Mathematical Excalibur

Volume 6, Number 2 April 2001 – May 2001

## Olympiad Corner

*The 2000 Canadian Mathematical Olympiad*

**Problem 1.** At 12:00 noon, Anne, Beth and Carmen begin running laps around a circular track of length three hundred meters, all starting from the same point on the track. Each jogger maintains a constant speed in one of the two possible directions for an indefinite period of time. Show that if Anne's speed is different from the other two speeds, then at some later time Anne will be at least one hundred meters from each of the other runners. (Here, distance is measured along the shorter of the two arcs separating two runners).

**Problem 2.** A *permutation* of the integers 1901, 1902, …, 2000 is a sequence $a_1$, $a_2$, …, $a_{100}$ in which each of those integers appears exactly once. Given such a permutation, we form the sequence of partial sums

$$s_1 = a_1,\ s_2 = a_1 + a_2,$$
$$s_3 = a_1 + a_2 + a_3, ..., s_{100} = a_1 + a_2 + \cdots + a_{100}.$$

***(continued on page 4)***



**Acknowledgment**: Thanks to Elina Chiu, MATH Dept, HKUST for general assistance.

**On-line:** http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is *June 30, 2001*.

For individual subscription for the next five issues for the 01-02 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

Dr. Kin-Yin Li
Department of Mathematics
Hong Kong University of Science and Technology
Clear Water Bay, Kowloon, Hong Kong

Fax: 2358-1643
Email: makyli@ust.hk

# Base n Representations

*Kin Y. Li*

When we write down a number, it is understood that the number is written in base 10. We learn many interesting facts at a very young age. Some of these can be easily explained in terms of base 10 representation of a number. Here is an example.

***Example 1.*** Show that a number is divisible by 9 if and only if the sum of its digits is divisible by 9. How about divisibility by 11?

***Solution.*** Let $M = d_m 10^m + \cdots + d_1 10 + d_0$, where $d_i$ = 0, 1, 2, …, 9. The binomial theorem tells us $10^k = (9+1)^k = 9N_k + 1$. So

$$M = d_m(9N_m + 1) + \cdots + d_1(9+1) + d_0$$
$$= 9(d_m N_m + \cdots + d_1) + (d_m + \cdots + d_1 + d_0).$$

Therefore, $M$ is a multiple of 9 if and only if $d_m + \cdots + d_1 + d_0$ is a multiple of 9.

Similarly, we have $10^k = 11N'_k + (-1)^k$. So $M$ is divisible by 11 if and only if $(-1)^m d_m + \cdots - d_1 + d_0$ is divisible by 11.

*Remarks*. In fact, we can also see that the remainder when $M$ is divided by 9 is the same as the remainder when the sum of the digits of $M$ is divided by 9. Recall the notation $a \equiv b \pmod{c}$ means $a$ and $b$ have the same remainder when divided by $c$. So we have $M \equiv d_m + \cdots + d_1 + d_0$ (mod 9).

The following is an IMO problem that can be solved using the above remarks.

***Example 2.*** (*1975 IMO*) Let $A$ be the sum of the decimal digits of $4444^{4444}$, and $B$ be the sum of the decimal digits of $A$. Find the sum of the decimal digits of $B$.

***Solution.*** Since $4444^{4444} < (10^5)^{4444} = 10^{22220}$, so $A < 22220 \times 9 = 199980$. Then $B < 1 + 9 \times 5 = 46$ and the sum of the decimal digits of $B$ is at most 3+9=12. Now $4444 \equiv 7$ (mod 9) and $7^3 = 343 \equiv 1$ (mod 9) imply $4444^3 \equiv 1$ (mod 9). Then $4444^{4444} = (4444^3)^{1481} 4444 \equiv 7$ (mod 9). By the remarks above, $A$, $B$ and the sum of the decimal digits of $B$ also have remainder 7 when divided by 9. So the sum of the decimal digits of $B$ being at most 12 must be 7.

Although base 10 representations are common, numbers expressed in other bases are sometimes useful in solving problems, for example, base 2 is common. Here are a few examples using other bases.

***Example 3.*** (*A Magic Trick*) A magician asks you to look at four cards. On the first card are the numbers 1, 3, 5, 7, 9, 11, 13, 15; on the second card are the numbers 2, 3, 6, 7, 10, 11, 14, 15; on the third card are the numbers 4, 5, 6, 7, 12, 13, 14, 15; on the fourth card are the numbers 8, 9, 10, 11, 12, 13, 14, 15. He then asks you to pick a number you saw in one of these cards and hand him all the cards that have that number on them. Instantly he knows the number. Why?

***Solution.*** For $n$ = 1, 2, 3, 4, the numbers on the $n$-th card have the common feature that their $n$-th digits from the end in base 2 representation are equal to 1. So you are handing the base 2 representation of your number to the magician. As the numbers are less than $2^4$, he gets your number easily.

*Remarks.* A variation of this problem is the following. A positive integer less than $2^4$ is picked at random. What is the least number of yes-no questions you can ask that always allow you to know the number? Four questions are enough as you can ask if each of the four digits of the number in base 2 is 1 or not. Three questions are not enough as there are 15 numbers and three questions can only provide $2^3 = 8$ different yes-no combinations.

***Example 4.*** (*Bachet's Weight Problem*) Give a set of distinct integral weights that allowed you to measure any object having weight $n = 1, 2, 3, \ldots, 40$ on a balance. Can you do it with a set of no more than four distinct integral weights?

***Solution.*** Since the numbers 1 to 40 in base 2 have at most 6 digits, we can do it with the set 1, 2, 4, 8, 16, 32. To get a set with fewer weights, we observe that we can put weights from this set on both sides of the balance! Consider the set of weights 1, 3, 9, 27. For example to determine an object with weight 2, we can put it with a weight of 1 on one side to balance a weight of 3 on the other side. Note the sum of 1, 3, 9, 27 is 40. For any integer $n$ between 1 and 40, we can write it in base 3. If the digit 2 appears, change it to $3-1$ so that $n$ can be written as a *unique* sum and difference of 1, 3, 9, 27. For example, 22 = $2 \cdot 9$ + 3 + 1 = (3 – 1)9 + 3 + 1 = 27 – 9 + 3 + 1 suggests we put the weights of 22 with 9 on one side and the weights of 27, 3, 1 on the other side.

***Example 5.*** (*1983 IMO*) Can you choose 1983 pairwise distinct nonnegative integers less than $10^5$ such that no three are in arithmetic progression?

***Solution.*** Start with 0, 1 and at each step add the *smallest* integer which is not in arithmetic progression with any two preceding terms. We get 0, 1, 3, 4, 9, 10, 12, 13, 27, 18, … . In base 3, this sequence is

$$0, 1, 10, 11, 100, 101, 110, 111, 1000, 1001, \ldots$$

(Note this sequence is the nonnegative integers in base 2.) Since 1982 in base 2 is 11110111110, so switching this from base 3 to base 10, we get the 1983[th] term of the sequence is $87843 < 10^5$. To see this sequence works, suppose $x, y, z$ with $x < y < z$ are three terms of the sequence in arithmetic progression. Consider the rightmost digit in base 3 where $x$ differs from $y$, then that digit for $z$ is a 2, a contradiction.

***Example 6.*** Let $[r]$ be the greatest integer less than or equal to $r$. Solve the equation

$$[x]+[2x]+[4x]+[8x]$$
$$+[16x]+[32x]=12345.$$

***Solution.*** If $x$ is a solution, then since $r-1<[r]\le r$, we have $63x - 6 < 12345 \le 63x$. It follows that $195 < x < 196$. Now write the number $x$ in base 2 as $11000011.abcde\ldots$, where the digits $a, b, c, d, e, \ldots$ are 0 or 1. Substituting this into the equation, we will get $12285 + 31a + 15b + 7c + 3d + e = 12345$. Then $31a + 15b + 7c + 3d + e = 60$, which is impossible as the left side is at most 31 + 15 + 7 + 3 + 1 = 57. Therefore, the equation has no solution.

***Example 7.*** (*Proposed by Romania for 1985 IMO*) Show that the sequence $\{a_n\}$ defined by $a_n = [n\sqrt{2}]$ for $n$ = 1, 2, 3, … (where the brackets denote the greatest integer function) contains an infinite number of integral powers of 2.

***Solution.*** Write $\sqrt{2}$ in base 2 as $b_0.b_1b_2b_3\ldots$, where each $b_i = 0$ or 1. Since $\sqrt{2}$ is irrational, there are infinitely many $b_k = 1$. If $b_k = 1$, then in base 2, $2^{k-1}\sqrt{2} = b_0 \cdots b_{k-1}.b_k \cdots$ . Let $m = [2^{k-1}\sqrt{2}]$, then

$$2^{k-1}\sqrt{2}-1<[2^{k-1}\sqrt{2}]=m<2^{k-1}\sqrt{2}-\frac{1}{2}.$$

Multiplying by $\sqrt{2}$ and adding $\sqrt{2}$, we get $2^k < (m+1)\sqrt{2} < 2^k + \frac{\sqrt{2}}{2}$. Then $[(m+1)\sqrt{2}] = 2^k$.

***Example 8.*** (*American Mathematical Monthly, Problem 2486*) Let $p$ be an odd prime number. For any positive integer $k$, show that there exists a positive integer $m$ such that the rightmost $k$ digits of $m^2$, when expressed in the base $p$, are all 1's.

***Solution.*** We prove by induction on $k$. For $k = 1$, take $m = 1$. Next, suppose $m^2$ in base $p$, ends in $k$ 1's, i.e.

$$m^2 = 1 + p + \cdots + p^{k-1} + (ap^k + \cdots).$$

This implies $m$ is not divisible by $p$. Let gcd stand for greatest common divisor (or highest common factor). Then gcd($m$, $p$) = 1. Now

$$(m+cp^k)^2 = m^2 + 2mcp^k + c^2p^{2k}$$
$$= 1 + p + \cdots + p^{k-1} + (a+2mc)p^k + \cdots.$$

Since $\gcd(2m, p) = 1$, there is a positive integer $c$ such that $(2m)c \equiv 1 - a \pmod{p}$. This implies $a + 2mc$ is of the form $1 + Np$ and so $(m+cp^k)^2$ will end in at least $(k+1)$ 1's as required.

***Example 9.*** Determine which binomial coefficients $C_r^n = \dfrac{n!}{r!(n-r)!}$ are odd.

***Solution.*** We remark that modulo arithmetic may be extended to polynomials with integer coefficients. For example, $(1+x)^2 = 1 + 2x + x^2 \equiv 1 + x^2$ (mod 2). If $n = a_m + \cdots + a_1$, where the $a_i$'s are distinct powers of 2. We have $(1+x)^{2^k} \equiv 1 + x^{2^k}$ (mod 2) by induction on $k$ and so

$$(1+x)^n \equiv (1+x^{a_m})\cdots(1+x^{a_1}) \pmod{2}.$$

The binomial coefficient $C_r^n$ is odd if and only if the coefficient of $x^r$ in $(1+x^{a_m})\cdots(1+x^{a_1})$ is 1, which is equivalent to $r$ being 0 or a sum of one or more of the $a_i$'s. For example, if $n = 21 = 16 + 4 + 1$, then $C_r^n$ is odd for $r$ = 0, 1, 4, 5, 16, 17, 20, 21 only.

***Example 10.*** (*1996 USAMO*) Determine (with proof) whether there is a subset $X$ of the integers with the following property: for any integer $n$ there is exactly one solution of $a + 2b = n$ with $a, b \in X$.

This is a difficult problem. Here we will try to lead the reader to a solution. For a problem that we cannot solve, we can try to change it to an easier problem. How about changing the problem to positive integers, instead of integers? At least we do not have to worry about negative integers. That is still not too obvious how to proceed. So can we change it to an even simpler problem? How about changing 2 to 10?

***(continued on page 4)***

# Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, home (or **email**) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science & Technology, Clear Water Bay, Kowloon*. The deadline for submitting solutions is ***June 30, 2001***.

**Problem 126.** Prove that every integer can be expressed in the form $x^2 + y^2 - 5z^2$, where *x, y, z* are integers.

**Problem 127.** For positive real numbers *a, b, c* with *a + b + c = abc*, show that

$$\frac{1}{\sqrt{1+a^2}} + \frac{1}{\sqrt{1+b^2}} + \frac{1}{\sqrt{1+c^2}} \le \frac{3}{2},$$

and determine when equality occurs. (*Source: 1998 South Korean Math Olympiad*)

**Problem 128.** Let *M* be a point on segment *AB*. Let *AMCD*, *BEHM* be squares on the same side of *AB*. Let the circumcircles of these squares intersect at *M* and *N*. Show that *B, N, C* are collinear and *H* is the orthocenter of $\Delta ABC$. (*Source: 1979 Henan Province Math Competition*)

**Problem 129.** If *f(x)* is a polynomial of degree 2*m*+1 with integral coefficients for which there are 2*m*+1 integers $k_1, k_2, \ldots, k_{2m+1}$ such that $f(k_i) = 1$ for $i = 1, 2, \cdots, 2m+1$, prove that *f(x)* is not the product of two nonconstant polynomials with integral coefficients.

**Problem 130.** Prove that for each positive integer *n*, there exists a circle in the *xy*-plane which contains exactly *n* lattice points in its interior, where a *lattice point* is a point with integral coordinates. (*Source: H. Steinhaus, Zadanie 498, Matematyka 10 (1957), p. 58*)

*****************

## *Solutions*

*****************

**Problem 121.** Prove that any integer greater than or equal to 7 can be written as a sum of two relatively prime integers, both greater than 1.
(Two integers are *relative prime* if they share no common positive divisor other than 1. For example, 22 and 15 are relatively prime, and thus 37 = 22 + 15 represents the number 37 in the desired way.) (*Source: Second Bay Area Mathematical Olympaid*)

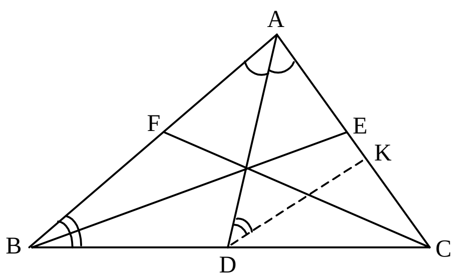
***Solution.*** **CHAO Khek Lun Harold** (St. Paul's College, Form 6), **CHIU Yik Yin** (St. Joseph's Anglo-Chinese School, Form 5), **CHONG Fan Fei** (Queen's College, Form 4), **CHUNG Tat Chi** (Queen Elizabeth School, Form 4), **LAW Siu Lun** (Ming Kei College, Form 6), **NG Cheuk Chi** (Tsuen Wan Public Ho Chuen Yiu Memorial College), **WONG Wing Hong** (La Salle College, Form 3) & **YEUNG Kai Sing** (La Salle College, Form 4).

For an integer $n \ge 7$, $n$ is either of the form $2j + 1$ ($j > 2$) or $4k$ ($k > 1$) or $4k + 2$ ($k > 1$). If $n = 2j + 1$, then $j$ and $j + 1$ are relatively prime and $n = j + (j + 1)$. If $n = 4k$, then $2k - 1$ $(>1)$ and $2k + 1$ are relatively prime and $n = (2k - 1) + (2k + 1)$. If $n = 4k + 2$, then $2k - 1$ and $2k + 3$ are relatively prime and $n = (2k - 1) + (2k + 1)$.

*Other commended solvers:* **HON Chin Wing** (Pui Ching Middle School, Form 6), **LEUNG Wai Ying** (Queen Elizabeth School, Form 6), **NG Ka Chun Bartholomew** (Queen Elizabeth School, Form 6) & **WONG Tak Wai Alan** (University of Toronto).

**Problem 122.** Prove that the product of the lengths of the three angle bisectors of a triangle is less than the product of the lengths of the three sides. (*Source: 1957 Shanghai Junior High School Math Competition*).

***Solution.*** **YEUNG Kai Sing** (La Salle College, Form 4).

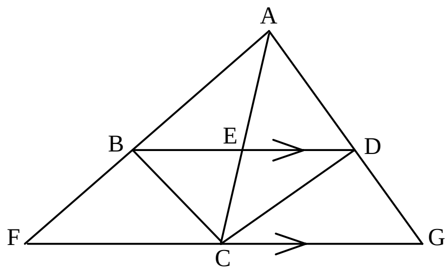



Let *AD*, *BE* and *CF* be the angle bisectors of $\Delta ABC$, where *D* is on *BC*, *E* is on *CA* and *F* is on *AB*. Since $\angle ADC = \angle ABD + \angle BAD > \angle ABD$, there is a point *K* on *CA* such that $\angle ADK = \angle ABD$. Then $\Delta ABD$ is similar to $\Delta ADK$. So *AB/AD* = *AD/AK*. Then $AD^2 = AB \cdot AK < AB \cdot CA$. Similarly, $BE^2 < BC \cdot AB$ and $CF^2 < CA \cdot BC$. Multiplying these in-equalities and taking square roots, we get $AD \cdot BE \cdot CF < AB \cdot BC \cdot CA$.

*Other commended solvers:* **CHAO Khek Lun Harold** (St. Paul's College, Form 6), **CHIU Yik Yin** (St. Joseph's Anglo-Chinese School, Form 5), **HON Chin Wing** (Pui Ching Middle School, Form 6) & **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).

**Problem 123.** Show that every convex quadrilateral with area 1 can be covered by some triangle of area at most 2. (*Source: 1989 Wuhu City Math Competition*)

***Solution.*** **CHAO Khek Lun Harold** (St. Paul's College, Form 6), **CHUNG Tat Chi** (Queen Elizabeth School, Form 4) & **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).




Let *ABCD* be a convex quadrilateral with area 1. Let *AC* meet *BD* at *E*. Without loss of generality, suppose $AE \ge EC$. Construct $\Delta AFG$, where lines *AB* and *AD* meet the line parallel to *BD* through *C* at *F* and *G* respectively. Then $\Delta ABE$ is similar to $\Delta AFC$. Now $AE \ge EC$ implies $AB \ge BF$. Let $[XY \cdots Z]$ denote the area of polygon $XY \cdots Z$, then $[ABC] \ge [FBC]$. Similarly, $[ADC] \ge [GDC]$. Since $[ABC] + [ADC] = [ABCD] = 1$, so $[AFG] = [ABCD] + [FBC] + [GDC] \le 2[ABCD] = 2$ and $\Delta AFG$ covers *ABCD*.

**Problem 124**. Find the least integer *n* such that among every $n$ distinct numbers $a_1, a_2, \ldots, a_n$, chosen from [1,1000], there always exist $a_i, a_j$ such that

$$0 < a_i - a_j < 1 + 3\sqrt[3]{a_i a_j}.$$

(*Source: 1990 Chinese Team Training Test*)

***Solution.*** **CHAO Khek Lun Harold** (St. Paul's College, Form 6), **CHUNG Tat Chi** (Queen Elizabeth School, Form 4) & **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).

For $n \le 10$, let $a_i = i^3 \quad (i = 1, 2, \cdots, n)$. Then the inequality cannot hold since $0 < i^3 - j^3$ implies $i - j \ge 1$ and so $i^3 - j^3 = (i-j)^3 + 3ij(i-j) \ge 1 + 3ij$. For $n = 11$, divide [1,1000] into intervals $[k^3 + 1, (k+1)^3]$ for $k = 0, 1, \ldots, 9$. By pigeonhole principle, among any 11 distinct numbers $a_1, a_2, \cdots, a_{11}$ in [1, 1000], there always exist $a_i, a_j$, say $a_i > a_j$, in the same interval. Let $x = \sqrt[3]{a_i}$ and $y = \sqrt[3]{a_j}$, then $0 < x - y < 1$ and $0 < a_i - a_j = x^3 - y^3 = (x-y)^3 + 3xy(x-y) < 1 + 3xy = 1 + 3\sqrt[3]{a_i a_j}$.

*Other commended solvers:* **NG Cheuk Chi** (Tsuen Wan Public Ho Chuen Yiu Memorial College), **NG Ka Chun Bartholomew** (Queen Elizabeth School, Form 6), **WONG Wing Hong** (La Salle College, Form 3) & **YEUNG Kai Sing** (La Salle College, Form 4).

**Problem 125.** Prove that

$$\tan^2 1^\circ + \tan^2 3^\circ + \tan^2 5^\circ + \cdots + \tan^2 89^\circ$$

is an integer.

***Solution.*** **CHAO Khek Lun** (St. Paul's College, Form 6).

For $\theta = 1^\circ, 3^\circ, 5^\circ, \ldots, 89^\circ$, we have $\cos\theta \ne 0$ and $\cos 90\theta = 0$. By de Moivre's theorem, $\cos 90\theta + i \sin 90\theta = (\cos\theta + i\sin\theta)^{90}$. Taking the real part of both sides, we get

$$0 = \sum_{k=0}^{45} (-1)^k C_{2k}^{90} \cos^{90-2k}\theta \sin^{2k}\theta.$$

Dividing by $\cos^{90}\theta$ on both sides and letting $x = \tan^2\theta$, we get

$$0 = \sum_{k=0}^{45} (-1)^k C_{2k}^{90} x^k.$$

So $\tan^2 1^\circ, \tan^2 3, \tan^2 5^\circ, \ldots, \tan^2 89^\circ$ are the 45 roots of this equation. Therefore, their sum is $C_{88}^{90} = 4005$.

## Olympiad Corner

***(continued from page 1)***

How many of these permutations will have no terms of the sequence $s_1, \ldots, s_{100}$ divisible by three?

**Problem 3.** Let $A = (a_1, a_2, \cdots, a_{2000})$ be a sequence of integers each lying in the interval [-1000, 1000]. Suppose that the entries in $A$ sum to 1. Show that some nonempty subsequence of $A$ sums to zero.

**Problem 4.** Let $ABCD$ be a convex quadrilateral with

$$\angle CBD = 2\angle ADB,$$
$$\angle ABD = 2\angle CDB$$

and $AB = CB$.

Prove that $AD = CD$.

**Problem 5.** Suppose that the real numbers $a_1, a_2, \cdots, a_{100}$ satisfy

$$a_1 \ge a_2 \ge \cdots \ge a_{100} \ge 0,$$
$$a_1 + a_2 \le 100$$

and $a_3 + a_4 + \cdots + a_{100} \le 100.$

Determine the maximum possible value of $a_1^2 + a_2^2 + \cdots + a_{100}^2$, and find all possible sequences $a_1, a_2, \cdots, a_{100}$ which achieve this maximum.

## Base n Representations

***(continued from page 2)***

Now try an example, say $n = 12345$. We can write $n$ in more than one ways in the form $a + 10b$. Remember we want $a$, $b$ to be unique in the set $X$. Now for $b$ in $X$, $10b$ will *shift* the digits of $b$ to the left one space and fill the last digit with a 0. Now we can try writing $n = 12345 = 10305 + 10(204)$. So if we take $X$ to be the positive integers whose even position digits from the end are 0, then the problem will be solved for $n = a + 10b$. How about $n = a + 2b$? If the reader examines the reasoning in the case $a + 10b$, it is easy to see the success comes from separating the digits and observing that multiplying by 10 is a shifting operation in base 10. So for $a + 2b$, we take $X$ to be the set of positive integers whose base 2 even position digits from the end are 0, then the problem is solved for positive integers.

How about the original problem with *integers*? It is tempting to let $X$ be the set of positive or negative integers whose base 2 even position digits from the end are 0. It does not work as the example $1 + 2 \cdot 1 = 3 = 5 + 2(-1)$ shows uniqueness fails. Now what other ways can we describe the set $X$ we used in the last paragraph? Note it is also the set of positive integers whose base 4 representations have only digits 0 or 1. How can we take care of uniqueness and negative integers at the same time? One idea that comes close is the Bachet weights.

The brilliant idea in the official solution of the 1996 USAMO is do things in base $(-4)$. That is, show every integer has a unique representation as $\sum_{i=0}^{k} c_i(-4)^i$, where each $c_i = 0, 1, 2$ or 3 and $c_k \ne 0$. Then let $X$ be the set of integers whose base $(-4)$ representations have only $c_i = 0$ or 1 will solve the problem.

To show that an integer $n$ has a base $(-4)$ representation, find an integer $m$ such that $4^0 + 4^2 + \cdots + 4^{2m} \ge n$ and express

$$n + 3\ (4^1 + 4^3 + \cdots + 4^{2m-1})$$

in base 4 as $\sum_{i=0}^{2m} b_i 4^i$. Now set $c_{2i} = b_{2i}$ and $c_{2i-1} = 3 - b_{2i-1}$. Then $n = \sum_{i=0}^{2m} c_i (-4)^i$.

To show the uniqueness of base $(-4)$ representation of $n$, suppose $n$ has two distinct representations with digits $c_i$'s and $d_i$'s. Let $j$ be the smallest integer such that $c_j \ne d_j$. Then

$$0 = n - n = \sum_{i=j}^{k} (c_i - d_i)(-4)^i$$

would have a nonzero remainder when divided by $4^{j+1}$, a contradiction.