Mathematical Excalibur

Volume 6, Number 1 *January 2001 – March 2001*

Olympiad Corner

17^{th} Balkan Mathematical Olympiad, 3-9 May 2000:

Time allowed: 4 hours 30 minutes

Problem 1. Find all the functions f : $\mathbf{R} \to \mathbf{R}$ with the property:

$$f(xf(x)+f(y))=(f(x))^2+y,$$

for any real numbers x and y.

Problem 2. Let ABC be a nonisosceles acute triangle and E be an interior point of the median AD, $D \in (BC)$. The point F is the orthogonal projection of the point E on the straight line BC. Let M be an interior point of the segment EF, N and P be the orthogonal projections of the point M on the straight lines AC and AB, respectively. Prove that the two straight lines containing the bisectrices of the angles PMN and PEN have no common point.

Problem 3. Find the maximum number of rectangles of the dimensions $1\times 10\sqrt{2}$, which is possible to cut off from a rectangle of the dimensions 50×90, by using cuts parallel to the edges of the initial rectangle.

(continued on page 2)

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is *April 15, 2001*.

For individual subscription for the next five issues for the 01-02 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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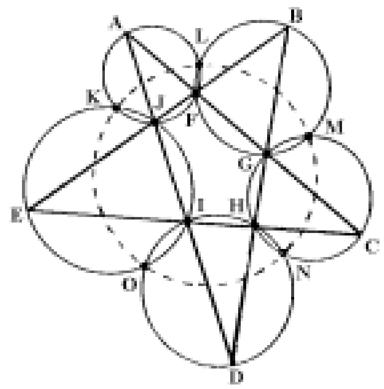
Concyclic Problems

Kin Y. Li

Near Christmas last year, I came across two beautiful geometry problems. I was informed of the first problem by a reporter, who was covering President Jiang Zemin's visit to Macau. While talking to students and teachers, the President posed the following problem.

For any pentagram ABCDE obtained by extending the sides of a pentagon FGHIJ, prove that neighboring pairs of the circumcircles of ΔAJF, BFG, CGH, DHI, EIJ intersect at 5 concyclic points K, L, M, N, O as in the figure.



The second problem came a week later. I read it in the Problems Section of the November issue of the *American Mathematical Monthly*. It was proposed by Floor van Lamoen, Goes, The Netherlands. Here is the problem.

A triangle is divided by its three medians into 6 smaller triangles. Show that the circumcenters of these smaller triangles lie on a circle.

To get the readers appreciating these problems, here I will say, *stop reading, try to work out these problems and come back to compare your solutions with those given below!*

Here is a guided tour of the solutions. The first step in enjoying geometry problems is to draw accurate pictures with compass and ruler!

Now we look at ways of getting solutions to these problems. Both are concyclic problems with more than 4 points. Generally, to do this, we show the points are concyclic four at a time. For example, in the first problem, if we can show *K, L, M, N* are concyclic, then by similar reasons, *L, M, N, O* will also be concyclic so that all five points lie on the circle passing through *L, M, N*.

There are two common ways of showing 4 points are concyclic. One way is to show the sum of two opposite angles of the quadrilateral with the 4 points as vertices is 180°. Another way is to use the converse of the intersecting chord theorem, which asserts that if lines WX and YZ intersect at P and $PW\cdot PX$ = $PY\cdot PZ$, then W, X, Y, Z are concyclic. (The equation implies ΔPWY, PZX are similar. Then $\angle PWY = \angle PZX$ and the conclusion follows.)

For the first problem, as the points *K, L, M, N, O* are on the circumcirles, checking the sum of opposite angles equal 180° is likely to be easier as we can use the theorem about angles on the same segment to move the angles. To show *K, L, M, N* are concyclic, we consider showing $\angle LMN$ + $\angle LKN = 180^\circ$. Since the sides of $\angle LMN$ are in two circumcircles, it may be wise to break it into two angles LMG and GMN. Then the strategy is to *change* these to other angles *closer* to $\angle LKN$.

Now $\angle LMG = 180^\circ - \angle LFG = \angle LFA$ = $\angle LKA$. (So far, we are on track. We *bounced* $\angle LMG$ to $\angle LKA$, which shares a side with $\angle LKN$.) Next, $\angle GMN$ = $\angle GCN$ = $\angle ACN$. Putting these together, we have

$$\begin{aligned}&\angle LMN+\angle LKN\\ &=\angle LKA+\angle ACN+\angle LKN\\ &=\angle AKN+\angle ACN.\end{aligned}$$

Now if we can only show *A, K, N, C* are concyclic, then we will get 180° for the displayed equations above and we will finish. However, life is not that easy. This turned out to be the hard part. If you draw the circle through *A, C, N*, then you see it goes through *K* as expected and surprisingly, it also goes through another point, *I*. With this discovery, there is new hope. Consider the arc through *B, I, O*. On the two sides of this arc, you can see there are *corresponding point pairs* (*A, C*), (*K, N*), (*J, H*), (*F, G*). So to show *A, K, N, C* are concyclic, we can first try to show *N* is on the circle through *A, C, I*, then in that argument, if we interchange *A* with *C, K,* with *N* and so on, we should also get *K* is on the circle through *C, A, I*. Then *A, K, N, C* (and *I*) will be concyclic and we will finish.

Wishful thinking like this sometimes works! Here are the details:

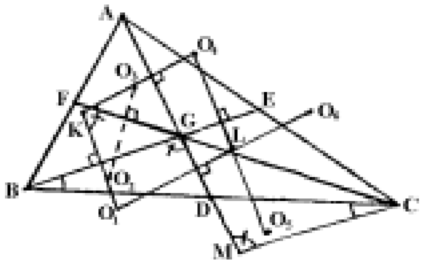
$$\angle ACN = \angle GCN = 180^\circ - \angle GHN$$

$$= \angle NHD = \angle NID = 180^\circ - \angle AIN .$$

So *N* is on the circle *A, C, I*. *Interchanging letters*, we get *similarly K* is on circle *C, A, I*. So *A, K, N, C* (and *I*) are concyclic. Therefore, *K, L, M, N, O* are indeed concyclic.

(*History*. My friend C.J. Lam did a search on the electronic database JSTOR and came across an article titled *A Chain of Circles Associated with the 5-Line* by J.W. Clawson published in the *American Mathematical Monthly*, volume 61, number 3 (March 1954), pages 161-166. There the problem was attributed to the nineteenth century geometer Miquel, who published the result in Liouville's Journal de Mathematiques, volume 3 (1838), pages 485-487. In that paper, Miquel proved his famous theorem that for four pairwise intersecting lines, taking three of the lines at a time and forming the circles through the three intersecting points, the four circles will always meet at a common point, which nowadays are referred to as the *Miquel point*. The first problem was then deduced as a corollary of this Miquel theorem.)

For the second problem, as the 6 circumcenters of the smaller triangles are not on any circles that we can see immediately, so we may try to use the converse of the intersecting chord theorem. For a triangle *ABC*, let *G, D, E, F* be the centroid, the midpoints of sides *BC, CA, AB*, respectively. Let O_1, O_2, O_3, O_4, O_5, O_6 be the circumcenters of triangles *DBG, BFG, FAG, AEG, ECG, CDG*, respectively.



Well, should we draw the 6 circumcircles? It would make the picture complicated. The circles do not seem to be helpful at this *early* stage. We give up on drawing the circles, but the circumcenters are important. So at least we should locate them. To locate the circumcenter of ΔFAG, for example, which two sides do we draw perpendicular bisectors? Sides *AG* and *FG* are the choices because they are also the sides of the other small triangles, so we can save some work later. Trying this out, we discover these perpendicular bisectors produce many parallel lines and parallelograms!

Since circumcenters are on perpendicular bisectors of chords, lines $O_3\,O_4$, $O_6\,O_1$ are perpendicular bisectors of *AG, GD*, respectively. So they are perpendicular to line *AD* and are $\frac{1}{2}$ *AD* units apart. Similarly, the two lines $O_1\,O_2$, $O_4\,O_5$ are perpendicular to line *BE* and are $\frac{1}{2}$ *BE* units apart. Aiming in showing O_1, O_2, O_3, O_4 are concyclic by the converse of the intersecting chord theorem, let *K* be the intersection of lines $O_1\,O_2$, $O_3\,O_4$ and *L* be the intersection of the lines $O_4\,O_5$, $O_6\,O_1$. Since the area of the parallelogram $K\,O_4\,L\,O_1$ is

$$\frac{1}{2}AD \cdot KO_4 = \frac{1}{2}BE \cdot KO_1,$$

we get $KO_1/KO_4 = AD/BE$.

Now that we get ratio of KO_1 and KO_4, we should examine KO_2 and KO_3. Trying to understand ΔKO_2O_3, we first find its angles. Since $KO_2 \perp BG$, $O_2O_3 \perp FG$ and $KO_3 \perp AG$, we see that $\angle KO_2O_3 = \angle BGF$ and $\angle KO_3O_2 = \angle FGA$. Then $\angle O_2KO_3 = \angle DGB$. At this point, you can see the angles of ΔKO_2O_3 equal the three angles with vertices at *G* on the left side of segment *AD*.

Now we try to put these three angles together in another way to form another triangle. Let *M* be the point on line *AG* such that *MC* is parallel to *BG*. Since $\angle MCG = \angle BGF$, $\angle MGC = \angle FGA$ (and $\angle GMC = \angle BGD$,) we see ΔKO_2O_3, *MCG* are similar.

The sides of ΔMCG are easy to compute in term of *AD, BE, CF*. As *AD* and *BE* occurred in the ratio of KO_1 and KO_4, this is just what we need! Observe that ΔMCD, *GBD* are congruent since $\angle MCD = \angle GBD$ (by *MC* parallel to *GB*), *CD = BD* and $\angle MDC = \angle GDB$. So

$$MG = 2GD = \frac{2}{3}AD,$$

$$MC = GB = \frac{2}{3}BE$$

(and $CG = \frac{2}{3}CF$. Incidentally, this means the three medians of a triangle can be put together to form a triangle! Actually, this is well-known and was the reason we considered ΔMCG .) We have KO_3/KO_2 = *MG/MC* = *AD/BE* = KO_1/KO_4.

So $KO_1 \cdot KO_2 = KO_3 \cdot KO_4$, which implies O_1, O_2, O_3, O_4 are concyclic. Similarly, we see that O_2, O_3, O_4, O_5 concyclic (using the parallelogram formed by the lines O_1O_2, O_4O_5, O_2O_3, O_5O_6 instead) and O_3, O_4, O_5, O_6 are concyclic.

Olympiad Corner

(continued from page 1)

Problem 4. We say that a positive integer *r* is a *power*, if it has the form $r = t^s$ where *t* and *s* are integers, $t \ge 2$, $s \ge 2$. Show that for any positive integer *n* there exists a set *A* of positive integers, which satisfies the conditions:

1. *A* has *n* elements;
2. any element of *A* is a power;
3. for any r_1, r_2, …, r_k $(2 \le k \le n)$ from *A* the number $\dfrac{r_1 + r_2 + \cdots + r_k}{k}$ is a power.

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, home (or **email**) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science & Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is ***April 15, 2001***.

Problem 121. Prove that any integer greater than or equal to 7 can be written as a sum of two relatively prime integers, both greater than 1.
(Two integers are *relative prime* if they share no common positive divisor other than 1. For example, 22 and 15 are relatively prime, and thus 37 = 22 + 15 represents the number 37 in the desired way.) (*Source: Second Bay Area Mathematical Olympaid*)

Problem 122. Prove that the product of the lengths of the three angle bisectors of a triangle is less than the product of the lengths of the three sides. (*Source: 1957 Shanghai Junior High School Math Competition*)

Problem 123. Show that every convex quadrilateral with area 1 can be covered by some triangle of area at most 2. (*Source: 1989 Wuhu City Math Competition*)

Problem 124. Find the least integer n such that among every n distinct numbers a_1, a_2, …, a_n, chosen from [1, 1000], there always exist a_i, a_j such that

$$0 < a_i - a_j < 1 + 3\sqrt[3]{a_i a_j}\ .$$

(*Source: 1990 Chinese Team Training Test*)

Problem 125. Prove that
$\tan^2 1^\circ + \tan^2 3^\circ + \tan^2 5^\circ + \cdots + \tan^2 89^\circ$
is an integer.

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Solutions

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Problem 116. Show that the interior of a convex quadrilateral with area A and perimeter P contains a circle of radius A/P.

Solution 1. **CHAO Khek Lun** (St. Paul's College, Form 6).

Draw four rectangles on the sides of the quadrilateral and each has height A/P pointing inward. The sum of the areas of the rectangles is A. Since at least one interior angle of the quadrilateral is less than 180°, at least two of the rectangles will overlap. So the union of the four rectangular regions does not cover the interior of the quadrilateral. For any point in the interior of the quadrilateral not covered by the rectangles, the distance between the point and any side of the quadrilateral is greater than A/P. So we can draw a desired circle with that point as center.

Solution 2. **CHUNG Tat Chi** (Queen Elizabeth School, Form 4) and **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).

Let $BCDE$ be a quadrilateral with area A and perimeter P. One of the diagonal, say BD is inside the quadrilateral. Then either ΔBCD or ΔBED will have an area greater than or equal to $A/2$. Suppose this is ΔBCD. Then $BCDE$ contains the incircle of ΔBCD, which has a radius of

$$\begin{aligned} &\frac{2[BCD]}{BC+CD+DB} \\ &> \frac{2[BCD]}{BC+CD+DE+EB} \\ &\ge \frac{A}{P}, \end{aligned}$$

where the brackets denote area. Hence, it contains a circle of radius A/P.
Comment: Both solutions do not need the convexity assumption.

Problem 117. The lengths of the sides of a quadrilateral are positive integers. The length of each side divides the sum of the other three lengths. Prove that two of the sides have the same length.

Solution. **CHAO Khek Lun** (St. Paul's College, Form 6) and **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).

Suppose the sides are a, b, c, d with $a < b < c < d$. Since $d < a + b + c < 3d$ and d divides $a + b + c$, we have $a + b + c = 2d$. Now each of a, b, c divides $a + b + c + d = 3d$. Let $x = 3d/a, y = 3d/b$ and $z = 3d/c$. Then $a < b < c < d$ implies $x > y > z > 3$. So $z \ge 4$, $y \ge 5$, $x \ge 6$. Then

$$2d = a + b + c \le \frac{3d}{6} + \frac{3d}{5} + \frac{3d}{4} < 2d,$$

a contradiction. Therefore, two of the sides are equal.

Problem 118. Let R be the real numbers. Find all functions $f : R \to R$ such that for all real numbers x and y,

$$f(xf(y) + x) = xy + f(x).$$

Solution 1. **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).

Putting $x = 1$, $y = -1 - f(1)$ and letting $a = f(y) + 1$, we get

$$f(a) = f(f(y) + 1) = y + f(1) = -1.$$

Putting $y = a$ and letting $b = f(0)$, we get

$$b = f(xf(a) + x) = ax + f(x),$$

so $f(x) = -ax + b$. Putting this into the equation, we have

$$a^2xy - abx - ax + b = xy - ax + b.$$

Equating coefficients, we get $a = \pm 1$ and $b = 0$, so $f(x) = x$ or $f(x) = -x$. We can easily check both are solutions.

Solution 2. **LEE Kai Seng** (HKUST).

Setting $x = 1$, we get

$$f(f(y)+1) = y + f(1).$$

For every real number a, let $y = a - f(1)$, then $f(f(y) + 1) = a$ and f is surjective. In particular, there is b such that $f(b) = -1$. Also, if $f(c) = f(d)$, then

$$\begin{aligned} c + f(1) &= f(f(c)+1) \\ &= f(f(d)+1) \\ &= d + f(1). \end{aligned}$$

So $c = d$ and f is injective. Taking $x = 1$, $y = 0$, we get $f(f(0) + 1) = f(1)$. Since f is injective, we get $f(0) = 0$.
For $x \ne 0$, let $y = -f(x)/x$, then

$$f(xf(y)+x) = 0 = f(0).$$

By injectivity, we get $xf(y) + x = 0$. Then

$$f(-f(x)/x) = f(y) = -1 = f(b)$$

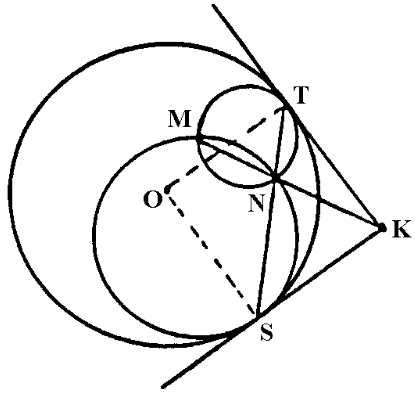
and so $-f(x)/x = b$ for every $x \ne 0$. That is, $f(x) = -bx$. Putting this into the given equation, we find $f(x) = x$ or $f(x) = -x$, which are checked to be solutions.

Other commended solvers: **CHAO Khek Lun** (St. Paul's College, Form 6) and **NG Ka Chun Bartholomew** (Queen Elizabeth School, Form 6).

Problem 119. A circle with center O is internally tangent to two circles inside it at points S and T. Suppose the two circles inside intersect at M and N with N

closer to *ST*. Show that $OM \perp MN$ if and only if *S, N, T* are collinear. (*Source: 1997 Chinese Senior High Math Competition*)

Solution. **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).



Consider the tangent lines at *S* and at *T*. (Suppose they are parallel, then *S, O, T* will be collinear so that *M* and *N* will be equidistant from *ST*, contradicting *N* is closer to *ST*.) Let the tangent lines meet at *K*, then $\angle OSK = 90^\circ = \angle OTK$ implies *O, S, K, T* lie on a circle with diameter *OK*. Also, $KS^2 = KT^2$ implies *K* is on the *radical axis MN* of the two inside circles. So *M, N, K* are collinear.

If *S, N, T* are collinear, then $\angle SMT$ = $\angle SMN + \angle TMN$ = $\angle NSK + \angle KTN$ = $180^\circ - \angle SKT$. So *M, S, K, T, O* are concyclic. Then $\angle OMN$ = $\angle OMK$ = $\angle OSK$ = 90°.

Conversely, if $OM \perp MN$, then $\angle OMK$ = $90^\circ = \angle OSK$ implies *M, S, K, T, O* are concyclic. Then

$$\begin{aligned}\angle SKT &= 180^\circ - \angle SMT\\ &= 180^\circ - \angle SMN - \angle TMN\\ &= 180^\circ - \angle NSK - \angle KTN.\end{aligned}$$

Thus, $\angle TNS = 360^\circ - \angle NSK - \angle SKT$ - $\angle KTN = 180^\circ$. Therefore, *S, N, T* are collinear.

Comments: For the meaning of *radical axis*, we refer the readers to pages 2 and 4 of *Math Excalibur, vol. 4, no. 3* and the corrections on page 4 of *Math Excalibur, vol. 4, no. 4*.

Other commended solvers: **CHAO Khek Lun** (St. Paul's College, Form 6).

Problem 120. Twenty-eight integers are chosen from the interval [104, 208]. Show that there exist two of them having a common prime divisor.

Solution 1. **CHAO Khek Lun** (St. Paul's College, Form 6), **CHAU Suk Ling** (Queen Elizabeth School, Form 6) and **CHUNG Tat Chi** (Queen Elizabeth School, Form 4).

Applying the inclusion-exclusion principle, we see there are 82 integers on [104, 208] that are divisible by 2, 3, 5 or 7. There remain 23 other integers on the interval. If 28 integers are chosen from the interval, at least 28 – 23 = 5 are among the 82 integers that are divisible by 2, 3, 5 or 7. So there will exist two that are both divisible by 2, 3, 5 or 7.

Solution 2. **CHAN Yun Hung** (Carmel Divine Grace Foundation Secondary School, Form 4), **KWOK Sze Ming** (Queen Elizabeth School, Form 5), **LAM Shek Ming** (La Salle College, Form 5), **LEUNG Wai Ying** (Queen Elizabeth School, Form 6), **WONG Tak Wai Alan** (University of Toronto) and **WONG Wing Hong** (La Salle College, Form 3).

There are 19 prime numbers on the interval. The remaining 86 integers on the interval are all divisible by at least one of the prime numbers 2, 3, 5, 7, 11 and 13 since 13 is the largest prime less than or equal to $\sqrt{208}$. So every number on the interval is a multiple of one of these 25 primes. Hence, among any 26 integers on the interval at least two will have a common prime divisor.

A Proof of the Majorization Inequality

Kin Y. Li

Quite a few readers would like to see a proof of the majorization inequality, which was discussed in the last issue of the *Mathematical Excalibur*. Below we will present a proof. We will first make one observation.

Lemma. Let $a < c < b$ and f be convex on an interval *I* with *a, b, c* on *I*. Then the following are true:

$$\frac{f(c)-f(a)}{c-a} \le \frac{f(b)-f(a)}{b-a}$$

and

$$\frac{f(b)-f(c)}{b-c} \le \frac{f(b)-f(a)}{b-a}.$$

Proof. Since $a < c < b$, we have $c = (1-t)a + tb$ for some $t \in (0, 1)$. Solving for *t*, we get $t = (c-a)/(b-a)$. Since *f* is convex on *I*,

$$f(c) \le (1-t)\,f(a) + t\,f(b)$$

$$= \frac{b-c}{b-a}f(a) + \frac{c-a}{b-a}f(b),$$

which is what we will get if we solve for $f(c)$ in the two inequalities in the statement of the lemma.

In brief the lemma asserts that the slopes of chords are increasing as the chords are moving to the right. Now we are ready to proof the majorization inequality.

Suppose

$$(x_1, x_2, \ldots, x_n) \succ (y_1, y_2, \ldots, y_n).$$

Since $x_i \ge x_{i+1}$ and $y_i \ge y_{i+1}$ for i = 1, 2, …, n – 1, it follows from the lemma that the slopes

$$m_i = \frac{f(x_i) - f(y_i)}{x_i - y_i}$$

satisfy $m_i \ge m_{i+1}$ for 1 ≤ i ≤ n – 1. (For example, if $y_{i+1} \le y_i \le x_{i+1} \le x_i$, then applying the lemma twice, we get

$$\begin{aligned}m_{i+1} &= \frac{f(x_{i+1}) - f(y_{i+1})}{x_{i+1} - y_{i+1}}\\ &\le \frac{f(x_{i+1}) - f(y_i)}{x_{i+1} - y_i}\\ &\le \frac{f(x_i) - f(y_i)}{x_i - y_i} = m_i\end{aligned}$$

and similarly for the other ways y_{i+1}, y_i, x_{i+1}, x_i are distributed.)

For k = 1, 2, …, n, let

$$X_k = x_1 + x_2 + \cdots + x_k$$

and

$$Y_k = y_1 + y_2 + \cdots + y_k.$$

Since $X_k \ge Y_k$ for k = 1, 2, …, n – 1 and $X_n = Y_n$, we get

$$\sum_{k=1}^{n}(X_k - Y_k)(m_k - m_{k+1}) \ge 0,$$

where we set $m_{n+1} = 0$ for convenience. Expanding the sum, grouping the terms involving the same m_k 's and letting $X_0 = 0 = Y_0$, we get

$$\sum_{k=1}^{n}(X_k - X_{k-1} - Y_k + Y_{k-1})m_k \ge 0,$$

which is the same as

$$\sum_{k=1}^{n}(x_k - y_k)m_k \ge 0.$$

Since $(x_k - y_k)m_k = f(x_k) - f(y_k)$, we get

$$\sum_{k=1}^{n}(f(x_k) - f(y_k))m_k \ge 0.$$

Transferring the $f(y_k)$ terms to the right, we get the majorization inequality.