Mathematical Excalibur

Volume 8, Number 5 *November 2003 – December 2003*

Olympiad Corner

The 2003 USA Mathematical Olympiad took place on May 1. Here are the problems.

Problem 1. Prove that for every positive integer n there exists an n-digit number divisible by 5^n all of whose digits are odd.

Problem 2. A convex polygon P in the plane is dissected into smaller convex polygons by drawing all of its diagonals. The lengths of all sides and all diagonals of the polygons P are rational numbers. Prove that the lengths of all sides of all polygons in the dissection are also rational numbers.

Problem 3. Let $n \neq 0$. For every sequence of integers $A = a_0, a_1, a_2, \ldots, a_n$ satisfying $0 \le a_i \le i$, for $i = 0, \ldots, n$, define another sequence $t(A) = t(a_0), t(a_1), t(a_2), \ldots, t(a_n)$ by setting $t(a_i)$ to be the number of terms in the sequence A that precede the terms a_i and are different from a_i. Show that, starting from any sequence A as above, fewer than n applications of the transformation t lead to a sequence B such that $t(B) = B$.

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Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***February 28, 2004***.

For individual subscription for the next five issues for the 03-04 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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集與子集族

梁達榮

眾所周知，如果 S 是一個含 n 個元素的集，則它有 2^n 個子集，(包含空集及 S 本身)。不過如果選取子集的條件有所限制，例如子集只能有最多 k 個元素，或者所選取的兩個子集都必須相交（或不相交）等，則所能選取的子集必相應減少。

反過來說，如果 S 含一固定數目的子集，而這些子集又適合某些條件，則 n 的值不可能太小，又或者可以推到這些子集必須含有一些共同元素等。

這一類問題，泛稱集與子集族的問題，已經有很多有趣的成果。另外這些問題很能考驗學生的分析能力，並且需要的數學知識較少，所以在數學比賽中亦經常出現。先舉一個較簡單的例子。

<u>例一</u>：(蘇聯數學競賽 1965) 有一個委員會共舉行了 40 次會議，每次會議共有 10 人參加。並且每 2 個委員最多共一起參加同一會議 1 次。試證該委會員組成人數必多於 60 人。

<u>證明</u>：每一個會議有 10 人參加，因此共有 $C_2^{10} = 45$ “對”委員。按條件每一對委員不會在其他會議中出現，即 40 個會議共產生 40 × 45 = 1,800 不同的委員對。

如果該委員會有 n 人，則有 $C_2^n = \dfrac{n(n-1)}{2}$ 不同的對。所以必有 $1800 \le \dfrac{n(n-1)}{2}$，解之即得 $n > 60$。

如果所有委員組成一個集，則每一個會議參加的 10 人就是這個集的一個子集。按條件即是說任意這樣的兩個子集的交集最多包含一個元素。現在這樣的子集共 40 個，可以推斷這個集不可能太小、考慮這些問題，一個可能的策略是，尋找一個適當的觀察量，再用兩個不同的角度估計這個量，在這個例子中我們考慮的是共同參加同一會議的委員對。

<u>例一的另一證明</u>：我們也可以從以下的角度考慮此一問題。因為有 40 次會議，每次有 10 人參加，所以共 400 “人次”參加這些會議。假設這個委員會的總人數不多於 60 人，因為 400/60 ≈ 6.67，則其中 1 人必參加 7 個或以上的會議。但是按照條件，參加這 7 個或以上會議的其他委員都不可能相同，因此共有 7 × 9 = 63 或以上不同的委員，矛盾！（留意在這裏用到鴿巢原理。）

有時候這一類問題可以另外的形式出現：

<u>例二</u>：(奧地利－波蘭數學競賽 1978) 有 1978 個集，每集含 40 個元素，並且任兩集剛好有 1 個共同元素。試證這 1978 個集必含有 1 個共同元素。

<u>證明</u>：設 A 為其中一個集，考慮其他 1977 個集，每一個集與 A 都有一個共同元素。由於 1977/40 ≈ 49.43，即是說，A 中必有一個元素 x 在另外 50 個集 $A_1, A_2, \ldots, A_{50}$ 內，且因條件所限，x 是 $A_1, A_2, \ldots, A_{50}$ 的惟一公共元。

考慮另外一個集 B，如果 x 不在 B 內，由於 B 和 $A_1, A_2, ..., A_{50}$ 都相交，且由條件所限，相交的元素都不同，則 B 最少有 51 個元素，這是不可能的。所以 x 在 B 內，且 B 是任意的，所以 x 在任一個集內，證畢。

這個結果可以這樣推廣，且證明完全相似：設有 n 個集，每一個集有 k 個元素，任意兩集剛好有一個共同元素。如果 $n > k^2 - k + 1$，則這 n 個集有一個共同元素。

考慮一個較為困難的例子:

例三：(俄羅斯數學競賽 1996) 由 1600 個議員組成 16000 個委員會，每個委員會由 80 個委員組成。試證明：一定存在兩個委員會，它們之間至少有 4 個相同的議員。

證明:這一次我們不考慮每一個委員會組成委員的對，反過來考慮每一個議員所參加委員會形成的對。設議員 1, 2, … , 1600 分別參加了 $k_1, k_2, \ldots, k_{1600}$ 個委員會，則總共有 $C_2^{k_1} + C_2^{k_2} + \ldots + C_2^{k_{1600}}$ 個委員會對。如果委員會的數目是 N, 則 $k_1 + k_2 + \ldots + k_{1600} = 80N$，(在題中 $N = 16000$，且每個委員會由 80 人組成。) 現在試圖估計這些委員會對

$$C_2^{k_1} + C_2^{k_2} + \cdots + C_2^{k_{1600}}$$
$$= \frac{\sum_{i=1}^{1600} k_i^2 - \sum_{i=1}^{1600} k_i}{2}$$
$$\geq \frac{(\sum_{i=1}^{1600} k_i)^2}{3200} - \frac{(\sum_{i=1}^{1600} k_i)}{2}$$
$$= \frac{(80N)^2}{3200} - \frac{80N}{2}$$
$$= 2N^2 - 40N = 2N(N-20)。$$

如果任兩個委員會最多有 3 個共同議員，則最多有

$$3C_2^N = \frac{3N(N-1)}{2}$$

個委員會對。因此

$$2N(N-20) \leq \frac{3}{2}N(N-1)。$$

即 $N \leq 77$，與 $N = 16,000$ 矛盾。

(留意在估計中用到 Cauchy-Schwarz Inequality。) 無獨有偶，我們有以下的例子:

例四：(IMO1998) 在一次比賽中，有 m 個比賽員和 n 個評判，其中 $n \geq 3$ 是一個奇數。每一個評判對每一個比賽員進行評審為合格或不合格。如果任一對評判最多對 k 個比賽員的評審一致，試證明

$$\frac{k}{m} \geq \frac{n-1}{2n}。$$

證明: 題目已經提醒我們，我們考慮的是評判所成的“對”，這些"對"評判員對某些比賽員的決定一致。對於比賽員 i，$1 \leq i \leq m$，如果有 x_i 個評判認為他合格，y_i 個評判認為他不合格，則評判一致的對是

$$C_2^{x_i} + C_2^{y_i}$$
$$= \frac{(x_i^2 + y_i^2) - (x_i + y_i)}{2}$$
$$\geq \frac{(x_i + y_i)^2}{4} - \frac{(x_i + y_i)}{2}$$
$$= \frac{1}{4}n^2 - \frac{n}{2} = \frac{1}{4}\left[(n-1)^2 - 1\right]。$$

因為 n 是奇數，而 $C_2^{x_i} + C_2^{y_i}$ 是整數，因為 $C_2^{x_i} + C_2^{y_i}$ 最少是 $\frac{1}{4}(n-1)^2$。現在因為有 n 個評判，而任一對評判最多對 k 個比賽員意見一致，因為一致的評判最多是 kC_2^n。所以

$$kC_2^n \geq \sum_{i=1}^{m}\left[C_2^{x_i} + C_2^{y_i}\right] \geq \frac{m(n-1)^2}{4}$$

，化簡結果即為所求。

現在考慮一個形式略為不同的題目。我們的對象是一些長為 n 的數列，這些數列只包括 0 或 1，兩個這樣的數列的“距離”定義為對應位置數字不同的個數。例如 1101011 和 1011000 為兩個長為 7 的數列，它們在位置 2,3,6,7 的數字不同，因此它們的距離是 4。用集的言語來描述是，有 7 個元素 1,2,3,4,5,6 和 7 的一個集，數列一在位置 1,2,4,6,7 非零，因此可想像是包括 1,2,4,6,7 的一個子集，數列二是包括 1,3 和 4 的子集，屬於數列一或數列二，但不同時屬於兩個數列的子集包括 2,3,6,7，稱為兩個子集的對稱差，而“距離”正好是對稱差所含元素的數目。現在可以考慮的是，給定 n 和距離的限制,，這樣的數列最多是多少。

例五：有 m 個包括 0 或 1，長為 n 的數列，如果任兩個數列間的距離最少為 d，試證明

$$m \leq \frac{2d}{2d-n}。$$

證明:現在要考慮的是任兩個數列中“相異對”的數目，因為有 C_2^m 對數列，而任一對數列的“相異對”或“距離最少是 d，因此總距離最少是 dC_2^m。將這些數列排起來成為 m 個橫行，每一直行 j, $1 \leq j \leq n$ 就對應著那些數列的 j 位置。如果 j 直行有 x_j 個 "0"，則有 $m - x_j$ 個 "1"，因此相異對有 $x_j(m - x_j)$ 個。觀察到

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon*. The deadline for submitting solutions is ***February 28, 2004***.

Problem 191. Solve the equation

$$x^3 - 3x = \sqrt{x+2}.$$

Problem 192. Inside a triangle *ABC*, there is a point *P* satisfies $\angle PAB = \angle PBC = \angle PCA = \varphi$. If the angles of the triangle are denoted by α, β and γ, prove that

$$\frac{1}{\sin^2\varphi} = \frac{1}{\sin^2\alpha} + \frac{1}{\sin^2\beta} + \frac{1}{\sin^2\gamma}.$$

Problem 193. Is there any perfect square, which has the same number of positive divisors of the form $3k + 1$ as of the form $3k + 2$? Give a proof of your answer.

Problem 194. (*Due to Achilleas Pavlos PORFYRIADIS, American College of Thessaloniki "Anatolia", Thessaloniki, Greece*) A circle with center O is internally tangent to two circles inside it, with centers O_1 and O_2, at points S and T respectively. Suppose the two circles inside intersect at points M, N with N closer to ST. Show that S, N, T are collinear if and only if $SO_1/OO_1 = OO_2/TO_2$.

Problem 195. (*Due to Fei Zhenpeng, Yongfeng High School, Yancheng City, Jiangsu Province, China*) Given n ($n > 3$) points on a plane, no three of them are collinear, x pairs of these points are connected by line segments. Prove that if

$$x \geq \frac{n(n-1)(n-2)+3}{3(n-2)},$$

then there is at least one triangle having these line segments as edges.

Find all possible values of integers $n > 3$ such that $\frac{n(n-1)(n-2)+3}{3(n-2)}$ is an integer and the minimum number of line segments guaranteeing a triangle in the above situation is this integer.

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Solutions

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Problem 186. (*Due to Fei Zhenpeng, Yongfeng High School, Yancheng City, Jiangsu Province, China*) Let α, β, γ be complex numbers such that

$$\alpha + \beta + \gamma = 1,$$
$$\alpha^2 + \beta^2 + \gamma^2 = 3,$$
$$\alpha^3 + \beta^3 + \gamma^3 = 7.$$

Determine the value of $\alpha^{21} + \beta^{21} + \gamma^{21}$.

Solution. **Helder Oliveira de CASTRO** (Colegio Objetivo, 3rd Grade, Sao Paulo, Brazil), **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 6), **CHUNG Ho Yin** (STFA Leung Kau Kui College, Form 7), **FOK Kai Tung** (Yan Chai Hospital No. 2 Secondary School, Form 7), **FUNG Chui Ying** (True Light Girls' College, Form 6), **Murray KLAMKIN** (University of Alberta, Edmonton, Canada), **LOK Kin Leung** (Tuen Mun Catholic Secondary School, Form 6), **SIU Ho Chung** (Queen's College, Form 5), **YAU Chi Keung** (CNC Memorial College, Form 7) and **YIM Wing Yin** (South Tuen Mun Government Secondary School, Form 4).

Using the given equations and the identities

$$(\alpha+\beta+\gamma)^2 = \alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\beta\gamma+\gamma\alpha),$$
$$(\alpha+\beta+\gamma)(\alpha^2+\beta^2+\gamma^2-\alpha\beta-\beta\gamma-\gamma\alpha) = \alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma,$$

we get $\alpha\beta + \beta\gamma + \gamma\alpha = -1$ and $\alpha\beta\gamma = 1$. These imply α, β, γ are the roots of $f(x) = x^3 - x^2 - x - 1 = 0$. Let $S_n = \alpha^n + \beta^n + \gamma^n$, then $S_1 = 1$, $S_2 = 3$, $S_3 = 7$ and for $n > 0$,

$$S_{n+3} - S_{n+2} - S_{n+1} - S_n = \alpha^n f(\alpha) - \beta^n f(\beta) - \gamma^n f(\gamma) = 0.$$

Using this recurrence relation, we find $S_4 = 11$, $S_5 = 21$, … , $S_{21} = 361109$.

Problem 187. Define $f(n) = n!$. Let

$$a = 0.f(1)f(2)f(3)\ldots.$$

In other words, to obtain the decimal representation of a write the numbers $f(1)$, $f(2)$, $f(3)$, … in base 10 in a row. Is a rational? Give a proof. (*Source: Israeli Math Olympiad*)

Solution. **Helder Oliveira de CASTRO** (Colegio Objetivo, 3rd Grade, Sao Paulo, Brazil), **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 6), **Murray KLAMKIN** (University of Alberta, Edmonton, Canada) and **Achilleas Pavlos PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece).

Assume a is rational. Then its decimal representation will eventually be periodic. Suppose the period has k digits. Then for every $n > 10^k$, $f(n)$ is nonzero and ends in at least k zeros, which imply the period cannot have k digits. We got a contradiction.

Problem 188. The line S is tangent to the circumcircle of acute triangle ABC at B. Let K be the projection of the orthocenter of triangle ABC onto line S (i.e. K is the foot of perpendicular from the orthocenter of triangle ABC to S). Let L be the midpoint of side AC. Show that triangle BKL is isosceles. (*Source: 2000 Saint Petersburg City Math Olympiad*)

Solution. **SIU Ho Chung** (Queen's College, Form 5).

Let O, G and H be the circumcenter, centroid and orthocenter of triangle ABC respectively. Let T and R be the projections of G and L onto line S. From the Euler line theorem (cf. *Math Excalibur, vol.* 3, *no.* 1, *p.*1), we know that O, G, H are collinear, G is between O and H and $2\,OG = GH$. Then T is between B and K and $2\,BT = TK$.

Also, G is on the median BL and $2\,LG = BG$. So T is between B and R and $2\,RT = BT$. Then $2\,BR = 2\,(BT + RT) = TK + TB = BK$. So $BR = RK$. Since LR is perpendicular to line S, by Pythagorean theorem, $BL = LK$.

Other commended solvers: **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 6) and **Achilleas Pavlos PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece).

Problem 189. $2n + 1$ segments are marked on a line. Each of the segments intersects at least n other segments. Prove that one of these segments intersect all other segments. (*Source 2000 Russian Math Olympiad*)

Solution. **Achilleas Pavlos PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece).

We imagine the segments on the line as intervals on the real axis. Going from left to right, let I_i be the i-th segment we meet with $i = 1, 2, \ldots, 2n + 1$. Let I_i^l and I_i^r be the left and right endpoints of I_i respectively. Now I_1 contains $I_2^l, \ldots, I_{n+1}^l$. Similarly, I_2 which already intersects I_1 must contain $I_3^l, \ldots, I_{n+1}^l$ and so on. Therefore the segments $I_1, I_2, \ldots, I_{n+1}$ intersect each other.

Next let I_k^r be the rightmost endpoint among $I_1^r, I_2^r, \ldots, I_{n+1}^r$ $(1 \le k \le n+1)$. For each of the n remaining intervals $I_{n+2}, I_{n+3}, \ldots, I_{2n+1}$, it must intersect at least one of $I_1, I_2, \ldots, I_{n+1}$ since it has to intersect at least n intervals. This means for every $j \ge n + 2$, there is at least one $m \le n + 1$ such that $I_j^l \le I_m^r \le I_k^r$, then I_k intersects I_j and hence every interval.

Problem 190. (*Due to Abderrahim Ouardini*) For nonnegative integer n, let $[x]$ be the greatest integer less than or equal to x and

$$f(n) = \left[\sqrt{n} + \sqrt{n+1} + \sqrt{n+2}\right] - \left[\sqrt{9n+1}\right].$$

Find the range of f and for each p in the range, find all nonnegative integers n such that $f(n) = p$.

Combined Solution by the Proposer and **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 6).

For positive integer n, we claim that

$$\sqrt{9n+8} < g(n) < \sqrt{9n+9},$$

where

$$g(n) = \sqrt{n} + \sqrt{n+1} + \sqrt{n+2}.$$

This follows from

$$g(n)^2 = 3n + 3 + 2(\sqrt{n(n+1)} + \sqrt{(n+1)(n+2)} + \sqrt{(n+2)n})$$

and the following readily verified inequalities for positive integer n,

$$(n + 0.4)^2 < n(n + 1) < (n + 0.5)^2,$$

$$(n + 1.4)^2 < (n + 1)(n + 2) < (n + 1.5)^2$$

and $(n + 0.7)^2 < (n + 2)n < (n + 1)^2$. The claim implies the range of f is a subset of nonnegative integers.

Suppose there is a positive integer n such that $f(n) \ge 2$. Then

$$\sqrt{9n+9} > [g(n)] > 1 + \sqrt{9n+1}.$$

Squaring the two extremes and comparing, we see this is false for $n > 1$. Since $f(0) = 1$ and $f(1) = 1$, we have $f(n) = 0$ or 1 for all nonnegative integers n.

Next observe that

$$\sqrt{9n+8} < [g(n)] < \sqrt{9n+9}$$

is impossible by squaring all expressions. So $[g(n)] = [\sqrt{9n+8}]$.

Now $f(n) = 1$ if and only if $p = [g(n)]$ satisfies $[\sqrt{9n+1}] = p - 1$, i.e.

$$\sqrt{9n+1} < p \le \sqrt{9n+8}.$$

Considering squares (mod 9), we see that $p^2 = 9n + 4$ or $9n + 7$.

If $p^2 = 9n + 4$, then $p = 9k + 2$ or $9k + 7$. In the former case, $n = 9k^2 + 4k$ and $(9k + 1)^2 \le 9n + 1 = 81k^2 + 36k + 1 < (9k + 2)^2$ so that $[\sqrt{9n+1}] = 9k + 1 = p - 1$. In the latter case, $n = 9k^2 + 14k + 5$ and $(9k + 6)^2 \le 9n + 1 = 81k^2 + 126k + 46 < (9k + 7)^2$ so that $[\sqrt{9n+1}] = 9k + 6 = p - 1$.

If $p^2 = 9n + 7$, then $p = 9k + 4$ or $9k + 5$. In the former case, $n = 9k^2 + 8k + 1$ and $(9k + 3)^2 \le 9n + 1 = 81k^2 + 72k + 10 < (9k + 4)^2$ so that $[\sqrt{9n+1}] = 9k + 3 = p - 1$. In the latter case, $n = 9k^2 + 10k + 2$ and $(9k + 4)^2 \le 9n + 1 = 81k^2 + 90k + 19 < (9k + 5)^2$ so that $[\sqrt{9n+1}] = 9k + 4 = p - 1$.

Therefore, $f(n) = 1$ if and only if n is of the form $9k^2 + 4k$ or $9k^2 + 14k + 5$ or $9k^2 + 8k + 1$ or $9k^2 + 10k + 2$.

Olympiad Corner

(continued from page 1)

Problem 4. Let ABC be a triangle. A circle passing through A and B intersects segments AC and BC at D and E, respectively. Rays BA and ED intersect at F while lines BD and CF intersect at M. Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$.

Problem 5. Let a, b, c be positive real numbers. Prove that

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2} + \frac{(2b+c+a)^2}{2b^2+(c+a)^2} + \frac{(2c+a+b)^2}{2c^2+(a+b)^2} \le 8.$$

Problem 6. At the vertices of a regular hexagon are written six nonnegative integers whose sum is 2003. Bert is allowed to make moves of the following form: he may pick a vertex and replace the number written there by the absolute value of the difference between the numbers written at the two neighboring vertices. Prove that Bert can make a sequence of moves, after which the number 0 appears at all six vertices.

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(continued from page 2)

$$x_j(m-x_j) \le \left[\frac{x_j+(m-x_j)}{2}\right]^2 = \frac{m^2}{4}$$

，因此

$$dc_2^m \le \sum_{j=1}^n x_j(m-x_j) \le \sum_{j=1}^n \frac{m^2}{4} = \frac{nm^2}{4}。$$

化簡即得 $m \le \frac{2d}{2d-n}$ 。

例如 $n = 7$，$d = 4$，得 $\frac{2d}{2d-n} = 8$ ，

所以不可能構造 9 個長為 7，而相互間最少距離為 4 的數列。(讀者可試圖構造 8 個這樣的數列。) 這個例子實際上是編碼理論一個結果的特殊情況，這個結果一般稱為 Plotkin 限 (Plotkin Bound)。

集和子集族還有許多有趣的結果，有待研究和討論。