Mathematical Excalibur

Volume 4, Number 5 October 1999 - December 1999

Olympiad Corner

40th International Mathematical Olympiad, July 1999:

Time allowed: 4.5 Hours
Each problem is worth 7 points.

Problem 1. Determine all finite sets S of at least three points in the plane which satisfy the following condition: for any two distinct points A and B in S, the perpendicular bisector of the line segment AB is an axis of symmetry for S.

Problem 2. Let n be a fixed integer, with $n \ge 2$.

(a) Determine the least constant C such that the inequality

$$\sum_{1\le i<j\le n} x_i x_j \left(x_i^2 + x_j^2\right) \le C\left(\sum_{1\le i\le n} x_i\right)^4$$

holds for all real numbers x_1, x_2, …, $x_n \ge 0$.

(b) For this constant C, determine when equality holds.

Problem 3. Consider an $n \times n$ square board, where n is a fixed even positive integer. The board is divided into n^2 unit squares. We say that two different squares on the board are *adjacent* if they have a common side.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, MATH Dept, HKUST for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is December 15, 1999.

For individual subscription for the next five issues for the 99-00 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

Dr. Kin-Yin Li
Department of Mathematics
Hong Kong University of Science and Technology
Clear Water Bay, Kowloon, Hong Kong

Fax: 2358-1643
Email: makyli@ust.hk

費馬最後定理（二）

梁子傑
香港道教聯合會青松中學

在「數論」的研究之中，有一門分枝不可不提，它就是「橢圓曲線」(Elliptic Curve) $y^2 = x^3 + ax^2 + bx + c$（見 page 2 附錄）。

「橢圓曲線」並非橢圓形，它是計算橢圓周長時的一件「副產品」。但「橢圓曲線」本身卻有著一些非常有趣的數學性質，吸引著數學家的注視。

提到「橢圓曲線」，又不可不提「谷山 – 志村猜想」了。

1954 年，志村五郎 (Goro Shimura) 在東京大學結識了比他太一歲的谷山豊（Yutaka Taniyama, 1927 - 1958），之後，就開始了二人對「模形式」(modular form) 的研究。「模形式」，起源於法國數學家龐加萊（Henry Poincaré, 1854 - 1912）對「自守函數」的研究。所謂「自守函數」，可以說是「週期函數」的推廣，而「模形式」則可以理解為在複平面上的「週期函數」。

1955 年，谷山開始提出他的驚人猜想。三年後，谷山突然自殺身亡。其後，志村繼續谷山的研究，總結出以下的一個想法：「每條橢圓曲線，都可以對應一個模形式。」之後，人們就稱這猜想為「谷山 – 志村猜想」。

起初，大多數數學家都不相信這個猜想，但經過十多年的反覆檢算後，又沒有理據可以將它推翻。到了 70 年代，相信「谷山 – 志村猜想」的人越來越多，甚至以假定「谷山 – 志村猜想」成立的前提下進行他們的論證。

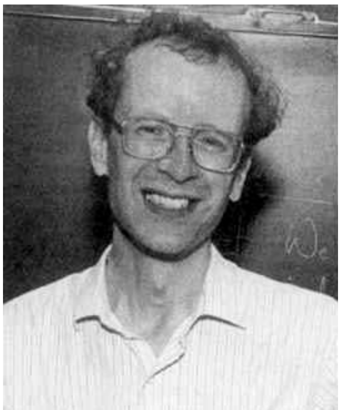
1984 年秋，德國數學家弗賴（Gerhand Frey），在一次數學會議上，提出了以下的觀點：

首先，假設「費馬最後定理」不成立。即能夠發現正整數 A、B、C 和 N，使得 $A^N + B^N = C^N$。於是利用這些數字構作橢圓曲線：$y^2 = x(x - A^N)(x + B^N)$。弗賴發現這條曲線有很多非常特別的性質，特別到不可能對應於任何一個「模形式」！換句話說，弗賴認為：如果「費馬最後定理」不成立，那麼「谷山 – 志村猜想」也是錯的！但倒轉來說，如果「谷山 – 志村猜想」成立，那麼「費馬最後定理」就必定成立！因此，弗賴其實是指出了一條證明「費馬最後定理」的新路徑：這就是去證明「谷山 – 志村猜想」！

可惜的是，弗賴在 1984 年的研究，並未能成功地證實他的觀點。不過，美國數學家里貝特（Kenneth Ribet），經過多次嘗試後，終於在 1986 年證實了有關的問題。

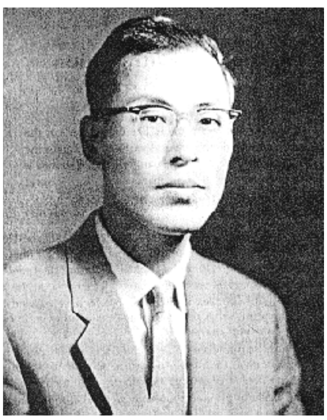
似乎，要證明「費馬最後定理」，現在祇需要證明「谷山 –志村猜想」就可以了。不過自從該猜想

Henri Poincaré



Andrew Wiles

Yutaka Taniyama



Goro Shimura

被提出以來，已經歷過差不多三十年的時間，數學家對這個證明，亦沒有多大的進展。不過，在這時候，英國數學家懷爾斯就開始他偉大而艱巨的工作。

懷爾斯（Andrew Wiles），出生於 1953 年。10 歲已立志要證明「費馬最後定理」。1975 年，開始在劍橋大學進行研究，專攻「橢圓曲線」和「岩澤理論」。在取得博士學位之後，就轉到美國的普林斯頓大學繼續工作。當他知道里貝特證實了弗賴的猜想後，就決定放棄當時手上的所有研究，專心於「谷山－志村猜想」的證明。由於他不想被人騷擾，他更決定要秘密地進行此項工作。

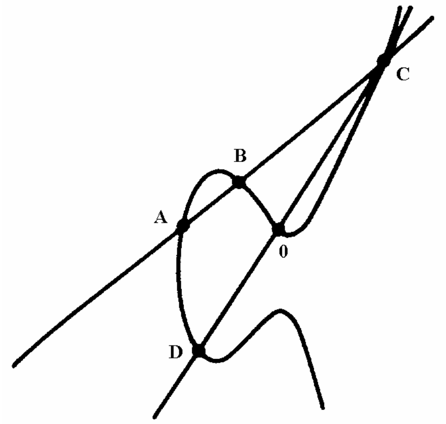
經過了七年的秘密工作後，懷爾斯認為他已證實了「谷山－志村猜想」，並且在 1993 年 6 月 23 日，在劍橋大學的牛頓研究所中，以「模形式、橢圓曲線、伽羅瓦表示論」為題，發表了他對「谷山－志村猜想」重要部份（即「費馬最後定理」）的證明。當日的演講非常成功，「費馬最後定理」經已被證實的消息，很快就傳遍世界。

不過，當懷爾斯將他長達二百頁的證明送給數論專家審閱時，卻發現當中出現漏洞。起初，懷爾斯以為很容易便可以將這個漏洞修補，但事與願遺，到了 1993 年的年底，他承認他的證明出現問題，而且要一段時間才可解決。

到了 1994 年的 9 月，懷爾斯終於突破了證明中的障礙，成功地完成了一項人類史上的創舉，證明了「費馬最後定理」。1995 年 5 月，懷爾斯的證明，發表在雜誌《數學年鑑》之中。到了 1997 年 6 月 27 日，懷爾斯更獲得價值五萬美元的「沃爾夫斯凱爾獎金」，實現了他的童年夢想，正式地結束了這個長達 358 年的數學證明故事。

附錄：橢圓曲線

「橢圓曲線」是滿足方程 $y^2 = x^3 + ax^2 + bx + c$ 的點所組成的曲線，其中 a, b, c 為有理數使 $x^3 + ax^2 + bx + c$ 有不同的根。在曲線上定一個有理點 O。不難證明，當直線穿過兩個曲線上的有理點 A, B 後，該直線必定與曲線再相交於第三個有理點 C。由 C 和 O 再得一點 D 如下圖。我們可以將曲線上的有理點以 $A + B = D$ 為定義看成一個「群」(group)。由於以上性質可以用來解答很多相關的問題，故此「橢圓曲線」就成為數學研究的一個焦點。現時，「橢圓曲線」的理論，主要應用於現代編寫通訊密碼的技術方面。



參考書目

《費瑪最後定理》
作者：賽門 辛
出版社：臺灣商務印書館

《費馬最後定理》
作者：艾克塞爾
出版社：時報出版

《費馬猜想》
作者：姚玉強
出版社：九章出版社

參考網頁

http://www.ams.org/new-in-math/fermat.html

http://www-history.mcs.st-and.ac.uk/~history/HistTopics/Fermat's_last_theorem.html

http://www.ams.org/notices/199710/barner.pdf

Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceeded by the solver's name, home address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is *December 4, 1999.*

Problem 91. Solve the system of equations:

$$\sqrt{3x}\left(1+\frac{1}{x+y}\right)=2$$

$$\sqrt{7y}\left(1-\frac{1}{x+y}\right)=4\sqrt{2}\,.$$

(This is the corrected version of problem 86.)

Problem 92. Let $a_1, a_2, ..., a_n$ $(n > 3)$ be real numbers such that $a_1 + a_2 + \ldots + a_n \geq n$ and $a_1^2 + a_2^2 + \ldots + a_n^2 \geq n^2$. Prove that $\max(a_1, a_2, ..., a_n) \geq 2$. (*Source: 1999 USA Math Olympiad*)

Problem 93. Two circles of radii R and r are tangent to line L at points A and B respectively and intersect each other at C and D. Prove that the radius of the circumcircle of triangle ABC does not depend on the length of segment AB. (*Source: 1995 Russian Math Olympiad*)

Problem 94. Determine all pairs (m, n) of positive integers for which $2^m + 3^n$ is a square.

Problem 95. Pieces are placed on an $n \times n$ board. Each piece "attacks" all squares that belong to its row, column, and the northwest-southeast diagonal which contains it. Determine the least number of pieces which are necessary to attack all the squares of the board. (*Source: 1995 Iberoamerican Math Olympiad*)

****

Solutions

****

Problem 86. Solve the system of equations:

$$\sqrt{3x}\left(1+\frac{1}{x+y}\right)=2$$

$$\sqrt{7y}\left(1+\frac{1}{x+y}\right)=4\sqrt{2}\,.$$

(*Source: 1996 Vietnamese Math Olympiad*)

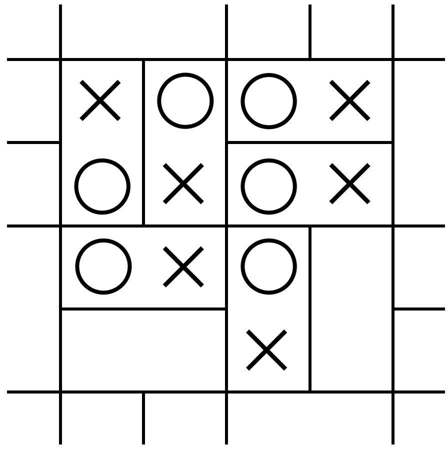
Solution. CHAO Khek Lun Harold (St. Paul's College, Form 5), **FAN Wai Tong Louis** (St. Marks' School, Form 7), **NG Ka Wing Gary** (STFA Leung Kau Kui College, Form 7) and **NG Lai Ting** (True Light Girls' College, Form 7).

Clearly, x and y are nonzero. Dividing the second equation by the first equation, we then simplify to get $y = 24x/7$. So $x + y = 31x/7$. Substituting this into the first equation, we then simplifying, we get $x - (2/\sqrt{3})\sqrt{x} + 7/31 = 0$. Applying the quadratic formula to find $\sqrt{x}$, then squaring, we get $x = (41 \pm 2\sqrt{310})/93$. Then $y = 24x/7 = (328 \pm 16\sqrt{310})/217$, respectively. By direct checking, we see that both pairs (x, y) are solutions.

Other recommended solvers: **CHAN Hiu Fai Philip** (STFA Leung Kau Kui College, Form 6), **CHAN Kwan Chuen** (HKSYC & IA Wong Tai Shan Memorial School, Form 4), **CHUI Man Kei** (STFA Leung Kau Kui College, Form 5), **HO Chung Yu** (HKU), **LAW Siu Lun Jack** (Ming Kei College, Form 5), **LEUNG Yiu Ka** (STFA Leung Kau Kui College, Form 4), **KU Hong Tung** (Carmel Divine Grace Foundation Secondary School, Form 6), **SUEN Yat Chung** (Carmel Divine Grace Foundation Secondary School, Form 6), **TANG Sheung Kon** (STFA Leung Kau Kui College, Form 5), **WONG Chi Man** (Valtorta College, Form 5), **WONG Chun Ho Terry** (STFA Leung Kau Kui College, Form 5), **WONG Chung Yin** (STFA Leung Kau Kui College), **WONG Tak Wai Alan** (University of Waterloo, Canada), **WU Man Kin Kenny** (STFA Leung Kau Kui College) and **YUEN Pak Ho** (Queen Elizabeth School, Form 6).

Problem 87. Two players play a game on an infinite board that consists of 1×1 squares. Player I chooses a square and marks it with an O. Then, player II chooses another square and marks it with X. They play until one of the players marks a row or a column of 5 consecutive squares, and this player wins the game. If no player can achieve this, the game is a tie. Show that player II can prevent player I from winning. (*Source: 1995 Israeli Math Olympiad*).

Solution. CHAO Khek Lun Harold (St. Paul's College, Form 5).



Divide the board into 2×2 blocks. Then bisect each 2×2 block into two 1×2 tiles so that for every pair of blocks sharing a common edge, the bisecting segment in one will be horizontal and the other vertical. Since every five consecutive squares on the board contain a tile, after player I choose a square, player II could prevent player I from winning by choosing the other square in the tile.

Problem 88. Find all positive integers n such that $3^{n-1} + 5^{n-1}$ divides $3^n + 5^n$. (*Source: 1996 St. Petersburg City Math Olympiad*).

Solution. CHAO Khek Lun Harold (St. Paul's College, Form 5), **HO Chung Yu** (HKU), **NG Ka Wing Gary** (STFA Leung Kau Kui College, Form 7), **NG Lai Ting** (True Light Girls' College, Form 7), **SHUM Ho Keung** (PLK No.1 W.H. Cheung College, Form 6) and **TSE Ho Pak** (SKH Bishop Mok Sau Tseng Secondary School, Form 5).

For such an n, since

$$3(3^{n-1} + 5^{n-1}) < 3^n + 5^n < 5(3^{n-1} + 5^{n-1}),$$

so $3^n + 5^n = 4(3^{n-1} + 5^{n-1})$. Cancelling, we get $5^{n-1} = 3^{n-1}$. This forces $n = 1$. Since 2 divides 8, $n = 1$ is the only solution.

Other recommended solvers: **CHAN Hiu Fai Philip** (STFA Leung Kau Kui College, Form 6), **CHAN Kwan Chuen** (HKSYC & IA Wong Tai Shan Memorial School, Form 4), **CHAN Man Wai** (St. Stephen's Girls' College, Form 5), **FAN Wai Tong Louis** (St. Mark's School, Form 7), **HON Chin Wing** (Pui Ching Middle School, Form 5), **LAW**

Siu Lun Jack (Ming Kei College, Form 5), **LEUNG Yiu Ka** (STFA Leung Kau Kui College, Form 4), **NG Ka Chun** (Queen Elizabeth School), **NG Tin Chi** (TWGH Chang Ming Thien College, Form 7), **TAI Kwok Fung** (Carmel Divine Grace Foundation Secondary School, Form 6), **TANG Sheung Kon** (STFA Leung Kau Kui College, Form 5), **TSUI Ka Ho Willie** (Hoi Ping Chamber of Commerce Secondary School, Form 6), **WONG Chi Man** (Valtorta College, Form 5), **WONG Chun Ho Terry** (STFA Leung Kau Kui College, Form 5), **WONG Tak Wai Alan** (University of Waterloo, Canada), **YU Ka Lok** (Carmel Divine Grace Foundation Secondary School, Form 6) and **YUEN Pak Ho** (Queen Elizabeth School, Form 6).

Problem 84. Let O and G be the circumcenter and centroid of triangle ABC, respectively. If R is the circumradius and r is the inradius of ABC, then show that $OG \le \sqrt{R(R-2r)}$. *(Source: 1996 Balkan Math Olympiad)*

Solution I. CHAO Khek Lun Harold (St. Paul's College, Form 5), **FAN Wai Tong Louis** (St. Mark's School, Form 7), **NG Lai Ting** (True Light Girls' College, Form 7) and **YUEN Pak Ho** (Queen Elizabeth School, Form 6)

Let line AG intersect side BC at A' and the circumcircle again at A''. Since $\cos BA'A + \cos CA'A = 0$, we can use the cosine law to get

$$A'A^2 = (2b^2 + 2c^2 - a^2)/4,$$

where a, b, c are the usual side lengths of the triangle. By the inter-secting chord theorem,

$$A'A \times A'A'' = A'B \times A'C = a^2/4.$$

Consider the chord through O and G interecting AA'' at G. By the intersecting chord theorem,

$$\begin{aligned}(R+OG)(R-OG) &= GA \times GA'' \\ &= (2A'A/3)(A'A/3 + A'A'') \\ &= (a^2+b^2+c^2)/9.\end{aligned}$$

Then

$$OG = \sqrt{R^2 - (a^2+b^2+c^2)/9}.$$

By the AM-GM inequality,

$$(a+b+c)(a^2+b^2+c^2) \ge$$

$$(3\sqrt[3]{abc})(3\sqrt[3]{a^2b^2c^2}) = 9abc.$$

Now the area of the triangle is $(ab \sin C)/2 = abc/(4R)$ (by the extended sine law) on one hand and $(a+b+c)r/2$ on the other hand. So, $a+b+c = abc/(2rR)$. Using this, we simplify the inequality to get $(a^2+b^2+c^2)/9 \ge 2rR$. Then

$$\sqrt{R^2-2rR} \ge \sqrt{R^2-(a^2+b^2+c^2)/9} = OG.$$

Solution II. NG Lai Ting (True Light Girls' College, Form 7)

Put the origin at the circumcenter. Let z_1, z_2, z_3 be the complex numbers corresponding to A, B, C, respectively on the complex plane. Then $OG^2 = \left|(z_1+z_2+z_3)/3\right|^2$. Using $|\omega|^2 = \omega\bar{\omega}$, we can check the right side equals $(3|z_1|^2 + 3|z_2|^2 + 3|z_3|^2 - |z_1-z_2|^2 - |z_2-z_3|^2 - |z_3-z_1|^2)/9$. Since $|z_1| = |z_2| = |z_3| = R$ and $|z_1-z_2| = c$, $|z_2-z_3| = a$, $|z_3-z_1| = b$, we get

$$OG^2 = (9R^2 - a^2 - b^2 - c^2)/9.$$

The rest is as in solution 1.

Problem 90. There are n parking spaces (numbered 1 to n) along a one-way road down which n drivers $d_1, d_2, \ldots, d_n$ in that order are traveling. Each driver has a favorite parking space and parks there if it is free; otherwise, he parks at the nearest free place down the road. (Two drivers may have the same favorite space.) If there is no free space after his favorite, he drives away. How many lists $a_1, a_2, \ldots, a_n$ of favorite parking spaces are there which permit all of the drivers to park? Here a_i is the favorite parking space number of d_i. (*Source: 1996 St. Petersburg City Math Olympiad*).

Solution: Call a list of favorite parking spaces $a_1, a_2, \ldots, a_n$ which permits all drivers to park a *good* list. To each good list, associate the list $b_2, \ldots, b_n$, where b_i is the difference (mod $n+1$) between the number a_i and the number of the space driver d_{i-1} took. Note from a_1 and $b_2, \ldots, b_n$, we can reconstruct $a_2, \ldots, a_n$. It follows that different good lists give rise to different lists of b_i's. Since there are $n+1$ possible choices for each b_i, there are $(n+1)^{n-1}$ possible lists of $b_2, \ldots, b_n$. For each of these lists of the b_i's, imagine the n parking spaces are arranged in a circle with an extra parking space put at the end. Let d_1 park anywhere temporarily and put $d_i (i>1)$ in the first available space after the space b_i away from the space taken by d_{i-1}. By shifting the position of d_1, we can ensure the extra parking space is not taken. This implies the corresponding list of $a_1, a_2, \ldots, a_n$ is good. So the number of good lists is $(n+1)^{n-1}$.

Comments: To begin the problem, one could first count the number of good lists in the cases $n = 2$ and $n = 3$. This will lead to the answer $(n+1)^{n-1}$. From the $n+1$ factor, it becomes natural to consider an extra parking space. The difficulty is to come up with the *one-to-one correspondence* between the good lists and the b_i lists. For this problem, only one incomplete solution with correct answer and right ideas was sent in by **CHAO Khek Lun Harold** (St. Paul's College, Form 5)

Olympiad Corner

(continued from page 1)

Problem 3. (cont'd) N unit squares on the board are marked in such a way that every square (marked or unmarked) on the board is adjacent to at least one marked square.
Determine the smallest possible value of N.

Problem 4. Determine all pairs (n, p) of positive integers such that p is a prime, $n \le 2p$, and $(p-1)^n + 1$ is divisible by n^{p-1}.

Problem 5. Two circles Γ_1 and Γ_2 are contained inside the circle Γ, and are tangent to Γ at the distinct points M and N, respectively. Γ_1 passes through the centre of Γ_2. The line passing through the two points of intersection of Γ_1 and Γ_2 meets Γ at A and B, respectively. The lines MA and MB meet Γ_1 at C and D, respectively.
Prove that CD is tangent to Γ_2.

Problem 6. Determine all functions $f: \mathbf{R} \to \mathbf{R}$ such that $f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$ for all $x, y \in \mathbf{R}$.