is worth 7 points.
Time Allowed: $4\frac{1}{2}$ hours.

Problem 1. In the plane the points with integer coordinates are the vertices of unit squares. The squares are coloured alternately black and white (as on a chessboard). For any pair of positive integers m and n, consider a right-angled triangle whose vertices have integer coordinates and whose legs, of lengths m and n, lie along edges of the squares. Let S_1 be the total area of the black part of the triangle and S_2 be the total area of the white part. Let

$$f(m,n) = |S_1 - S_2|.$$

(a) Calculate $f(m,n)$ for all positive integers m and n which are either both even or both odd.

(b) Prove that $f(m,n) \le \frac{1}{2}\max\{m,n\}$ for all m and n.

(c) Show that there is no constant C such that $f(m,n) < C$ for all m and n.

(*continued on page 4*)

Acknowledgment: Thanks to Catherine NG, EEE Dept, HKUST for general assistance.

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Error Correcting Codes (Part II)

Tsz-Mei Ko

In Part I, we introduced the family of Hamming codes. In particular, the (7,4) Hamming code encodes 4-bit messages $p_1p_2p_3p_4$ into 7-bit codewords $p_1p_2p_3p_4p_5p_6p_7$ by appending three parity bits

$$p_5 = p_1 + p_2 + p_4 \pmod 2,$$
$$p_6 = p_1 + p_3 + p_4 \pmod 2,$$
$$p_7 = p_2 + p_3 + p_4 \pmod 2,$$

to the original message. Figure 1 shows the 16 possible codewords for the (7,4) Hamming code. To convey the message 0100, as an example, the sender would send 0100101. If there is a transmission error in position 4 so that the received sequence becomes 0101101, the receiver would still be able to recover the error by decoding the received sequence as the closest codeword. (Note that 0100101 is different from 0101101 in only one position while all other codewords are different from 0101101 in more than one position.)

Now, if we group the first six bits of a (7,4) Hamming codeword into two-bit pairs (p_1p_2, p_3p_4, p_5p_6) and use an arithmetic system called a 4-element field (Figure 2), we observe something interesting: the three points (1, p_1p_2), (2, p_3p_4) and (3, p_5p_6) form a straight line! For example, the first 6 bits of the codeword 0100101 forms the ordered triple (01, 00, 10) = (1, 0, 2) and (1,1), (2,0), (3,2) are three consecutive points on the straight line $f(x) = 2x + 3$ since

$$f(1) = 2(1) + 3 = 2 + 3 = 1;$$
$$f(2) = 2(2) + 3 = 3 + 3 = 0;$$
$$f(3) = 2(3) + 3 = 1 + 3 = 2;$$

by using the addition and multiplication tables given in Figure 2. This fact is also true for the other 15 codewords and their corresponding straight lines $f(x)$ are listed in Figure 3.

This "straight line" property can be utilized for decoding. As an example, assume that the received sequence is 0101101. The first 6 bits form the ordered triple (01, 01, 10) = (1, 1, 2). We observe that a straight line passing through (1,1) and (2,1) should pass through (3,1). That is (1,1), (2,1) and (3,2) do not lie on a straight line and thus there is a transmission error. For the (7,4) Hamming code which is capable of correcting one error, we

message $p_1p_2p_3p_4$	codeword $p_1p_2p_3p_4p_5p_6p_7$
0000	0000000
0001	0001111
0010	0010011
0011	0011100
0100	0100101
0101	0101010
0110	0110110
0111	0111001
1000	1000110
1001	1001001
1010	1010101
1011	1011010
1100	1100011
1101	1101100
1110	1110000
1111	1111111

Figure 1. The (7,4) Hamming Code.

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

×	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	1
3	0	3	1	2

Figure 2. Arithmetic Tables for a 4-Element Field.

codeword	p_1p_2	p_3p_4	p_5p_6	$f(x)$
0000000	0	0	0	0
0001111	0	1	3	$2x+2$
0010011	0	2	1	$3x+3$
0011100	0	3	2	$x+1$
0100101	1	0	2	$2x+3$
0101010	1	1	1	1
0110110	1	2	3	x
0111001	1	3	0	$3x+2$
1000110	2	0	3	$3x+1$
1001001	2	1	0	$x+3$
1010101	2	2	2	2
1011010	2	3	1	$2x$
1100011	3	0	1	$x+2$
1101100	3	1	2	$3x$
1110000	3	2	0	$2x+1$
1111111	3	3	3	3

Figure 3. The (7,4) Hamming Codewords form Straight Lines $f(x)$.

assume that only one of the three points is incorrect. That is, the original "straight line" $f(x)$ should pass through (1,1) and (2,1); (1,1) and (3,2); or (2,1) and (3,2) corresponding to $f(x) = 1$; $f(x) = 2x + 3$; or $f(x) = 3x$ respectively. Then the first 6 bits for the original codeword should be 010101, 010010 or 110110. Among these three possible solutions, only 010010 satisfies the equation for the last parity bit $p_7 = p_2 + p_3 + p_4 \pmod 2$. Thus we decode the received sequence 0101101 as 0100101 corresponding to the message 0100.

The above decoding procedure seems to be quite complicated. However, it can be generalized to construct (and decode) multiple-error correcting codes by using "polynomials" instead of "straight lines". Suppose we would like to transmit a message that contains k symbols $s_1 s_2 \cdots s_k$. We may use these k symbols to form a kth degree polynomial $f(x)$ such that $f(i) = s_i$ $(1 \le i \le k)$. To construct a code that can correct t errors, we may append $2t$ symbols $f(k+1)$, $f(k+2)$, ..., $f(k+2t)$ to the original message so that the encoded sequence contains $k + 2t$ symbols corresponding to $k + 2t$ consecutive points on a kth degree polynomial (Figure 4). If there are less than or equal to t errors during transmission, at least $k + t$ symbols would be received correctly. Then the receiver may simply check which $k + t$ symbols lie on a kth degree polynomial to decode the received sequence.

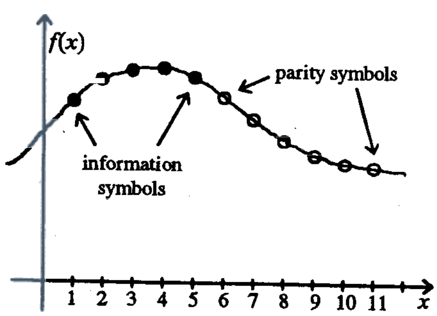

Figure 4. A Polynomial Code.

We use a (21,9) double error correcting code to illustrate the idea. Assume we would like to send a 9 bit message, say 101010100. We may first group the information bits into 3-bit symbols as (101, 010, 100) = (5, 2, 4). (In general, we may group the information bits into m-bit symbols where m cannot be too small. Otherwise, we cannot construct the polynomial $f(x)$. Why? Also m should not be too large to reduce the number of parity bits.) Then we use the three message symbols (5, 2, 4) to form a second degree polynomial $f(x)$ such that $f(1) = 5, f(2) = 2$ and $f(3) = 4$. That is

$$f(x) = \frac{5(x-2)(x-3)}{(1-2)(1-3)} + \frac{2(x-1)(x-3)}{(2-1)(2-3)} + \frac{4(x-1)(x-2)}{(3-1)(3-2)}.$$

Note that we have 8 kinds of symbols (since we group the bits into 3-bit symbols) and thus we need an 8-element field for our arithmetic. (Basically, a field is an arithmetic system that allows us to add, subtract, multiply and divide.) By using the 8-element field given in Figure 5, we can simplify $f(x)$ to obtain

$$f(x) = x^2 + 7x + 5.$$

Note that $f(1) = 5$, $f(2) = 2$ and $f(3) = 4$ as desired.

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	1	2	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

×	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	3	1	7	5
3	0	3	6	5	7	4	1	2
4	0	4	3	7	6	2	5	1
5	0	5	1	4	2	7	3	6
6	0	6	7	1	5	3	2	4
7	0	7	5	2	1	6	4	3

Figure 5. Arithmetic Tables for an 8-Element Field.

Now suppose we would like to construct a code that can correct two errors. We can append

$$f(4) = 4^2 + 7(4) + 3 = 6 + 1 + 3 = 4;$$
$$f(5) = 5^2 + 7(5) + 3 = 7 + 6 + 3 = 2;$$
$$f(6) = 6^2 + 7(6) + 3 = 2 + 4 + 3 = 5;$$
$$f(7) = 7^2 + 7(7) + 3 = 3 + 3 + 3 = 3;$$

to the message symbols. That is, we would transmit a 21 bit sequence (5,2,4,4,2,5,3) = 101010100100010101011. If there are transmission errors, say at positions 5 and 15, the received sequence becomes 101000100100011101011 = (5,0,4,4,3,5,3). (This code is actually capable of correcting two symbol errors instead of two bit errors.) Then the receiver would search for the 5 received symbols that are not corrupted. Among the $\binom{7}{5} = 21$ cases, only $f(1) = 5$, $f(3) = 4$, $f(4) = 4$, $f(6) = 5$, $f(7) = 3$, form a second degree polynomial. So the receiver uses these five points to reconstruct $f(x) = x^2 + 7x + 5$ and decode the received message as $(f(1), f(2), f(3)) = (5, 2, 4) = 101010100$.

The above idea, using polynomials to construct codes, was first proposed by Reed and Solomon in 1960. It is now widely used in electronics and communication systems including our compact discs.

38th IMO

Kin-Yin Li

For the first time in history, the International Mathematical Olympiad (IMO) was held in the southern hemisphere. Teams representing a record 82 countries and regions participated in the event at Mar del Plata, Argentina this year from July 18 to 31. The site was at a resort area bordered by the beautiful Atlantic Ocean. All through the period, the weather was nice and cool.

The Hong Kong team, like many southeast Asia teams, had to overcome thirty plus hours of flight time to arrive Argentina. With two short days of rest, the team members wrote the exams with jet lag. This year the team consisted of

Chan Chung Lam (*Bishop Hall Jubilee School*)
Cheung Pok Man (*STFA Leung Kau Kui College*)
Lau Lap Ming (*St. Paul's College*)
Leung Wing Chung (*Queen Elizabeth School*)
Mok Tze Tao (*Queen's College*)
Yu Ka Chun (*Queen's College*)

brought home 5 bronze medals and came in one mark behind Canada and one mark ahead of France. The top team was China with 6 gold, followed by Hungary, Iran, USA and Russia. As usual, problem 6 was the most difficult with 73% of the contestants getting zero, 90% getting less than half of the score for the problem.

The excursions were good. The hospitality was superb!!! The team members had a wild time playing the indoor games the day before the closing ceremony. One member of the team even admitted it was the best he has participated in three years. There were many fond memories.

There was a surprise ending on the way back. Due to the typhoon weather in Hong Kong, the team was stranded in Los Angeles for a day. Yes, the team took full advantage to tour the city, Hollywood, Beverly Hills, Rodeo Drive, in particular. The next day the team was stranded again in Taipei. It was unbelievably fortunate to have a chance to see these cities. What a bonus for a year's hard work!

Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, address, school affiliation and grade level. Please send submissions to *Dr. Kin-Yin Li, Dept of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is September 30, 1997.

Problem 61. Find the smallest positive integer which can be written as the sum of nine, the sum of ten and the sum of eleven consecutive positive integers.

Problem 62. Let $ABCD$ be a cyclic quadrilateral and let P and Q be points on the sides AB and AD respectively such that $AP = CD$ and $AQ = BC$. Let M be the point of intersection of AC and PQ. Show that M is the midpoint of PQ. (*Source*: 1996 Australian Mathematical Olympiad.)

Problem 63. Show that for $n \ge 2$, there is a permutation $a_1, a_2, \ldots, a_n$ of $1, 2, \ldots, n$ such that $|a_k - k| = |a_1 - 1| \ne 0$ for $k = 2, 3, \ldots, n$ if and only if n is even.

Problem 64. Show that it is impossible to place 1995 different positive integers along a circle so that for every two adjacent numbers, the ratio of the larger to the smaller one is a prime number.

Problem 65. All sides and diagonals of a regular 12-gon are painted in 12 colors (each segment is painted in one color). Is it possible that for any three colors there exist three vertices which are joined with each other by segments of these colors?

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Solutions

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Problem 56. Find all prime numbers p such that $2^p + p^2$ is also prime.

Solution: CHAN Lung Chak (St. Paul's Co-ed. College, Form 4), **CHAN Wing Sum** (HKUST), **LAW Ka Ho** (Queen Elizabeth School, Form 4), **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 4), **POON Man Wai** (St. Paul's College, Form 4), **TAM Siu Lung** (Queen Elizabeth School, Form 4), **WONG Chun Wai** (SKH Kei Hau Secondary School, Form 4), **Alan WONG Tak Wai** (University of Waterloo, Canada), **WONG Sui Kam** (Queen Elizabeth School, Form 4) and **Sam YUEN Man Long** (STFA Leung Kau Kui College, Form 4).

For $p = 2$, $2^p + p^2 = 8$ is not prime. For $p = 3$, $2^p + p^2 = 17$ is prime. For prime $p = 3n \pm 1 > 3$, we see that

$$2^p + p^2 = (3-1)^p + (3n \pm 1)^2$$

is divisible by 3 (after expansion) and is greater than 3. So $p = 3$ is the only such prime.

Problem 57. Prove that for real numbers $x, y, z > 0$,

$$\frac{x^2}{x+y} + \frac{y^2}{y+z} + \frac{z^2}{z+x} \ge \frac{x+y+z}{2}.$$

Solution 1: Note that

$$\begin{aligned} 4x^2 &= ((x+y)+(x-y))^2 \\ &= (x+y)^2 + 2(x+y)(x-y) + (x-y)^2 \\ &\ge (x+y)^2 + 2(x+y)(x-y). \end{aligned}$$

Dividing both sides by $4(x+y)$, we obtain

$$\frac{x^2}{x+y} \ge \frac{x+y}{4} + \frac{x-y}{2}.$$

In place of x, y, similar inequalities for y, z and z, x can be obtained. Adding these inequalities give the desired inequality.

Solution 2: Venus CHU Choi Yam (St. Paul's Co-ed. College, Form 6), **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 4), **POON Man Wai** (St. Paul's College, Form 4), **Alan WONG Tak Wai** (University of Waterloo, Canada).

The Cauchy-Schwarz inequality asserts that

$$\left(a_1^2 + a_2^2 + \cdots + a_k^2\right)\left(b_1^2 + b_2^2 + \cdots + b_k^2\right) \ge \left(a_1 b_1 + a_2 b_2 + \cdots + a_k b_k\right)^2$$

with equality if and only if $a_i b_j = a_j b_i$ for all i, j such that $1 \le i < j \le k$. Taking $k = 3$,

$$a_1 = \sqrt{x+y},\ a_2 = \sqrt{y+z},\ a_3 = \sqrt{z+x},$$

$$b_1 = \frac{x}{\sqrt{x+y}},\ b_2 = \frac{y}{\sqrt{y+z}},\ b_3 = \frac{z}{\sqrt{z+x}},$$

then dividing both sides by $2(x + y + z)$, we get the desired inequality.

Other commended solvers: **CHAN Wing Sum** (HKUST), **Alex CHUENG King Chung** (Po Leung Kuk 1983 Board of Director's College, Form 6), **Yves CHEUNG Yui Ho** (STFA Leung Kau Kui College, Form 5), **TAM Siu Lung** (Queen Elizabeth School, Form 4), and **Sam YUEN Man Long** (STFA Leung Kau Kui College, Form 4).

Problem 58. Let ABC be an acute-angled triangle with $BC > CA$. Let O be its circumcenter, H its orthocenter, and F the foot of its altitude CH. Let the perpendicular to OF at F meet the side CA at P. Prove that $\angle FHP = \angle BAC$. (*Source*: unused problem in the 1996 IMO.)

Solution: Official Solution.

Let Y be the midpoint of AC. Since $\angle OFP = \angle OYP = 90°$, points F, P, Y, O lie on a circle Γ_1 with center at the midpoint Q of OP. Now the nine point circle Γ_2 of ΔABC also passes through F and Y and has center at the midpoint N of OH. So FY is perpendicular to NQ. Since NQ is parallel to HP by the midpoint theorem, FY is perpendicular to HP. Then $\angle FHP = 90° - \angle YFH = 90° - \angle YCH = \angle BAC$.

Problem 59. Let n be a positive integer greater than 2. Find all real number solutions $(x_1, x_2, \cdots, x_n)$ to the equation

$$(1-x_1)^2 + (x_1-x_2)^2 + \cdots + (x_{n-1}-x_n)^2 + x_n^2 = \frac{1}{n+1}$$

(Source: 1975 British Mathematical Olympiad)

Solution 1: Official Solution.

Let
$$\begin{aligned} 1 - x_1 &= \frac{1}{n+1} + z_1, \\ x_1 - x_2 &= \frac{1}{n+1} + z_2, \ \ldots \\ x_{n-1} - x_n &= \frac{1}{n+1} + z_n, \\ x_n &= \frac{1}{n+1} + z_{n+1}. \end{aligned}$$

Adding the above $n + 1$ equations, we get

$$z_1 + z_2 + \cdots + z_{n+1} = 0.$$

(continued on page 4)

Problem Corner

(*continued from page 3*)

In terms of z_i, the given equation can then be simplified to

$$z_1^2 + z_2^2 + \cdots + z_{n+1}^2 = 0.$$

So all $z_i = 0$, which implies

$$x_i = \frac{n+1-i}{n+1} \text{ for } i = 1, 2, \ldots, n.$$

Solution 2: Venus CHU Choi Yam (St. Paul's Co-ed. College, Form 6), **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 4) and **POON Man Wai** (St. Paul's College, Form 4).

We use the Cauchy-Schwarz inequality as stated in Problem 57 Solution 2. Taking $k = n + 1$,

$$a_1 = 1 - x_1,\ a_2 = x_1 - x_2,\ \ldots,\ a_n = x_{n-1} - x_n,\ a_{n+1} = x_n,$$

$$b_1 = b_2 = \cdots = b_{n+1} = 1,$$

we see that we have equality. So $a_1 = a_2 = \cdots = a_{n+1}$ yielding the unique solution

$$x_i = \frac{n+1-i}{n+1} \text{ for } i = 1, 2, \ldots, n.$$

Problem 60. Find (without calculus) a fifth degree polynomial $p(x)$ such that $p(x) + 1$ is divisible by $(x - 1)^3$ and $p(x) - 1$ is divisible by $(x + 1)^3$.

Solution: LAW Ka Ho (Queen Elizabeth School, Form 4), **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 4), **POON Man Wai** (St. Paul's College, Form 4) and **TAM Siu Lung** (Queen Elizabeth School, Form 4).

Note that $(x - 1)^3$ divides $p(x) + 1$ and $p(-x) - 1$; so $(x - 1)^3$ divides their sum $p(x) + p(-x)$. Also $(x + 1)^3$ divides $p(x) - 1$ and $p(-x) + 1$; so $(x + 1)^3$ divides $p(x) + p(-x)$. Then $(x - 1)^3(x + 1)^3$ divides $p(x) + p(-x)$, which is of degree at most 5. So $p(x) + p(-x) = 0$ for all x. Then the even degree term coefficients of $p(x)$ are zero. Now

$$p(x) + 1 = (x - 1)^3(Ax^2 + Bx - 1).$$

Comparing the degree 2 and 4 coefficients, we get $3 + 3B - A = 0$ and $B - 3A = 0$, which implies $A = -3/8$ and $B = -9/8$. This yields

$$p(x) = -\frac{3}{8}x^5 + \frac{5}{4}x^3 - \frac{15}{8}x.$$

Other commended solvers: **CHAN Wing Sum** (HKUST), **OR Kin** (SKH Bishop Mok Sau Tseng Secondary School, Form 3), **SIN Ka Fai** (STFA Leung Kau Kui College, Form 4) and **Sam YUEN Man Long** (STFA Leung Kau Kui College, Form 4).

Olympiad Corner

(*continued from page 1*)

Problem 2. Angle A is the smallest in the triangle ABC. The points B and C divide the circumcircle of the triangle into two arcs. Let U be an interior point of the arc between B and C which does not contain A. The perpendicular bisectors of AB and AC meet the line AU at V and W, respectively. The lines BV and CW meet at T. Show that

$$AU = TB + TC.$$

Problem 3. Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying the conditions:

$$|x_1 + x_2 + \cdots + x_n| = 1$$

and $$|x_i| \le \frac{n+1}{2} \quad \text{for } i = 1, 2, \ldots, n.$$

Show that there exists a permutation $y_1, y_2, \ldots, y_n$ of $x_1, x_2, \ldots, x_n$ such that

$$|y_1 + 2y_2 + \cdots + ny_n| \le \frac{n+1}{2}.$$

Second day (July 25, 1997)
Each problem is worth 7 points.
Time Allowed: $4\frac{1}{2}$ hours.

Problem 4. An $n \times n$ matrix (square array) whose entries come from the set $S = \{1, 2, \ldots, 2n - 1\}$ is called a *silver* matrix if, for each $i = 1, \ldots, n$, the ith row and the ith column together contain all elements of S. Show that

(a) there is no silver matrix for $n = 1997$;

(b) silver matrices exist for infinitely many values of n.

Problem 5. Find all pairs (a,b) of integers $a \ge 1$, $b \ge 1$ that satisfy the equation

$$a^{b^2} = b^a$$

Problem 6. For each positive integer n, let $f(n)$ denote the number of ways of representing n as a sum of powers of 2 with nonnegative integer exponents. Representations which differ only in the ordering of their summands are considered to be the same. For instance, $f(4) = 4$ because the number 4 can be represented in the following four ways:

$$4;\ 2 + 2;\ 2 + 1 + 1;\ 1 + 1 + 1 + 1.$$

Prove that, for any integer $n \ge 3$,

$$2^{n^2/4} < f(2^n) < 2^{n^2/2}.$$

Above: A photo of the Hong Kong Team taken in front of the IMO97 score board. From left to right are: LEUNG Wing Chung, CHEUNG Pok Man, YU Ka Chun, LAU Lap Ming, CHAN Chung Lam, MOK Tze Tao, LUK Mee Lin (La Salle College, Deputy Leader), LI Kin Yin (HKUST Math Dept, Team Leader).