Mathematical Excalibur

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Olympiad Corner

The 10th Winter Camp, Taipei, Taiwan, February 14, 2001.

Problem 1. Determine all integers a and b which satisfy that

$$a^{13}+b^{90}=b^{2001}.$$

Problem 2. Let $\langle a_n\rangle$ be sequence of real numbers satisfying the recurrence relation

$$a_1=k,\ \ a_{n+1}=\left[\sqrt{2}a_n\right],\ \ n=1,2,\ldots.$$

where $[x]$ denotes the largest number which is less or equal than x. Find all positive integers k for which three exist three consecutive terms a_{i-1}, a_i, a_{i+1} satisfy $2a_i = a_{i-1}+a_{i+1}$.

Problem 3. A real number r is said to be *attainable* if there is a triple of positive real numbers (a, b, c) such that a, b, c are not the lengths of any triangle and satisfy the inequality

$$rabc > a^2b+b^2c+c^2a.$$

(a) Determine whether or not $\frac{7}{2}$ is *attainable*.

(b) Find all positive integer n such that n is *attainable*.

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On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***March 23, 2002***.

For individual subscription for the next five issues for the 01-02 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Vector Geometry

Kin Y. Li

A *vector* $\overrightarrow{XY}$ is an object having a magnitude (the length XY) and a direction (from X to Y). Vectors are very useful in solving certain types of geometry problems. First, we will mention some basic concepts related to vectors. Two vectors are considered the same if and only if they have the same magnitudes and directions. A vector $\overrightarrow{OX}$ from the origin O to a point X is called a *position vector*. For convenience, often a position vector $\overrightarrow{OX}$ will simply be denoted by X, when the position of the origin is understood, so that the vector $\overrightarrow{XY} = \overrightarrow{OY}-\overrightarrow{OX}$ will simply be $Y – X$. The length of the position vector $\overrightarrow{OX} = X$ will be denoted by $|X|$. We have the triangle inequality $|X+Y|\le|X|+|Y|$, with equality if and only if $X = tY$ for some $t \ge 0$. Also, $|cX| = |c||X|$ for number c.

For a point P on the line XY, in terms of position vectors, $P = tX + (1 – t)Y$ for some real number t. If P is on the segment XY, then $t = PY/XY \in [0, 1]$.

Next, we will present some examples showing how vectors can be used to solve geometry problems.

Example 1. (*1980 Leningrad High School Math Olympiad*) Call a segment in a convex quadrilateral a *midline* if it joins the midpoints of opposite sides. Show that if the sum of the midlines of a quadrilateral is equal to its semiperimeter, then the quadrilateral is a parallelogram.

Solution. Let $ABCD$ be such a convex quadrilateral. Set the origin at A. The sum of the lengths of the midlines is

$$\frac{|B+C-D|+|D+C-B|}{2}$$

and the semiperimeter is

$$\frac{|B|+|C-D|+|D|+|C-B|}{2}.$$

So

$$|B+C-D|+|D+C-B| = |B|+|C-D|+|D|+|C-B|$$

By triangle inequality, $|B|+|C-D|\ge|B+C-D|$, with equality if and only if $B=t(C-D)$ (or $AB\|CD$). Similarly, $|D|+|C-B|\ge|D+C-B|$, with equality if and only if $AD\|BC$. For the equation to be true, both triangle inequalities must be equalities. In that case, $ABCD$ is a parallelogram.

Example 2. (*Crux Problem 2333*) D and E are points on sides AC and AB of triangle ABC, respectively. Also, DE is not parallel to CB. Suppose F and G are points of BC and ED, respectively, such that $BF:FC = EG:GD = BE:CD$. Show that GF is parallel to the angle bisector of $\angle BAC$.

Solution. Set the origin at A. Then $E = pB$ and $D = qC$ for some $p, q \in (0, 1)$. Let $t = \frac{BF}{FC}$, then $F = \frac{tC+B}{t+1}$ and $G = \frac{tD+E}{t+1} = \frac{tqC+pB}{t+1}$.

Since $BE = tCD$, so $(1 – p)|B| = t(1 – q)|C|$. Thus,

$$F-G=\frac{t(1-q)}{t+1}C+\frac{1-p}{t+1}B$$
$$=\frac{(1-p)|B|}{t+1}\left(\frac{C}{|C|}+\frac{B}{|B|}\right).$$

This is parallel to $\frac{C}{|C|}+\frac{B}{|B|}$, which is in the direction of the angle bisector of $\angle BAC$.

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The *dot product* of two vectors X and Y is the number $X \cdot Y = |X||Y| \cos\theta$, where θ is the angle between the vectors. Dot product has the following properties:

(1) $X \cdot Y = Y \cdot X, (X+Y)\cdot Z = X \cdot Z + Y \cdot Z$ and $(cX)\cdot Y = c(X \cdot Y)$.

(2) $|X|^2 = X \cdot X$, $|X \cdot Y| \le |X||Y|$ and $OX \perp OY$ if and only if $X \cdot Y = 0$.

Example 3. (*1975 USAMO*) Let *A*, *B*, *C*, *D* denote four points in space and *AB* the distance between *A* and *B*, and so on. Show that

$$AC^2 + BD^2 + AD^2 + BC^2 \ge AB^2 + CD^2.$$

Solution. Set the origin at *A*. The inequality to be proved is

$$C\cdot C + (B-D)\cdot(B-D) + D\cdot D + (B-C)\cdot(B-C) \ge B\cdot B + (C-D)\cdot(C-D).$$

After expansion and regrouping, this is the same as $(B-C-D)\cdot(B-C-D) \ge 0$, with equality if and only if $B - C = D = D - A$, i.e. is *BCAD* is a parallelogram.

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For a triangle *ABC*, the position vectors of its centroid is

$$G = \frac{A+B+C}{3}.$$

If we take the circumcenter *O* as the origin, then the position of the orthocenter is $H = A + B + C$ as $\overrightarrow{OH} = 3\overrightarrow{OG}$. Now for the incenter *I*, let *a*, *b*, *c* be the side lengths and *AI* intersect *BC* at *D*. Since $BD{:}CD = c{:}b$ and $DI{:}AI = \frac{ca}{b+c}{:}c = a{:}b+c$, so $D = \frac{bB+cC}{b+c}$ and $I = \frac{aA+bB+cC}{a+b+c}$.

Example 4. (*2nd Balkan Math Olympiad*) Let *O* be the center of the circle through the points *A, B, C*, and let *D* be the midpoint of *AB*. Let *E* be the centroid of triangle *ACD*. Prove that the line *CD* is perpendicular to line *OE* if and only if *AB* = *AC*.

Solution. Set the origin at *O*. Then

$$D = \frac{A+B}{2},$$

$$E = \frac{A+C+D}{3} = \frac{3A+B+2C}{6},$$

$$D - C = \frac{A+B-2C}{2}.$$

Hence $CD \perp OE$ if and only if $(A+B-2C)\cdot(3A+B+2C) = 0$. Since $A \cdot A = B \cdot B = C \cdot C$, this is equivalent to $A\cdot(B-C) = A\cdot B - A\cdot C = 0$, which is the same as $OA \perp BC$, i.e. *AB* = *AC*.

Example 5. (*1990 IMO Unused Problem, Proposed by France*) Given ΔABC with no side equal to another side, let *G*, *I* and *H* be its centroid, incenter and orthocenter, respectively. Prove that $\angle GIH > 90°$.

Solution. Set the origin at the circumcenter. Then

$$H = A + B + C, \quad G = \frac{A+B+C}{3},$$

$$I = \frac{aA+bB+cC}{a+b+c}.$$

We need to show $(G-I)\cdot(H-I) = G\cdot H + I\cdot I - I\cdot(G+H) < 0$. Now $A \cdot A = B\cdot B = C\cdot C = R^2$ and $2B\cdot C = B\cdot B + C\cdot C - (B-C)\cdot(B-C) = 2R^2 - a^2$, … . Hence,

$$G\cdot H = \frac{(A+B+C)\cdot(A+B+C)}{3}$$

$$= 3R^2 - \frac{a^2+b^2+c^2}{3},$$

$$I\cdot I = \frac{(aA+bB+cC)\cdot(aA+bB+cC)}{(a+b+c)^2}$$

$$= R^2 - \frac{abc}{a+b+c},$$

$$I\cdot(G+H) = \frac{4(aA+bB+cC)\cdot(A+B+C)}{3(a+b+c)}$$

$$= 4R^2 - \frac{2[a^2(b+c)+b^2(c+a)+c^2(a+b)]}{3(a+b+c)}.$$

Thus, it is equivalent to proving $(a+b+c)(a^2+b^2+c^2) + 3abc > 2[a^2(b+c) + b^2(c+a) + c^2(a+b)]$, which after expansion and regrouping will become $a(a-b)(a-c) + b(b-c)(b-a) + c(c-a)(c-b) > 0$. To obtain this inequality, without loss of generality, assume $a \ge b \ge c$. Then $a(a-b)(a-c) \ge b(a-b)(b-c)$ so that the sum of the first two terms is nonnegative. As the third term is also nonnegative, the above inequality is true.

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The *cross product* of two vectors *X* and *Y* is a vector $X \times Y$ having magnitude $|X||Y| \sin\theta$, where θ is the angle between the vectors, and direction perpendicular to the plane of *X* and *Y* satisfying the right hand rule. Cross product has the following properties:

(1) $X\times Y = -Y\times X$, $(X+Y)\times Z = X\times Z + Y\times Z$ and $(cX)\times Y = c(X\times Y)$.

(2) $\frac{|X\times Y|}{2}$ is the area of triangle *XOY*. When $X, Y \ne O$, $X\times Y = 0$ if and only if *X*, *O*, *Y* are collinear.

Example 6. (*1984 Annual Greek High School Competition*) Let $A_1\ A_2\ A_3\ A_4\ A_5\ A_6$ be a convex hexagon having its opposite sides parallel. Prove that triangles $A_1\ A_3\ A_5$ and $A_2\ A_4\ A_6$ have equal areas.

Solution. Set the origin at any point. As the opposite sides are parallel, $(A_1 - A_2)\times(A_4 - A_5) = 0$, $(A_3 - A_2)\times(A_5 - A_6) = 0$ and $(A_3 - A_4)\times(A_6 - A_1) = 0$. Expanding these equations and adding them, we get $A_1\times A_3 + A_3\times A_5 + A_5\times A_1 = A_2\times A_4 + A_4\times A_6 + A_6\times A_2$. Now

$$[A_1\ A_3\ A_5] = \frac{|(A_1-A_3)\times(A_1-A_5)|}{2}$$

$$= \frac{|A_1\times A_3 + A_3 + A_5 + A_5\times A_1|}{2}.$$

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, home (or **email**) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is ***March 23, 2002***.

Problem 141. Ninety-eight points are given on a circle. Maria and José take turns drawing a segment between two of the points which have not yet been joined by a segment. The game ends when each point has been used as the endpoint of a segment at least once. The winner is the player who draws the last segment. If José goes first, who has a winning strategy? (*Source: 1998 Iberoamerican Math Olympiad*)

Problem 142. *ABCD* is a quadrilateral with $AB \parallel CD$. *P* and *Q* are on sides *AD* and *BC* respectively such that $\angle APB = \angle CPD$ and $\angle AQB = \angle CQD$. Prove that *P* and *Q* are equal distance from the intersection point of the diagonals of the quadrilateral. (*Source: 1994 Russian Math Olympiad, Final Round*)

Problem 143. Solve the equation cos cos cos cos *x* = sin sin sin sin *x*. (*Source: 1994 Russian Math Olympiad, 4th Round*)

Problem 144. (*Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain*) Find all (non-degenerate) triangles *ABC* with consecutive integer sides *a, b, c* and such that *C = 2A*.

Problem 145. Determine all natural numbers $k > 1$ such that, for some distinct natural numbers *m* and *n*, the numbers $k^m + 1$ and $k^n + 1$ can be obtained from each other by reversing the order of the digits in their decimal representations. (*Source: 1992 CIS Math Olympiad*)

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Solutions

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Problem 136. For a triangle *ABC*, if sin*A*, sin*B*, sin*C* are rational, prove that cos*A*, cos*B*, cos*C* must also be rational. If cos*A*, cos*B*, cos*C* are rational, must at least one of sin*A*, sin*B*, sin*C* be rational?

Solution. **CHAN Wai Hong** (STFA Leung Kau Kui College, Form 6), **CHAO Khek Lun Harold** (St. Paul's College, Form 7), **CHIU Yik Yin** (St. Joseph's Anglo-Chinese School, Form 6), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **LO Chi Fai** (STFA Leung Kau Kui College, Form 6), **WONG Tak Wai Alan** (University of Toronto), **WONG Tsz Wai** (Hong Kong Chinese Women's Club College, Form 6) and **WONG Wing Hong** (La Salle College, Form 4).

If sin*A*, sin*B*, sin*C* are rational, then by cosine law and sine law,

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}-\frac{a}{b}\frac{a}{c}\right)$$

$$= \frac{1}{2}\left(\frac{\sin B}{\sin C}+\frac{\sin C}{\sin B}-\frac{\sin A}{\sin B}\frac{\sin A}{\sin C}\right)$$

is rational. Similarly, cos*B* and cos*C* are rational. In the case of an equilateral triangle, cos*A* = cos*B* = cos*C* = cos 60° = $\frac{1}{2}$ is rational, but sin*A* = sin*B* = sin*C* = $\sin 60^\circ = \frac{\sqrt{3}}{2}$ is irrational.

Other commended solvers: **LEE Tsun Man Clement** (St. Paul's College, Form 3), **LOONG King Pan Campion** (STFA Leung Kau Kui College, Form 6), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6) and **TANG Chun Pong** (La Salle College, Form 4).

Problem 137. Prove that for every positive integer *n*,

$$(\sqrt{3}+\sqrt{2})^{1/n} + (\sqrt{3}-\sqrt{2})^{1/n}$$

is irrational.

Solution. **CHAO Khek Lun Harold** (St. Paul's College, Form 7) and **LEUNG Wai Ying** (Queen Elizabeth School, Form 7).

Let $x = (\sqrt{3}+\sqrt{2})^{1/n}$. Since $(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2}) = 1$, $x^{-1} = (\sqrt{3}-\sqrt{2})^{1/n}$. If $x + x^{-1}$ is rational, then $x^2 + x^{-2} = (x + x^{-1})^2 - 2$ is also rational. Since

$$x^{k+1} + x^{-(k+1)} = (x + x^{-1})\left(x^k + x^{-k}\right) - \left(x^{k-1} + x^{-(k-1)}\right),$$

by math induction, $x^n + x^{-n} = 2\sqrt{3}$ would be rational, a contradiction. Therefore, $x + x^{-1}$ is irrational.

Other commended solvers: **CHAN Wai Hong** (STFA Leung Kau Kui College, Form 6), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6) and **WONG Wing Hong** (La Salle College, Form 4).

Problem 138. (*Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain*) If *a + b* and *a – b* are relatively prime integers, find the greatest common divisor (or the highest common factor) of $2a + (1 + 2a)(a^2 - b^2)$ and $2a(a^2 + 2a - b^2)(a^2 - b^2)$.

Solution. **CHAO Khek Lun Harold** (St. Paul's College, Form 7) and **LEUNG Wai Ying** (Queen Elizabeth School, Form 7).

Let (*r, s*) denote the greatest common divisor (or highest common factor) of *r* and *s*. If (*r, s*) = 1, then for any prime *p* dividing *rs*, either *p* divides *r* or *p* divides *s*, but not both. In particular *p* does not divide *r* + *s*. So (*r* + *s, rs*) = 1. Let *x* = *a* + *b* and *y* = *a* – *b*. Then

$$2a + (1 + 2a)(a^2 - b^2)$$
$$= x + y + (1 + x + y)xy$$
$$= (x + y + xy) + (x + y)xy$$

and

$$2a(a^2 + 2a - b^2)(a^2 - b^2)$$
$$= (x + y)(xy + x + y)xy.$$

Now (*x, y*) = 1 implies (*x* + *y, xy*) = 1. Repeating this twice, we get

$$(x + y + xy, (x + y)\, xy) = 1$$

and

$$((x + y + xy + (x + y)xy,$$
$$(x + y + xy)(x + y)xy) = 1.$$

So the answer to the problem is 1.

Other commended solvers: **LEE Tsun Man Clement** (St. Paul's College, Form 3), **POON Yiu Keung** (HKUST, Math Major, Year 1), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6), **TANG Chun Pong** (La Salle College, Form 4), **WONG Chun Ho** (STFA Leung Kau Kui College, Form 7) and **WONG Wing Hong** (La Salle College, Form 4).

Problem 139. Let a line intersect a pair of concentric circles at points *A, B, C, D* in that order. Let *E* be on the outer circle and *F* be on the inner circle such that chords *AE* and *BF* are parallel. Let

G and H be points on chords BF and AE that are the feet of perpendiculars from C to BF and from D to AE, respectively. Prove that $EH = FG$. (*Source: 1958 Shanghai City Math Competition*)

Solution. **WONG Tsz Wai** (Hong Kong Chinese Women's Club College, Form 6).

Let M be the midpoint of BC (and AD). Since $\angle DHA = 90^\circ$, $\angle ADH = \angle DHM$. Since $BF \parallel AE$, $\angle BAE = \angle FEA$ by symmetry with respect to the diameter perpendicular to BF and AE. Now $\angle FEA = \angle BAE = 90^\circ - \angle ADH = 90^\circ - \angle DHM = \angle AHG$. So $EF \parallel HG$. Since $EH \parallel FG$ also, $EFGH$ is a parallelogram. Therefore, $EH = FG$.

Other commended solvers: **CHAO Khek Lun Harold** (St. Paul's College, Form 7), **CHUNG Tat Chi** (Queen Elizabeth School, Form 5), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6) and **WONG Chun Ho** (STFA Leung Kau Kui College, Form 7).

Problem 140. A convex pentagon has five equal sides. Prove that the interior of the five circles with the five sides as diameters do not cover the interior of the pentagon.

Solution. **LEUNG Wai Ying** (Queen Elizabeth School, Form 7).

Let the pentagon be $A_1A_2A_3A_4A_5$ and $2r$ be the common length of the sides. Let M_{ij} be the midpoint of A_iA_j and C_i be the circle with diameter A_iA_{i+1} for i = 1, 2, 3, 4, 5 (with $A_6 = A_1$). Since $540 - 3\cdot 60 = 2\cdot 180$ and $\angle A_i <$ 180°, there are at least 3 interior angles (in particular, two adjacent angles) greater than 60°. So we may suppose $\angle A_1, \angle A_2 > 60^\circ$. Since $A_3A_4 = A_5A_4$, we get $A_4M_{35} \perp A_3A_5$. Then M_{35} is on C_3, C_4 and the points on the ray from A_4 to M_{35} lying beyond M_{35} is outside C_3, C_4.

Next, since $\angle A_1 > 60^\circ$ and $A_1A_2 = A_1A_5$, A_2A_5 is the longest side of $\Delta A_1A_2A_5$. By the midpoint theorem, $M_{23}M_{35} = \dfrac{A_2A_5}{2} > \dfrac{A_1A_2}{2} = r$ so that M_{35} is outside C_2. Similarly, M_{35} is outside C_5. If M_{35} is not outside C_1, then $A_2M_{35} < A_1A_2 = A_2A_3$ and $\angle A_1M_{35}A_2 \geq 90^\circ$. Since $A_3M_{35} < A_3A_4 = A_2A_3$ also, A_2A_3 must be the longest side of $\Delta A_2A_3M_{35}$. Then $\angle A_2M_{35}A_3 > 60^\circ$. Similarly, $\angle A_1M_{35}A_5 > 60^\circ$. Then, we have $\angle A_1M_{35}A_2 < 60^\circ$, a contradiction. So M_{35} is outside C_1, too.

For i = 1, 2, 5 let $d_i = M_{35}M_{i,i+1} - r > 0$. Let d be the distance from M_{35} to the intersection point of the pentagon with the ray from A_4 to M_{35} lying beyond M_{35}. Choose a point X beyond M_{35} on the ray from A_4 to M_{35} with $XM_{35} < d, d_1, d_2$ and d_5. Then X is inside the pentagon and is outside C_3, C_4. Also, for $i = 1, 2, 5$,

$$XM_{i,i+1} > M_{35}M_{i,i+1} - XM_{35}$$
$$= r + d_i - XM_{35} > r$$

so that X is outside C_1, C_2, C_5.

Comments: The point M_{35} is enough for the solution as it is not in the interior of the 5 circles. The point X is better as it is not even on any of the circles.

Olympiad Corner

(continued from page 1)

Problem 4. Let O be the center of excircle of ΔABC touching the side BC internally. Let M be the midpoint of AC, P the intersection point of MO and BC. Prove that $AB = BP$, if $\angle BAC = 2\angle ACB$.

Problem 5. Given that 21 regular pentagons $P_1, P_2, \ldots, P_{21}$ are such that for any $k \in \{1, 2, 3, \ldots, 20\}$, all the vertices of P_{k+1} are the midpoints of the sides of P_k. Let S be the set of the vertices of $P_1, P_2, \ldots, P_{21}$. Determine the largest positive integer n for which there always exist four points A, B, C, D from S such that they are the vertices of an isosceles trapezoid and with the same color if we use n kinds of different colors to paint the element of S.

Vector Geometry

(continued from page 2)

Similarly,

$$[A_2A_4A_6] = \frac{|A_2\times A_4 + A_4\times A_6 + A_6\times A_2|}{2}.$$

So $[A_1A_3A_5] = [A_2A_4A_6]$.

Example 7. (*1996 Balkan Math Olympiad*) Let $ABCDE$ be a convex pentagon and let M, N, P, Q, R be the midpoints of sides AB, BC, CD, DE, EA, respectively. If the segments AP, BQ, CR, DM have a common point, show that this point also lies on EN.

Solution. Set the origin at the commom point. Since, A, P and the origin are collinear,

$$0 = A\times P = A\times\left(\frac{C+D}{2}\right) = \frac{A\times C + A\times D}{2}.$$

So $A\times C = D\times A$. Similarly, $B\times D = E\times B$, $C\times E = A\times C$, $D\times A = B\times D$. Then $E\times B = C\times E$. So $E\times N = E\times \left(\dfrac{B+C}{2}\right) = 0$, which implies E, N and the origin are collinear.

Example 8. (*16th Austrian Math Olympiad*) A line interesects the sides (or sides produced) BC, CA, AB of triangle ABC in the points A_1, B_1, C_1, respectively. The points A_2, B_2, C_2 are symmetric to A_1, B_1, C_1 with respect to the midpoints of BC, CA, AB, respectively. Prove that A_2, B_2, C_2 are collinear.

Solution. Set the origin at a vertex, say C. Then $A_1 = c_1B, B_1 = c_2A, C_1 = A + c_3(B - A)$ for some constants c_1, c_2, c_3. Since A_1, B_1, C_1, are collinear,

$$0 = (B_1 - A_1)\times(C_1 - A_1)$$
$$= (c_1 - c_1c_2 - c_1c_3 + c_2c_3)A\times B.$$

Since

$$A_2 = B - A_1 = (1 - c_1)B,$$
$$B_2 = A - B_1 = (1 - c_2)A$$

and

$$C_2 = (A + B) - C_1 = c_3A + (1 - c_3)B,$$

so A_2, B_2, C_2, are collinear if and only if

$$0 = (B_2 - A_2)\times(C_2 - A_2)$$
$$= (c_1 - c_1c_2 - c_1c_3 + c_2c_3)A\times B,$$

which is true.