Mathematical Excalibur

Volume 4, Number 1 December, 1997 - February, 1998

Olympiad Corner

International Mathematics Tournament of the Towns, Spring 1997:

Junior A-Level Paper

Problem 1. One side of a triangle is equal to one third of the sum of the other two. Prove that the angle opposite the first side is the smallest angle of the triangle. (3 points)

Problem 2. You are given 25 pieces of cheese of different weights. Is it always possible to cut one of the pieces in two parts and put the 26 pieces in two packets so that

(i) each packet contains 13 pieces;

(ii) the total weights of the two packets are equal;

(iii) the two parts of the piece which has been cut are in different packets?

(5 points)

Problem 3. In a chess tournament, each of $2n$ players plays every other player once in each of two rounds. A win is worth 1 point and a draw is worth ½point. Prove that if for every player, the total score in the first round differs from that in the second round by at least n points, then the difference is exactly n points for every player. (5 points)

(*continued on page 4*)

Acknowledgment: Thanks to Catherine NG, EEE Dept, HKUST for general assistance.

On-line: http: //www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is April 15, 1998.

For individual subscription for the three remaining issues for the 97-98 academic year, send us three stamped self-addressed envelopes. Send all correspondence to:

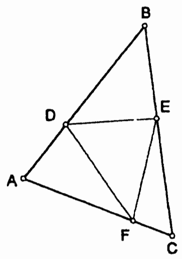
Dr. Kin-Yin Li
Department of Mathematics
Hong Kong University of Science and Technology
Clear Water Bay, Kowloon, Hong Kong

Fax: 2358-1643
Email: makyli@uxmail.ust.hk

老師不教的幾何（五）

張百康

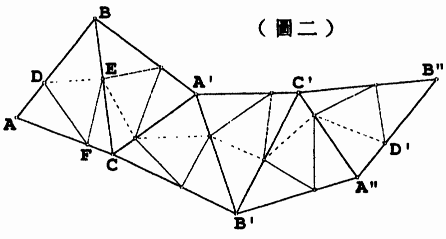
圖一顯示了一個銳角三角形ABC和它的一個內切三角形DEF。



（圖一）

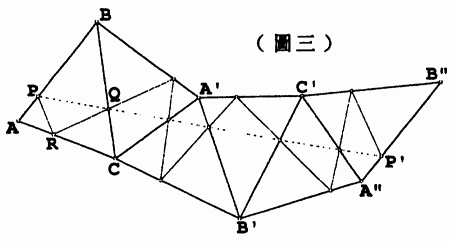
大家想一想：隨意在ΔABC上作內切三角形，哪一個的周界最短？這問題早在十八世紀時，由數學家Fagnano最先提出，並且用微分方法，經過繁複的運算和簡化，求得一個最短周長的內切三角形。因此這問題又名Fagnano問題。

經過整整一個世紀，才有另一位數學家Schwarz找到一個漂亮的初等幾何解法：如圖二所示，Schwarz將ΔABC以它的邊輪流作鏡面反射，得到六個相連的全等三角形。



（圖二）

圖二的虛折線$DE\cdots D'$全長剛好是ΔDEF周長的兩倍。將D、E、F沿ΔABC的三邊移動，是否可以找到另一個內切三角形PQR，使虛折線成一直線PP'（圖三）？



（圖三）

如果我們用一線段連D和D'，這線段肯定較折線$DE\cdots D'$短。但PP'和DD'的長度又如何比較呢？破綻正正在於爲甚麼Schwarz要作不多不少的五次鏡面反射。大家細心看一看圖二的AB和$A''B''$，它們不但等長，而且好像互相平行呢。Schwarz巧妙地用旋轉的觀念來證明AB平行$A''B''$：

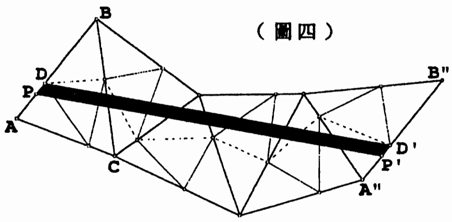
AB繞點B旋轉$2\angle B$得$A'B$，再繞點A'旋轉$2\angle A$得$A'B'$；$A'B'$繞點B'旋轉$-2\angle B$得$A''B$，再繞點A''旋轉$-2\angle A$得$A''B''$。因此AB和$A''B''$的夾角是

$$2\angle B+2\angle A-2\angle B-2\angle A=0,$$

也就是說，AB平行$A''B''$。

由於$AD=A''D'$和$AP=A''P'$，因此$PDD'P'$是平行四邊形（圖四）。換言之，ΔPQR的周長$(=\frac{1}{2}PP')$是所有ΔABC的內切三角形中最短的。

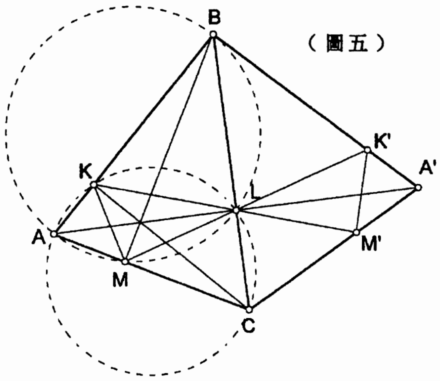
（圖四）



這ΔPQR究竟有甚麼特性？大家不妨再看一遍圖三，不難發現ΔPQR好像是ΔABC的垂足三角形(orthic triangle)。各同學可利用圖五證明圖中的垂足三角形KLM的角$\angle KLM$被高AL平分，關鍵在於圖中的一些四點共圓特性，留待各同學自行理解。

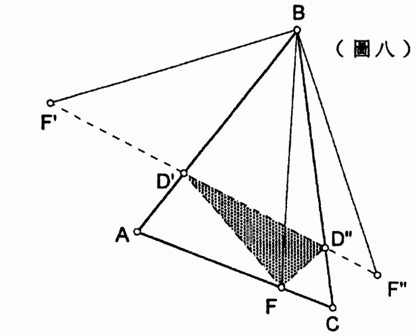
由於$\Delta A'BC$是ΔABC以BC爲鏡面的反射影象，所以高AL和$A'L$成一直線，並且$A'L$也平分角$K'LM$。由此推知，KL和LM也成一直線。餘此類推，如按圖二

連續進行鏡面反射，則ΔKLM就是我們要找尋的最短周長內切三角形。



（圖五）

接著，我們改變F的位置，找尋上述這種$\Delta FD'D''$中周長最短者，便是我們要找的最短周長內切三角形。



（圖八）

利用鏡面反射的對稱性質可知：圖八中的$BF' = BF = BF''$，$\angle F'BD' = \angle FBD'$及$\angle FBD'' = \angle F''BD''$，因此

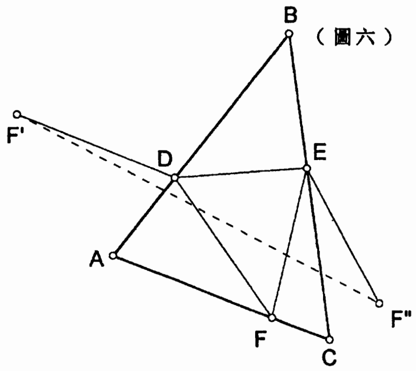
$$F'F'' = 2BF \sin B。$$

其中只有BF可改變，而BF長度的最小值是當它是ΔABC的高，即F是垂足。同理可知另外兩頂點D和E也必定是垂足方可使ΔDEF成為周長最短的內切三角形。

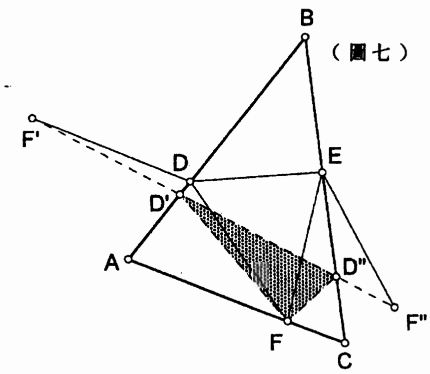
青出於藍勝於藍，Schwarz 看過學生 Fejér 的證明後，也讚賞不已。

Schwarz 的證明固然巧妙，但好戲還在後頭。在1900年，當時還在柏林唸書的匈牙利數學家 Fejér 找到一個比 Schwarz 的證明還精簡的證法：

分別以ΔABC的邊BA和BC作鏡面，找到F的影像F'和F''（圖六）。由鏡面反射的性質可知：$FD = F'D$及$FE = F''E$，因此折線$F'DEF''$全長等於內切三角形DEF的周長。明顯地，祇要F點不變，不管其餘兩點D和E在AB和BC上如何移動，所得的內切三角形周長肯定大於直線$F'F''$的長度。設$F'F''$與AB及BC分別交於點D'和D''（圖七），則$\Delta FD'D''$的周長是所有一頂點在F的內切三角形中最短的。



（圖六）



（圖七）

A Proof for The Lambek and Moser Theorem

Two sequences $f(n)$ and $f^*(n)$ are called inverse sequences if

$$f^*(n) = k, \text{ where } f(k) < n \le f(k+1).$$

Two sequences $F(n)$ and $G(n)$ are called complementary sequences if $F(n)$ and $G(n)$ together contain each natural number exactly once. (c.f. vol. 3 no. 4)

Theorem: $f(n)$ and $f^*(n)$ are inverse sequences if and only if $F(n) = f(n) + n$ and $G(n) = f^*(n) + n$ are complementary sequences (with the minor conditions that (i) $f(n)$ and $f^*(n)$ are non-decreasing sequences of non-negative integers; (ii) $F(n)$ and $G(n)$ are strictly increasing sequences of positive integers.)

Proof: We will first prove the converse. Let $F(n)$ and $G(n)$ be strictly increasing sequences of positive integers such that F and G are complementary. For example,

$$F(n) = \overbrace{1,2,3,\quad 6,\ \ 8,}^{r}\quad 10,\ 11,\ \cdots$$
$$G(n) = \qquad\underbrace{4,5,\ \ 7,\ \ 9,}_{s}\qquad\quad 12,\ \cdots$$

(Note the inserted spaces in the above illustration so that the natural numbers are in increasing order from left to right in relative position.) Let N be a natural number. Let r and s be the number of terms in $F(n)$ and $G(n)$ that are $\le N$ respectively. (In the above illustration, $N = 9$, $r = 5$ and $s = 4$.) Note that $r + s = N$.

Now consider $f(n) = F(n) - n$ and $f^*(n) = G(n) - n$.

$$f(n) = \overbrace{0,0,0,\quad 2,\ 3,}^{r}\quad 4,\ 4,\ \cdots$$
$$f^*(n) = \qquad\underbrace{3,3,\ \ 4,\ \ 5,}_{s}\qquad 7,\ \cdots$$

We observe that

$$f^*(s) = G(s) - s = N - s = r.$$

That is,

$f^*(s) =$ the number of terms in f appear on the left hand side (in position) of the term $f^*(s)$.

Likewise,

$f(r) =$ the number of terms in f^* appear on the left hand side (in position) of the term $f(r)$.

Since the term $f(r)$ appear on the left hand side of $f^*(s)$, $f(r) < s$. We may similarly show that $f(r+1) \ge s$ and thus

$$f(r) < s \le f(r+1).$$

That is, $f^*(n)$ is the frequency distribution of $f(n)$ and thus $f(n)$ and $f^*(n)$ are inverse sequences. The fact that $f(r) < s \le f(r+1)$ can also be proved formally as follows.

$$f(r) = F(r) - r < N - r = s;$$
$$f(r+1) = F(r+1) - (r+1) > N - (r+1) = s-1,$$
$$f(r+1) \ge s.$$

We will now show that if $f(n)$ is a non-decreasing sequence of non-negative integers and $f^*(n)$ is the frequency distribution function of $f(n)$, then $F(n) = f(n) + n$ and $G(n) = g(n) + n$ are complementary. Given the sequence $f(n)$, we can first construct the sequence $F(n) = f(n) + n$. Let $H(n)$ be the complementary sequence of $F(n)$ and let $h(n) = H(n) - n$. From the converse proof, $h(n)$ must be the frequency distribution of $f(n)$. Since the frequency distribution of a given function is unique, $h(n) = f^*(n)$ and thus

$$G(n) = f^*(n) + n = h(n) + n = H(n)$$

is the complementary sequence of $F(n)$.

Q.E.D.

Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, address, school affiliation and grade level. Please send submissions to *Dr. Kin-Yin Li, Dept of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is April 15, 1998.

Problem 71. Find all real solutions of the system

$$x+\log\left(x+\sqrt{x^2+1}\right)=y,$$
$$y+\log\left(y+\sqrt{y^2+1}\right)=z,$$
$$z+\log\left(z+\sqrt{z^2+1}\right)=x.$$

(*Source:* 1995 Israel Math Olympiad.)

Problem 72. Is it possible to write the numbers 1, 2, … , 121 in an 11×11 table so that any two consecutive numbers be written in cells with a common side and all perfect squares lie in a single column? (*Source:* 1995 Russian Math Olympiad.)

Problem 73. Prove that if a and b are rational numbers satisfying the equation $a^5+b^5=2a^2b^2$, then $1-ab$ is the square of a rational number. (*Source:* 26th British Math Olympiad.)

Problem 74. Points A_2, B_2, C_2 are the midpoints of the altitudes AA_1, BB_1, CC_1 of acute triangle ABC, respectively. Find the sum of $\angle B_2A_1C_2$, $\angle C_2B_1A_2$, $\angle A_2C_1B_2$. (*Source:* 1995 Russian Math Olympiad.)

Problem 75. Let $P(x)$ be any polynomial with integer coefficients such that $P(21) = 17$, $P(32) = -247$, $P(37) = 33$. Prove that if $P(N) = N + 51$, for some integer N, then $N = 26$. (*Source:* 23rd British Math Olympiad.)

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Solutions

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Problem 66.

(a) Find the first positive integer whose square ends in three 4′s.

(b) Find all positive integers whose squares end in three 4′s.

(c) Show that no perfect square ends with four 4′s.

(*Source:* 1995 British Mathematical Olympiad.)

Solution: Andy CHAN Kin Hang (Bishop Hall Jubilee School, Form 4) and **SHUM Ho Keung** (PLK No. 1 W. H. Cheung College, Form 5).

(a) Since $21^2 < 444 < 22^2$ and $1444 = 38^2$, the first such positive integer is 38.

(b) Assume n is such an integer. Then

$$n^2 - 1444 = (n-38)(n+38)$$

is divisible by $1000 = 2^3 5^3$. This implies at least one of $n-38$, $n+38$ is divisible by 4. Since their difference is 76, hence both must be divisible by 4. Since 76 is not divisible by 5, hence one of $n-38$, $n+38$ is divisible by $4 \cdot 5^3 = 500$. Then $n = 500k \pm 38$ for some nonnegative integer k. Conversely, for such n,

$$n^2 = 1000(250k^2 \pm 38k) + 1444$$

always ends in three 4's.

(c) Since $250k^2 \pm 38k$ is even, no perfect square ends with four 4's.

Other commended solvers: **KWOK Chi Hang** (Valtorta College, Form 6), **LAI Chi Fung, Brian** (Queen Elizabeth School, Form 5), **LAW Ka Ho** (Queen Elizabeth School, Form 5), **LI Fung** (HK Taoist Association Ching Chung Secondary School, Form 7), **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 5) and **WONG Shu Fai** (Valtorta College, Form 6).

Problem 67. Let Z and R denote the integers and real numbers, respectively. Find all functions $f: Z \to R$ such that

$$f\left(\frac{x+y}{3}\right)=\frac{f(x)+f(y)}{2}$$

for all integers x, y such that $x + y$ is divisible by 3. (*Source:* a modified problem from the 1995 Iranian Mathematical Olympiad.)

Solution: CHAN Wing Sum (City U) and **TSANG Sai Wing** (Valtorta College, Form 7).

For all integer n,

$$f(0)+f(3n)=2f(n)=f(n)+f(2n).$$

This implies

$$f(n)=f(2n)=\frac{f(3n)+f(3n)}{2}=f(3n).$$

So $f(n) = f(0)$ for all integer n. It is also clear that all constant functions are solutions.

Other commended solvers: **Andy CHAN Kin Hang** (Bishop Hall Jubilee School, Form 4), **CHING Wai Hung** (STFA Leung Kau Kui College, Form 6), **LAW Ka Ho** (Queen Elizabeth School, Form 5), **LI Fung** (HK Taoist Association Ching Chung Secondary School, Form 7), **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 5) and **WONG Hau Lun** (STFA Leung Kau Kui College, Form 6).

Problem 68. If the equation

$$ax^2 + (c-b)x + (e-d) = 0$$

has real roots greater than 1, show that the equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

has at least one real root. (*Source*: 1995 Greek Mathematical Olympiad.)

Solution: CHAN Wing Chiu (La Salle College, Form 5).

Suppose

$$p(x) = ax^4 + bx^3 + cx^2 + dx + e$$

has no real root. Let $y > 1$ be a root of $ay^2 + (c-b)y + (e-d) = 0$ and $z=\sqrt{y}$. Since

$$p(x) = ax^4+(c-b)x^2+(e-d)+(x-1)(bx^2+d),$$

we get

$$p(z) = (z-1)(bz^2+d)$$

and

$$p(-z) = (-z-1)(bz^2+d).$$

Now $z > 1$ implies one of $p(z)$, $p(-z)$ is positive, while the other is negative. Therefore, $p(x)$ has a root between z and $-z$, a contradiction.

Problem 69. *ABCD* is a quadrilateral such that $AB = AD$ and $\angle B = \angle D = 90°$. Points F and E are chosen on BC and CD, respectively, so that $DF \perp AE$. Prove that $AF \perp BE$. (*Source:* 1995 Russian Mathematical Olympiad.)

Solution 1: WONG Hau Lun (STFA Leung Kau Kui College, Form 6).

Let E' be the mirror image of E with

(*continued on page 4*)

Problem Corner

(*continued from page 3*)

respect to AC. Let X be the intersection of DF and AE. Let Y be the intersection of AF and BE. Since $\angle ADE = 90° = \angle AXD$, we have $\angle ADF = \angle DEA = \angle BE'A = 180° - \angle AE'F$. So A, D, F, E' are concyclic. Then $\angle AFD = \angle AE'D = \angle AEB$. So X, E, F, Y are concyclic. Therefore $\angle EYF = \angle EXF = 90°$.

Solution 2: CHING Wai Hung (STFA Leung Kau Kui College, Form 6).

Since $DF \perp AE$ and $DA \perp DE$, so

$$\begin{aligned} 0 &= \overrightarrow{DF} \cdot \overrightarrow{AE} \\ &= (\overrightarrow{DA} + \overrightarrow{AF}) \cdot \overrightarrow{AE} \\ &= \overrightarrow{DA} \cdot (\overrightarrow{AD} + \overrightarrow{DE}) + \overrightarrow{AF} \cdot \overrightarrow{AE} \end{aligned}$$

which simplifies to

$$\overrightarrow{AF} \cdot \overrightarrow{AE} = |\overrightarrow{AD}|^2 .$$

Since $BF \perp BA$, so

$$\begin{aligned} \overrightarrow{AF} \cdot \overrightarrow{BE} &= \overrightarrow{AF} \cdot (\overrightarrow{BA} + \overrightarrow{AE}) \\ &= (\overrightarrow{AB} + \overrightarrow{BF}) \cdot \overrightarrow{BA} + \overrightarrow{AF} \cdot \overrightarrow{AE} \\ &= -\left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{AD}\right|^2 \\ &= 0 \end{aligned}$$

which implies $AF \perp BE$.

Other commended solver: **TSANG Kam Wing** (Valtorta College, Form 5).

Problem 70. Lines $l_1, l_2, \cdots, l_k$ are on a plane such that no two are parallel and no three are concurrent. Show that we can label the C_2^k intersection points of these lines by the numbers $1, 2, \cdots, k-1$ so that in each of the lines $l_1, l_2, \cdots, l_k$ the numbers $1, 2, \cdots, k-1$ appear exactly once if and only if k is even. (*Source:* a modified problem from the 1995 Greek Mathematical Olympiad.)

Solution: Gary NG Ka Wing (STFA Leung Kau Kui College, Form 5).

If such labeling exists for an integer k, then the label 1 must occur once on each line and each point labeled 1 lies on exactly 2 lines. Hence there are $k/2$ 1's, i.e. k is even.

Conversely, if k is even, then the following labeling works: for $1 \le i < j \le k - 1$, give the intersection of lines l_i and l_j the label $i + j - 1$ when $i + j \le k$, the label $i + j - k$ when $i + j > k$. For the intersection of lines l_k and l_i ($i = 1, 2, \ldots, k - 1$), give the label $2i - 1$ when $2i \le k$ the label $2i - k$ when $2i > k$.

Comments: The official solution made use of the special symmetry of an odd number sided regular polygon to construct the labeling as follow: for k even, consider the $k - 1$ sided regular polygon with the vertices labeled $1, 2, \ldots, k - 1$. For $1 \le i < j \le k - 1$, the perpendicular bisector of the segment joining vertices i and j passes through a unique vertex, give the intersection of lines l_i and l_j the label of that vertex. For the intersection of lines l_k and l_i ($i = 1, 2, \ldots, k - 1$), give the label i.

Other commended solver: **LAW Ka Ho** (Queen Elizabeth School, Form 5).

Olympiad Corner

(*continued from page 1*)

Problem 4. $AC'BA'CB'$ is a convex hexagon such that $AB' = AC'$, $BC' = BA'$ and $CA' = CB'$. Moreover, $\angle A + \angle B + \angle C = \angle A' + \angle B' + \angle C'$. Prove that the area of triangle ABC is half of the area of the hexagon. (6 points)

Problem 5. Prove that the number

(a) 97^{97}; (4 points)

(b) 1997^{17} (4 points)

is not representable as a sum of cubes of several consecutive integers.

Problem 6. Let P be a point inside the triangle ABC with $AB = BC$, $\angle ABC = 80°$, $\angle PAC = 40°$, and $\angle ACP = 30°$. Find $\angle BPC$. (7 points)

Problem 7. You are given a balance and one copy of each ten weights of 1, 2, 4, 8, 16, 32, 64, 128, 256 and 512 grams. An object weighing M grams, where M is a positive integer, may be balanced in different ways by placing various combinations of the given weights on either pans of the balance.

(a) Prove that no object may be balanced in more than 89 ways. (5 points)

(b) Find a value of M such that an object weighing M grams can be balanced in 89 ways. (4 points)

Senior A-Level Paper

Problem 1. same as Junior A-Level Paper Problem 2. (4 points)

Problem 2. D is the point on BC and E is the point on CA such that AD and BE are the bisectors of $\angle A$ and $\angle B$ of triangle ABC. If DE is the bisector of $\angle ADC$, find $\angle A$. (5 points)

Problem 3. You are given 20 positive weights such that any object of integer weight m, $1 \le m \le 1997$, can be balanced by placing in it one pan of a balance and a subset of the weights on the other pan. What is the minimal value of the largest of the 20 weights if the weights are

(a) all integers; (3 points)

(b) not necessarily integers? (3 points)

Problem 4. A convex polygon G is placed inside a convex polygon F so that their boundaries have no common points. A segment s containing two points on the boundary of F is called a support chord for G if s contains a side or only a vertex of G. Prove that

(a) there exists a support chord for G whose midpoint lies on the boundary of G; (6 points)

(b) there exist at least two such chords. (2 points)

Problem 5. Prove that

$$\frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+c+a} \le 1,$$

where a, b and c are positive numbers such that $abc = 1$. (8 points)

Problem 6. Prove that if $F(x)$ and $G(x)$ are polynomials with coefficients 0 and 1 such that

$$F(x)G(x) = 1 + x + x^2 + \ldots + x^{n-1}$$

holds for some $n > 1$, then one of them is representable in the form

$$(1 + x + x^2 + \ldots + x^{k-1})T(x)$$

for some $k > 1$ and some polynomial $T(x)$ with coefficients 0 and 1. (8 points)

Problem 7. Several strips and a circle of radius 1 are drawn on the plane. The sum of the widths of the strips is 100. Prove that one can translate each strip parallel to itself so that together they cover the circle. (8 points)