POLYNOMIAL STRUCTURE OF GROMOV–WITTEN POTENTIAL OF QUINTIC 3-FOLDS

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ABSTRACT. We prove two structure theorems for the Gromov-Witten theory of the quintic threefolds, which together give an effective algorithm for the all genus Gromov-Witten potential functions of quintics. By using these structure theorems, we prove Yamaguchi-Yau's Polynomial Ring Conjecture in this paper and prove Bershadsky-Cecotti-Ooguri-Vafa's Feynman rule conjecture in the subsequent paper.

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0. INTRODUCTION

The Gromov-Witten (GW) theory was pioneered by the work of Witten [Wit92] and Candelas-Ossa-Green-Parkes [COGP91]. Its mathematical foundation was completed by Ruan-Tian [RT95] for semi-positive symplectic manifolds, and by Li-Tian [LT98] and Behrend-Fantechi [BF97] for projective manifolds.

One of the primary problem in GW-theory is to determine explicitly all genus generating functions F_g (defined in (0.3)) of the GW-invariants of the quintic Calabi-Yau (CY) threefold Q. For genus zero case, the mirror formula of Candelas et. al. for F_0 [COGP91] was mathematically proved by Givental [Gi99] and by Lian-Liu-Yau [LLY99].

For higher genus cases, via Strings theory and Mirror Symmetry Conjecture, Bershadsky-Cecotti-Ooguri-Vafa (BCOV) deduced a *Feynman rule* [BCOV93] , which effectively calculates F_g, from lower genus GW-invariants and a finite *ambiguity*. BCOV's Feynman rule is a cornerstone in the GW theory of CY threefolds.

The BCOV theory is primarily a B-model theory. Later, the work of Yamaguchi-Yau [YY04] focus on its mathematical consequences to the GW-invariants of CY threefolds. Relying on the BCOV's conjecture for the quintic threefold, Yamaguchi-Yau identified a finitely generated ring $\mathcal{R}$, such that the GW generating functions F_g all lie in $\mathcal{R}$ and satisfy a class of partial differential equations with respect to the generators of $\mathcal{R}$. These two conjectures are commonly called the *Polynomial Structure* (or *Finite Generation*) and the *Yamaguchi-Yau Equations* (or *Holomorphic Anomaly Equations*) .

In this paper, we will prove Yamaguchi-Yau's Polynomial Structure Conjecture. This conjecture reduces the problem of computing infinitely many GW invariants into a finite problem (for each fixed g). The proof is based on two structure theorems (Theorem 3 and Theorem 4), which together determine the quintic GW potential F_g for each genus g via (a) lower genus potentials $\{F_h\}_{h<g}$, (b) twisted GW potentials of points (which are computable) and (c) a degree $g-1$ polynomial of q (determined by $g-1$ unknown $[0,1]$-theory invariants).

By using these structure theorems, we will also prove BCOV's Feynman rule conjecture in our subsequent paper [NMSP3]. As a consequence, we will prove the Yamaguchi-Yau equations for the quintic threefolds via the quantization form of BCOV's rule.

0.1. **The statement of the theorem.** We recall the I-functions of the quintic threefolds:

$$I(q,z)=\sum_{i=0}^{3} I_i(q)H^i z^{1-i} := z\sum_{d=0}^{\infty} q^d \frac{\prod_{m=1}^{5d}(5H+mz)}{\prod_{m=1}^{d}(H+mz)^5}. \tag{0.1}$$

Definition 0.1. *Following* [YY04]*, we let* $D:=q\frac{d}{dq}$ *and introduce*

$$I_{11}=1+D\Big(\frac{I_1}{I_0}\Big),\quad A_k:=\frac{D^kI_{11}}{I_{11}},\quad B_k:=\frac{D^kI_0}{I_0},\quad \text{and}\quad Y=\frac{1}{1-5^5q}. \tag{0.2}$$

We usually brief $A=A_1, B=B_1$*. We define the ring of five generators to be*

$$\mathcal{R}=\mathbb{Q}[A,B,B_2,B_3,Y].$$

The following key property was proved in [YY04, ZZ08]:

Lemma 0.2. *The ring* $\mathcal{R}$ *contains all* A_k*,* B_k *for* $k\ge 1$*, and is closed under the operator* D*.*

We consider the genus g GW generating function (potential) F_g of the quintic threefolds:

$$F_g(\mathrm{Q})=\begin{cases} \frac{5}{6}\log \mathrm{Q}^3+\sum_{d\ge1}N_{g,d}\cdot \mathrm{Q}^d, & g=0\,;\\ -\frac{25}{12}\log\mathrm{Q}+\sum_{d\ge1}N_{g,d}\cdot\mathrm{Q}^d, & g=1\,;\\ \sum_{d\ge0}N_{g,d}\cdot\mathrm{Q}^d, & g>1\,.\end{cases} \tag{0.3}$$

where $N_{g,d}$ is genus g degree d GW invariant of quintics, and the log term is from degree zero "unstable" contributions. We introduce the "normalized" GW potential

$$P_{g,n}:=\frac{(5Y)^{g-1}I_{11}^{\,n}}{I_0^{2g-2}}\cdot\big(\mathrm{Q}\frac{d}{d\mathrm{Q}}\big)^nF_g(\mathrm{Q})\Big|_{\mathrm{Q}:=q\,e^{I_1/I_0}}.$$

The main result of this paper is a proof, using the NMSP theory developed in [NMSP1], of Yamaguchi-Yau's Polynomial Structure Conjectrue.

Theorem 1 (Yamaguchi-Yau's Polynomial Structure Conjecture)**.** *For any* $2g-2+n>0$*,* $P_{g,n}$ *lies in* $\mathcal{R}$*.*

We comment that in the proof of this theorem, we have given a canonical presentation of $P_{g,n}$ as elements in $\mathcal{R}$ (without using the algebraic independence of the five generators).

Example 0.3. *It is known that, by genus* 0 *mirror theorem (c.f. Theorem A.1) and genus* 1 *mirror theorem* [Zi09][1]*, one has*

$$P_{0,3}=1,\qquad P_{1,1}=-\tfrac{1}{2}A-\tfrac{31}{3}B+\tfrac{1}{12}Y-\tfrac{13}{6}\quad\in\quad\mathcal{R}. \tag{0.4}$$

As a corollary, all F_g are germs of analytic functions.

Theorem 2. *The GW potential* $F_g(q)$ *are germs of analytic functions in* q*, near* $q=0$*.*

0.2. **Applying NMSP theory.** In this paper we will only consider NMSP fields with n scheme markings decorated with $(1,\rho)$. We write such numerical data as $(g,n,\mathbf{d})=(g,n,(d_0,d_\infty))$; we denote the moduli space of stable NMSP fields of numerical data $(g,n,\mathbf{d})$ to be $\mathcal{W}_{g,n,\mathbf{d}}$. Its closed points are (cf. Section 1, or [NMSP1, Definition 2.1])

$$\xi=(\mathcal{C},\Sigma^{\mathcal{C}},\mathcal{L},\mathcal{N},\varphi_1,\cdots,\varphi_5,\rho,\mu_1,\cdots,\mu_{\mathrm{N}},\nu), \tag{0.5}$$

consisting of pointed twisted curves, a pair of invertible sheaves and various fields.

The stack $\mathcal{W}_{g,n,\mathbf{d}}$ is a G-stack, where $G=(\mathbb{C}^*)^{\mathrm{N}}$. For $\sigma\in G$, $\sigma\cdot\xi$ is by scaling its μ field: $(\mu_i)^\sigma=(\sigma_i\mu_i)$. Let $\Sigma_i\subset\Sigma^{\mathcal{C}}$ be the i-th marking, the restriction

$$\xi\longmapsto[\varphi_1,\cdots,\varphi_5,(\mu_1/\nu),\cdots,(\mu_{\mathrm{N}}/\nu)]\big|_{\Sigma_i} \tag{0.6}$$

[1] Zinger's theorem can be recovered by applying the genus 1 NMSP-theory with one insertion (see [NMSP3]).

defines a G-equivariant evaluation $\mathrm{ev}_i : \mathcal{W}_{g,n,\mathbf{d}} \longrightarrow \mathbb{P}^{4+\mathrm{N}}$, where G acts on $\mathbb{P}^{4+\mathrm{N}}$ via scaling the last N-homogeneous coordinates:

$$(g_1, \cdots, g_{\mathrm{N}}) \cdot [x_1, \cdots, x_5, u_1, \cdots, u_{\mathrm{N}}] = [x_1, \cdots, x_5, g_1 u_1, \cdots, g_{\mathrm{N}} u_{\mathrm{N}}].$$

We define the NMSP correlators, for $\tau_i(z) \in H^*_G(\mathbb{P}^{4+\mathrm{N}})[z]$ with $1 \le i \le n$, to be [2]

$$\left\langle \otimes_{i=1}^n \tau_i(\psi_i) \right\rangle^M_{g,n,d_\infty} = \sum_{d \ge 0} (-1)^{d+1-g} q^d \int_{[\mathcal{W}_{g,n,(d,d_\infty)}]^{\mathrm{vir}}} \prod_{i=1}^n \mathrm{ev}_i^* \tau_i(\psi_i) \in H^*_G(\mathrm{pt})[\![q]\!]. \tag{0.7}$$

A few words on the notation is in order: $\mathcal{W}_{g,n,\mathbf{d}}$ is a DM stack, has a perfect obstruction theory, and has a properly supported (cosection localized) virtual cycle $[\mathcal{W}_{g,n,\mathbf{d}}]^{\mathrm{vir}} \in A_*(\mathcal{W}^-_{g,n,\mathbf{d}})$. [3] Thus the integration does make sense. For $d_\infty = 0$, we denote $\langle - \rangle^M_{g,n} := \langle - \rangle^M_{g,n,0}$.

We let $\zeta_{\mathrm{N}} = e^{2\pi\sqrt{-1}/\mathrm{N}}$; let $\mathbb{F} = \mathbb{Q}(\zeta_{\mathrm{N}})(t)$, and for q the Novikov variable we let $\mathbb{A} = \mathbb{F}[\![q]\!]$, which is the coefficient ring to be used in this paper.

The equivariant cohomology ring $H^*_G(\mathrm{pt})$ is generated(over $\mathbb{Q}$) by *standard generators* t_α, $\alpha \in [\mathrm{N}] := \{1, \cdots, \mathrm{N}\}$. Our convention is that after equivariant integration we *always* make the substitution $t_\alpha \mapsto -\zeta^\alpha_{\mathrm{N}} t$, $t \in \mathbb{F}$. This makes (0.7) lies in $\mathbb{A} = \mathbb{F}[\![q]\!]$.

As the virtual dimension of $\mathcal{W}_{g,n,(d,d_\infty)}$ is $\mathrm{N}(d+1-g) + d_\infty + n$, (0.7) is a polynomial in q with coefficients in $\mathbb{F}$, after the mentioned substitution $t_\alpha \mapsto -\zeta^\alpha_{\mathrm{N}} t$.

The virtual localization gives

$$[\mathcal{W}_{g,n,\mathbf{d}}]^{\mathrm{vir}} = \sum_{\Theta \in G^{\mathrm{fl}}_{g,n,\mathbf{d}}} \frac{[F_\Theta]^{\mathrm{vir}}}{e(N^{\mathrm{vir}}_\Theta)}, \tag{0.8}$$

where F_Θ are open and closed substacks of $\left(\mathcal{W}_{g,n,\mathbf{d}}\right)^G$, indexed by the set of flat decorated graphs $G^{\mathrm{fl}}_{g,n,\mathbf{d}}$ (cf. Section 1, or [NMSP1, Sect. 4.2]). And $[F_\Theta]^{\mathrm{vir}} \in A_*(F_\Theta \cap \mathcal{W}^-_{g,n,\mathbf{d}})$.

We now introduce the space of insertions. The fixed locus $(\mathbb{P}^{4+\mathrm{N}})^G$ is a union of $\mathbb{P}^4$ and N isolated points $\mathrm{Npt} = \cup_{\alpha=1}^{\mathrm{N}} \{\mathrm{pt}_\alpha\}$. We introduce

$$\aleph := Q \cup \mathrm{Npt} \subset \mathbb{P}^4 \cup \mathrm{Npt} = (\mathbb{P}^{4+\mathrm{N}})^G \tag{0.9}$$

where $Q \subset \mathbb{P}^4 \subset \mathbb{P}^{4+\mathrm{N}}$ is the Fermat quintic threefold.

Definition 0.4. *We define the state space*

$$\mathcal{H} = H^*(\aleph, \mathbb{F}). \tag{0.10}$$

We denote $\mathcal{H}^{\mathrm{ev}} := H^{\mathrm{ev}}(\aleph, \mathbb{F}) \subset \mathcal{H}$ *its even degree part.*

The restriction of $(\mathbb{P}^{4+\mathrm{N}})^G$ to $\aleph$ (0.9) gives a ring homomorphism

$$H^*_G(\mathbb{P}^{4+\mathrm{N}}) = \mathbb{Q}(t)[p]/\big(p^5 \cdot \textstyle\prod_\alpha (p + t_\alpha)\big) \longrightarrow \mathcal{H}^{\mathrm{ev}} \tag{0.11}$$

that sends[4] p to $p|_Q = H \in H^2(Q)$ and $p|_{\mathrm{pt}_\alpha} = -t_\alpha \in H^2(\mathrm{pt}_\alpha)$. Further, the map is surjective and its kernel is spanned by $p^4 \cdot \prod_\alpha (p + t_\alpha)$. We have the natural decomposition

$$\mathcal{H} = \mathcal{H}^{\mathrm{ev}} \oplus H^3(Q).$$

The description of the degeneracy loci $\mathcal{W}^-_{g,n,\mathbf{d}}$ in [NMSP1] makes

$$ev_i(\mathcal{W}^-_{g,n,\mathbf{d}}) \subset (x_1^5 + \cdots + x_5^5 = 0) \subset \mathbb{P}^{4+\mathrm{N}}.$$

Therefore, over each G fixed loci $F_\Theta \subset \mathcal{W}^-_{g,n,\mathbf{d}}$, the evaluation map (0.6) restricts to

$$\mathrm{ev}_i : F_\Theta \cap \mathcal{W}^-_{g,n,\mathbf{d}} \longrightarrow \aleph \subset \mathbb{P}^{4+\mathrm{N}}.$$

[2] For $\tau_i(\psi_i) = \sum_{k \ge 0} \tau_{i,k} \psi_i^k$, we denote $\mathrm{ev}_i^* \tau_i(\psi_i) = \sum_{k \ge 0} (\mathrm{ev}_i^* \tau_{i,k}) \cdot \psi_i^k$.

[3] Here $\mathcal{W}^-_{g,n,\mathbf{d}}$ is the locus where the support of the virtual cycle $[\mathcal{W}_{g,n,\mathbf{d}}]^{\mathrm{vir}}$ belongs to, which is a proper G-invariant substack of $\mathcal{W}_{g,n,\mathbf{d}}$ (cf. [NMSP1] and [CLLL15].)

[4] For coordinate $[x_1, \cdots, x_5, u_1, \cdots, u_{\mathrm{N}}]$ of $\mathbb{P}^{4+\mathrm{N}}$, let D_i be the equivariant divisor defined by $x_i = 0$, for $i = 1, \cdots, 5$. Then $p := [D_i] \in H^2_G(\mathbb{P}^{4+\mathrm{N}})$.

Thus we can integrate over $[F_\Theta]^{\mathrm{vir}}$ the pullbacks ev_i^* of classes in $\mathcal{H}$. This enables us to extend the insertions in (0.7) from $\mathcal{H}^{\mathrm{ev}}$ to $\mathcal{H}$ via localization.

Definition 0.5. *For $\tau_i(z) \in \mathcal{H}[z] = \mathcal{H} \otimes \mathbb{F}[z]$,*

$$\big\langle \otimes_i^n \tau_i(\psi_i) \big\rangle_{g,n,d_\infty}^M := \sum_{d\geq 0} (-1)^{d+1-g} q^d \sum_{F_\Theta} \int_{[F_\Theta]^{\mathrm{vir}}} \frac{1}{e(N_{F_\Theta}^{\mathrm{vir}})} \prod_{i=1}^n \mathrm{ev}_i^* \tau_i(\psi_i), \tag{0.12}$$

where the sum is over all $\Theta \in G_{g,n,\mathbf{d}}^{\mathrm{fl}}$ (cf. [NMSP1]*). By localization,* (0.12) *reproduces* (0.7) *via* (0.11)*. In case $d_\infty = 0$, we also abbreviate $\langle - \rangle_{g,n}^M := \langle - \rangle_{g,n,0}^M$.*

0.3. **NMSP $[0,1]$-theory via localization.** Graphs in $G_{g,n,\mathbf{d}}^{\mathrm{fl}}$ has some distinguished features. Among them is that the vertices of any graph in $G_{g,n,\mathbf{d}}^{\mathrm{fl}}$ are labeled by levels 0, 1 and ∞. Accordingly, we say a graph is supported on $[0,1]$ if all its vertices are of levels 0 or 1.

Definition 0.6. *Let $G_{g,n,d}^{[0,1]}$ be the set of graphs in $G_{g,n,(d,0)}^{\mathrm{fl}}$ that are supported on $[0,1]$. We introduce $[\mathcal{W}_{g,n,\mathbf{d}}]^{[0,1]} = \sum_{\Theta \in G_{g,n,d}^{[0,1]}} \frac{[F_\Theta]^{\mathrm{vir}}}{e(N_\Theta^{\mathrm{vir}})}$.*

Definition 0.7. *For any $2g-2+n>0$, $\tau_i(z) \in \mathcal{H}[\![z]\!]$ $(i = 1, \cdots, n)$, we define*

$$\big[\otimes_i^n \tau_i(\psi_i)\big]_{g,n}^{[0,1]} = \sum_{d\geq 0} (-1)^{d+1-g} q^d \big(\mathrm{pr}_{g,n}^W\big)_* \Big(\prod_{i=1}^n \mathrm{ev}_i^* \tau_i(\psi_i) \cdot [\mathcal{W}_{g,n,\mathbf{d}}]^{[0,1]} \Big),$$

taking values in $H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{A})$, where $\mathrm{pr}_{g,n}^W : \mathcal{W}_{g,n,\mathbf{d}} \to \overline{\mathcal{M}}_{g,n}$ is the projection.

Further, for $2g-2+n>0$ we define the NMSP-$[0,1]$ *correlators by* [5]

$$\big\langle \otimes_i^n \tau_i(\psi_i) \big\rangle_{g,n}^{[0,1]} = \int_{\overline{\mathcal{M}}_{g,n}} \big[\otimes_i^n \tau_i(\psi_i)\big]_{g,n}^{[0,1]} \ \in \ \mathbb{A}. \tag{0.13}$$

The NMSP $[0,1]$-theory takes a concise form when phrased as an R-matrix action on the Cohomological Field Theory (CohFT), introduced in [Gi01a, Te12, PPZ15]. [6]

Theorem 3. *The* NMSP $[0,1]$*-theory gives a CohFT $\Omega^{[0,1]}(-) := [-]^{[0,1]}$ with the state space $\mathcal{H}$* (0.10)*, related to the local CohFT $\Omega^{\aleph}(-) := [-]^{\aleph}$ with the same state space via*

$$\Omega^{[0,1]} = R.\Omega^{\aleph}.$$

Here the insertions "$-$" all lie in $\mathcal{H}$, the correlator $[-]_{g,n}^{\aleph}$ is defined in (1.10) *and the R-matrix is defined via the factorization between the local and global S-matrices* (3.1)*.*

The precise meaning of this theorem will be stated in Section 3 (Theorem 3.1). In the form of CohFT this theorem will be rephrased in [NMSP3, Sect 3]. Furthermore, the NMSP-$[0,1]$ theory has a similar degree bound as the total NMSP theory:

Theorem 4 (Polynomiality of NMSP-$[0,1]$ correlators)**.** *For $2g-2+n>0$ and $m_i \in [0, \mathrm{N}+3]$, the $[0,1]$-correlator*

$$t^{\sum(n-m_i-k_i)-\mathrm{N}(g-1)} \big\langle p^{m_1} \bar\psi_1^{k_1}, \cdots, p^{m_n} \bar\psi_n^{k_n} \big\rangle_{g,n}^{[0,1]} \in \mathbb{Q}[q'] \tag{0.14}$$

is a polynomial in $q' := -q/t^{\mathrm{N}}$, of degree no more than $g-1+\frac{1}{\mathrm{N}}\big(3g-3+\sum_{i=1}^n m_i\big)$.

We remark that Theorem 3 gives us explicit relations between global and local generating functions via the R-matrix. For each $g>0$, Theorem 3 and Theorem 4 together determine the GW potential F_g (of quintics) after knowing the lower genus GW potential $F_{h<g}$, up to $g-1$ ambiguity (unkonwns), which improves the numerical version of the algorithm in [CLLL16].

Our subsequent paper [NMSP3] is built upon these techniques and results.

[5] The unstable $[0,1]$ correlators will be defined later in Theorem 2.1.

[6] See also [NMSP3, Sect. 2] for the generalized definition that we will use.

The organization of this paper is as follows. In §1, we recall NMSP localizations and use Givental's theory to study $g = 0$ generating functions of NMSP-$[0,1]$ theory. In §2, we package NMSP localization graphs to "bipartite graphs" and decompose NMSP correlators into $[0,1]$ correlators with $(1,\infty]$ contributions as insertions (Theorem 2.1). In §3, we package $[0,1]$ theory by using the stabilization to prove the R-matrix action formula (Theorem 3, or more precisely Theorem 3.1). In §4, we prove $[0,1]$'s polynomiality (Theorem 4) by using NMSP's polynomiality established in [NMSP1] and Theorem 2.1 in §2. In §5, we prove Yamaguchi-Yau's polynomial structure (Theorem 1) by using Theorem 3, Theorem 4 and the polynomiality of R-matrix. In §6, we prove a key lemma for behavior of the R-matrix at pt_α.

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1. The NMSP theory and genus zero structures

We recall the technical details of the NMSP moduli space necessary for this paper. An NMSP field (of the Fermat quintic) of n $(1,\rho)$ markings is a ξ in (0.5) so that $\Sigma^{\mathcal{C}} \subset \mathcal{C}$ is a genus g, n-pointed twisted curve where all markings are scheme points over which the field ρ must vanishes; $\mathcal{L}$ and $\mathcal{N}$ are invertible sheaves of $\mathcal{O}_{\mathcal{C}}$-modules so that $\mathcal{L} \oplus \mathcal{N}$ is representable; and the fields are

$$\varphi_i \in H^0(\mathcal{L}),\ \rho \in H^0(\mathcal{L}^{-5} \otimes \omega_{\mathcal{C}}),\ \mu_1, \cdots, \mu_{\mathrm{N}} \in H^0(\mathcal{L} \otimes \mathcal{N}),\ \text{and } \nu \in H^0(\mathcal{N}),$$

satisfying properties that (φ, μ), (ρ, ν), and (μ, ν) are nowhere vanishing(where $\mu := (\mu_1, \cdots, \mu_n)$). It is called stable if $\mathrm{Aut}(\xi)$ is finite. It is called an MSP field if $\mathrm{N} = 1$.

The field ξ has numerical data: the genus g, the number of markings n, and the degrees $d_0 = \deg \mathcal{L} \otimes \mathcal{N}$ and $d_\infty = \deg \mathcal{N}$. Given $(g, n, \mathbf{d})$, $\mathbf{d} = (d_0, d_\infty)$, the moduli of stable NMSP fields of given data forms a DM stack $\mathcal{W}_{g,n,\mathbf{d}}$, acted on by $G = (\mathbb{C}^*)^{\mathrm{N}}$ via tautological scaling of the N factors of the μ fields. It has a cosection localized G-equivariant virtual cycle $[\mathcal{W}_{g,n,\mathbf{d}}]^{\mathrm{vir}}$.

We apply the virtual localization formula [GP99, CKL17]. For insertions $\tau_i(z) \in \mathcal{H}^{\mathrm{ev}}[z]$,

$$\int_{[\mathcal{W}_{g,n,\mathbf{d}}]^{\mathrm{vir}}} \prod_{i=1}^{n} \mathrm{ev}_i^* \tau_i(\psi_i) = \sum \int_{[F_\Theta]^{\mathrm{vir}}} \frac{\prod_{i=1}^n \mathrm{ev}_i^* \tau_i(\psi_i)}{e_G(N_\Theta^{\mathrm{vir}})},$$

where the summation is over all *regular* graphs $\Theta \in G^{\mathrm{reg}}_{g,n,\mathbf{d}}$. [7] Note that the virtual localization says the sum should be over all $\Theta \in G^{\mathrm{fl}}_{g,n,\mathbf{d}}$, and it is proved in [NMSP1, Thm 1.3] that only regular graphs $\Theta \in G^{\mathrm{reg}}_{g,n,\mathbf{d}} \subset G^{\mathrm{fl}}_{g,n,\mathbf{d}}$ will possibly have non-zero contribution to localizations.

To proceed, we list the relevant properties of a flat decorated graph $\Theta \in G^{\mathrm{fl}}_{g,n,\mathbf{d}}$. For its precise definition, see [NMSP1]. The graph Θ consists of vertices V, edges E, and legs L, plus decorations on them. (We will use $E(\Theta)$ to emphasize the dependence of $E(\Theta)$ on Θ; when Θ is understood, we abbreviate it to E.) We say an edge $e \in E$ is incident to $v \in V$ if e is attached to v, etc.. The decorations of Θ are level, hour, genus, and degrees.

1. **level**: Level is a map $\ell_\cdot : V \to \{0, 1, \infty\}$; we agree $v \in V_{\ell_v}$; when $e \in E$ is incident to $v, v' \in V$, we say $e \in E_{\ell_v \ell_{v'}}$.
2. **hour**: Hour is a map $\alpha_\bullet : V_\infty \cup V_1 \longrightarrow [\mathrm{N}]$: when $e \in E_{aa'}$ is incident to $v \in V_a$ and $v' \in V_{a'}$, then $\alpha_v \neq \alpha_{v'}$ when $a = a'$, and $\alpha_v = \alpha_{v'}$ when $a \neq a'$.
3. **genus and degree**: Genus is as usual; degree is a map $(d_{0\bullet}, d_{\infty\bullet}) : V \cup E \to (\mathbb{Q})^2$.

[7] Here "regular" roughly means no $0 - \infty$ edge(see below) and no node at infinity can have $2/5, 3/5, 4/5$ $\mathcal{L}$-monodromy weight. For precise definition of regular graphs, see [NMSP1, Sect. 5].

For $e\in E_{1\infty}$ we let (its hour) $\alpha_e=\alpha_v$ for v a vertex incident to e; it is well-defined. We let $E^\alpha_{1\infty}$ be the set of $e\in E_{1\infty}$ with hour α, likewise V^α_ℓ.

The total degree of a graph Θ is (d_0,d_∞), where $d_\bullet=\sum_{a\in V\cup E} d_{\bullet a}$; the total genus of Θ is $g=h^0(\Theta)+\sum g_v$. For $v\in V$, we denote by L_v (resp. E_v) be the set of legs (resp. edges) incident to v. We say v is unstable if $g_v=d_{0v}=d_{\infty v}=0$ and $|L_v|+|E_v|\le 2$.

When the graph is regular, among other things, its $E_{0\infty}=\emptyset$. Thus every $\Theta\in G^{\mathrm{reg}}_{g,n,\mathbf{d}}$ can be decomposed along level 1 vertices to its $[0,1]$-part and its $(1,\infty]$-part.

For the description of fixed loci's virtual cycles $[F_\Theta]^{\mathrm{vir}}$, its decomposition in terms of Θ's vertices and edges, and the formulas for normal factors $e_G(N^{\mathrm{vir}}_\Theta)$, we refer the readers to [NMSP1, Sect. 6]. The main feature is that, each vertex in V_0 associates curve countings in the quintic threefold, each vertex v in V_1 associates classical Hodge integrals on $\overline{\mathcal{M}}_{g_v,n_v}$, and each vertex v at infinity contributes invariants similar to quintic singularities' FJRW invariants.

1.1. Genus zero theory and the state spaces.

For $g=0$ and $d_\infty=0$, one can show[8]

$$\mathcal{W}_{0,n,(d,0)}\cong \overline{\mathcal{M}}_{0,n}(\mathbb{P}^{4+\mathrm{N}},d).$$

Let $\mathcal{L}_p:=\mathcal{O}_{\mathbb{P}^{4+\mathrm{N}}}(1)$ and $(\pi_d,f_d):\mathcal{C}\to\overline{\mathcal{M}}_{0,n}(\mathbb{P}^{4+\mathrm{N}},d)\times\mathbb{P}^{4+\mathrm{N}}$ be the universal family, then

$$[\mathcal{W}_{0,n,(d,0)}]^{\mathrm{vir}}=(-1)^{d+1}e_T(\pi_*f^*\mathcal{L}_p^{\otimes 5})\cap[\overline{\mathcal{M}}_{0,n}(\mathbb{P}^{4+\mathrm{N}},d)],$$

together with their torus localization formulae.

This allows us to apply Givental's theory to our NMSP theory.

Principle (★). *The genus zero* NMSP *theory equals the genus zero $\mathcal{L}_p^{\otimes 5}$-twisted G-equivariant GW theory of $\mathbb{P}^{4+\mathrm{N}}$. In particular, for $\tau_i(z)\in\mathcal{H}^{\mathrm{ev}}[z]$*

$$\big\langle\otimes_i^n\tau_i(\psi_i)\big\rangle^M_{0,n}=\sum_{d\ge 0}q^d\int_{\overline{\mathcal{M}}_{g=0,n}(\mathbb{P}^{4+\mathrm{N}},d)}e_G(\pi_{d*}f_d^*\mathcal{L}_p^{\otimes 5})\prod_{i=1}^n\tau_i(\psi_i). \tag{1.1}$$

We now define the unit and the pairing on the vector space $\mathcal{H}$:

Definition 1.1. *The state space is the triple $\big(\mathcal{H},(\ ,\ )^M,\mathbf{1}\big)$, which is an inner product space with unit. The unit $\mathbf{1}$ is defined by*

$$\mathbf{1}=\mathbf{1}_Q+\sum\mathbf{1}_\alpha\in H^0_G(\aleph,\mathbb{F})$$

where $\mathbf{1}_Q$ and $\mathbf{1}_\alpha$ are units of $H^*(Q)$ and $H^*(\mathrm{pt}_\alpha)$ respectively, and the pairing is defined by [9]

$$(x,y)^M:=q^0\ \textit{coefficient of}\ \langle\mathbf{1},x,y\rangle^M_{0,3},\quad\forall x,y\in\mathcal{H}. \tag{1.4}$$

Let p be the equivariant hyperplane class $(x_1=0)$ in $H^2_G(\mathbb{P}^{4+\mathrm{N}})$. Then

$$H^*_G(\mathbb{P}^{4+\mathrm{N}})=\mathbb{Q}(t)[p]/\big(p^5\prod_\alpha(p+t_\alpha)\big),$$

---

[8] For any (family of) $(\mathcal{C},\mathcal{L},\cdots)\in\mathcal{W}_{0,n,(d,d_\infty)}$, one shows every connected component of $(\nu=0)$ contributes a positive number to d_∞. This uses that on each such component $(\mu_1,\cdots,\mu_{\mathrm{N}})$ and ρ are nowhere zero. If further $d_\infty=0$, ν must be nowhere zero and $\mathcal{N}\cong\mathcal{O}$, and these imply $\rho=0$ because $g_{\mathcal{C}}=0$.

[9] The pairing can be further calculated by applying localization (0.12) to (1.4):

$$(\cdot,\cdot)^M=(\cdot|_Q,\cdot|_Q)^{Q,\mathrm{tw}}+\sum_\alpha(\cdot|_{\mathrm{pt}_\alpha},\cdot|_{\mathrm{pt}_\alpha})^{\mathrm{pt}_\alpha,\mathrm{tw}},$$

where $\cdot|_Q:\mathcal{H}\to\mathcal{H}_Q:=H^*(Q)$ and $\cdot|_\alpha:\mathcal{H}\to\mathcal{H}_{\mathrm{pt}_\alpha}:=H^*(\mathrm{pt}_\alpha)$ are projections, and

$$(x,y)^{Q,\mathrm{tw}}:=\int_Q\frac{xy}{\prod_{\alpha=1}^{\mathrm{N}}(H+t_\alpha)}=\int_Q\frac{xy}{-t^{\mathrm{N}}},\qquad x,y\in\mathcal{H}_Q \tag{1.2}$$

$$(x,y)^{\mathrm{pt}_\alpha,\mathrm{tw}}:=\frac{5}{t_\alpha^4\prod_{\beta:\beta\ne\alpha}(t_\beta-t_\alpha)}xy=\frac{-5}{\mathrm{N}t_\alpha^3t^{\mathrm{N}}}xy,\qquad x,y\in\mathcal{H}_{\mathrm{pt}_\alpha}. \tag{1.3}$$

over which the pullback pairing equals (by applying (1.1))

$$(x,y)^M = \int_{\mathbb{P}^{4+\mathrm{N}}} x\cup y\cup 5p, \qquad \forall x,y\in H^*_G(\mathbb{P}^{4+\mathrm{N}}).$$

Restricting to the even cohomology $\mathcal{H}^{\mathrm{ev}}$ with basis $\{p^i\}_{i=0}^{\mathrm{N}+3}$, the pairing is given by

$$\eta_{ij} := (p^i,p^j)^M = \int_{\mathbb{P}^{4+\mathrm{N}}} 5p\cup p^i\cup p^j = \begin{cases} 5t^{\mathrm{N}} & \text{if } i+j=2\mathrm{N}+3\\ 5 & \text{if } i+j=\mathrm{N}+3,\\ 0 & \text{otherwise}\end{cases}$$

Convention 1.2. *In the remainder of this paper, we view p^i as its image in $\mathcal{H}^{\mathrm{ev}}$ under* (0.11)*.*

By using the above convention, we obtain

- For $\alpha\in[\mathrm{N}]$, for $j=0,\cdots,3$, we have the following identification:
$$\mathbf{1}_\alpha = \frac{p^4}{t_\alpha^4}\prod_{\beta\neq\alpha}\frac{t_\beta+p}{t_\beta-t_\alpha} \quad \text{and} \quad H^j = \frac{p^j}{t^{\mathrm{N}}}(t^{\mathrm{N}}-p^{\mathrm{N}}) \ \text{ with } H^0=\mathbf{1}_Q.$$
Furthermore, the Poincare dual of $\{\mathbf{1}_Q = H^0, H, H^2, H^3\}\cup\{\mathbf{1}_\alpha\}_{\alpha\in[\mathrm{N}]}$ is given by
$$\{\frac{-t^{\mathrm{N}}}{5}H^3, \frac{-t^{\mathrm{N}}}{5}H^2, \frac{-t^{\mathrm{N}}}{5}H, \frac{-t^{\mathrm{N}}}{5}H^0\}\cup\{\mathbf{1}^\alpha = \frac{\mathrm{N}t_\alpha^3 t^{\mathrm{N}}}{(-5)}\mathbf{1}_\alpha\}_{\alpha\in[\mathrm{N}]}.$$
- Let $\{\phi_i := p^i\}_{i=0}^{\mathrm{N}+3}$, which is a basis of $\mathcal{H}^{\mathrm{ev}}$, then its dual basis $\{\phi^i\}\subset\mathcal{H}^{\mathrm{ev}}$ are given by
$$\{\phi^0,\cdots,\phi^{\mathrm{N}+3}\} = \Big\{\frac{p^3}{5}(p^{\mathrm{N}}-t^{\mathrm{N}}), \frac{p^2}{5}(p^{\mathrm{N}}-t^{\mathrm{N}}), \frac{p}{5}(p^{\mathrm{N}}-t^{\mathrm{N}}), \frac{1}{5}(p^{\mathrm{N}}-t^{\mathrm{N}}), \frac{p^{\mathrm{N}-1}}{5}, \frac{p^{\mathrm{N}-2}}{5},\cdots,\frac{p^0}{5}\Big\}.$$

1.2. **Twisted classes and correlators for local theory.** In the localization formula of the NMSP moduli (see [NMSP1] for details), the local contributions at vertices in V_0 (quintic) or V_1 (N points) are given by corresponding twisted classes, which are the contributions of the original theory (GW theory of the quintic/point) twisted by the virtual normal factors.

Definition 1.3 (Twisted classes)**.** *We introduce the quintic twisted classes*

$$[\overline{\mathcal{M}}_{g,n}(Q,d)]^{\mathrm{tw}} := \prod_\alpha e_T\big(R\pi_*\mathrm{ev}^*_{n+1}\mathcal{O}(1)\cdot t_\alpha\big)^{-1}\cap[\overline{\mathcal{M}}_{g,n}(Q,d)]^{\mathrm{vir}},$$

and the pt_α twisted classes

$$[\overline{\mathcal{M}}_{g,n}]^{\alpha,\mathrm{tw}} := (-1)^{1-g}\frac{5t_\alpha\cdot e_T(\mathbb{E}^\vee_{g,n}\otimes(-t_\alpha))^5}{(-t_\alpha)^5\cdot e_T(\mathbb{E}_{g,n}\otimes 5t_\alpha)}\frac{\prod_{\beta\neq\alpha}e_T(\mathbb{E}^\vee_{g,n}\otimes(t_\beta-t_\alpha))}{\prod_{\beta\neq\alpha}(t_\beta-t_\alpha)}\cap[\overline{\mathcal{M}}_{g,n}].$$

We define the topological twisted classes $[\overline{\mathcal{M}}_{g,n}(Q,d)]^{\mathrm{top}}$ and $[\overline{\mathcal{M}}_{g,n}]^{\alpha,\mathrm{top}}$ to be the top degree components (topological parts) of the corresponding twisted classes.

By dimension reason, one see

Lemma 1.4. *For the quintic threefold Q, assuming $\mathrm{N}>3$*

$$[\overline{\mathcal{M}}_{g,n}(Q,d)]^{\mathrm{top}} = (-t^{\mathrm{N}})^{-(d+1-g)}[\overline{\mathcal{M}}_{g,n}(Q,d)]^{\mathrm{vir}} = [\overline{\mathcal{M}}_{g,n}(Q,d)]^{\mathrm{tw}}; \tag{1.5}$$

for the points pt_α $(\alpha=1,\cdots,\mathrm{N})$,

$$[\overline{\mathcal{M}}_{g,n}]^{\alpha,\mathrm{top}} = \big(\tfrac15\mathrm{N}(-t_\alpha)^{3+\mathrm{N}}\big)^{g-1}[\overline{\mathcal{M}}_{g,n}]. \tag{1.6}$$

We now define correlators.

Definition 1.5. *For $\tau_i(\psi)=\sum_k\tau_{i,k}\psi^k\in\mathcal{H}[\psi]$: (1) We define*

$$\langle\otimes_i^n\tau_i(\psi_i)\rangle^{Q,\mathrm{tw}}_{g,n} := \sum_{d\geq0}q^d\int_{[\overline{\mathcal{M}}_{g,n}(Q,d)]^{\mathrm{tw}}}\prod_i\Big(\sum_{k\geq0}\psi_i^k\cup\mathrm{ev}_i^*(\tau_{i,k}|_Q)\Big)\in\mathbb{A}. \tag{1.7}$$

Replacing "Q,tw" by "Q", and replacing "$[\overline{\mathcal{M}}_{g,n}(Q,d)]^{\mathrm{tw}}$" by "$[\overline{\mathcal{M}}_{g,n}(Q,d)]$", the formula (1.7) defines $\langle\otimes_i^n\tau_i(\psi_i)\rangle_{g,n}^Q$. Similarly, we define

$$\langle\otimes_i^n\tau_i(\psi_i)\rangle_{g,n}^{\mathrm{pt}_\alpha,\mathrm{tw}} := \sum_{d\ge 0} q^d \int_{[\overline{\mathcal{M}}_{g,n}]^{\alpha,\mathrm{tw}}} \prod_i \left(\tau_i(\psi_i)|_{\mathrm{pt}_\alpha}\right) \in \mathbb{A}, \tag{1.8}$$

and we define $\langle\otimes_i^n\tau_i(\psi_i)\rangle_{g,n}^{\mathrm{pt}_\alpha}$ by replacing "$[\overline{\mathcal{M}}_{g,n}]^{\alpha,\mathrm{tw}}$" with $[\overline{\mathcal{M}}_{g,n}]$. In the end, we define

$$\langle -\rangle_{g,n}^{\mathrm{loc}} := \langle -\rangle_{g,n}^{Q,\mathrm{tw}} + \textstyle\sum_\alpha \langle -\rangle_{g,n}^{\mathrm{pt}_\alpha,\mathrm{tw}}. \tag{1.9}$$

(2) In the stable cases ($2g-2+n>0$), we also define the class version correlators

$$[\otimes_i^n\tau_i(\psi_i)]_{g,n}^{Q,\mathrm{tw}} := \sum_{d\ge 0} q^d (\mathrm{pr}_{g,n}^Q)_*\Big(\prod_i\sum_{k\ge 0}\psi_i^k\cup \mathrm{ev}_i^*(\tau_{i,k}|_Q)\cap[\overline{\mathcal{M}}_{g,n}(Q,d)]^{\mathrm{tw}}\Big);$$

$$[\otimes_i^n\tau_i(\psi_i)]_{g,n}^{\mathrm{pt}_\alpha,\mathrm{tw}} := \prod_i\left(\tau_i(\psi_i)|_{\mathrm{pt}_\alpha}\right)\cap[\overline{\mathcal{M}}_{g,n}]^{\alpha,\mathrm{tw}}.$$

where $\mathrm{pr}_{g,n}^Q : \overline{\mathcal{M}}_{g,n}(Q,d)\to\overline{\mathcal{M}}_{g,n}$ denotes the forgetful map. Removing "tw" in the above two lines define $[-]_{g,n}^Q$ and $[-]_{g,n}^{\mathrm{pt}_\alpha}$; replacing "$\mathrm{tw}$" by "$\mathrm{top}$" defines $[-]_{g,n}^{Q,\mathrm{top}}$ and $[-]_{g,n}^{\mathrm{pt}_\alpha,\mathrm{top}}$. In the end, for $\aleph := Q\cup\mathrm{Npt}$ we define

$$[-]_{g,n}^{\aleph} := [-]_{g,n}^{Q,\mathrm{top}} + \textstyle\sum_\alpha [-]_{g,n}^{\mathrm{pt}_\alpha,\mathrm{top}}. \tag{1.10}$$

1.3. **J-functions, S-matrices and the mirror formula.** Let "$\bullet$" be "M" or "loc", we define

$$J^\bullet(\mathbf{t},z)\in\mathbb{A}\otimes\mathcal{H}[\![z,z^{-1}]\!] \quad\text{and}\quad S_\tau^\bullet(z)\in\mathbb{A}\otimes\mathrm{End}\,\mathcal{H}[\![z^{-1}]\!],$$

with $\mathbf{t}\in\mathbb{A}\otimes\mathcal{H}[\![z]\!]$ and $\tau\in\mathbb{A}\otimes\mathcal{H}$ to be

$$J^\bullet(\mathbf{t},z) := z+\mathbf{t}(-z)+\sum_{\alpha,n}\frac{1}{n!}e^\alpha\Big\langle\frac{e_\alpha}{z-\psi},\mathbf{t}(\psi)^n\Big\rangle_{0,n+1}^\bullet \tag{1.11}$$

$$S_\tau^\bullet(z)x := x+\sum_{\alpha,n}\frac{1}{n!}e^\alpha\Big\langle\frac{x}{z-\psi},e_\alpha,\tau^n\Big\rangle_{0,n+2}^\bullet, \tag{1.12}$$

where $\{e_\alpha\}$ is a basis of $\mathcal{H}$, and $\{e^\alpha\}$ is its dual basis under the pairing $(,)^M$. We remark that whenever $x\in\mathcal{H}^{\mathrm{ev}}$, $S_{\tau=0}^M(z)x$ is even, namely $S_{\tau=0}^M(z)x\in\mathbb{A}\otimes\mathcal{H}^{\mathrm{ev}}[\![z,z^{-1}]\!]$.

Notice that for each d, the coefficient of q^d in $J^\bullet(\mathbf{t},z)$ is rational in z whenever $\mathbf{t}$ is rational in z, so is $S_\tau^\bullet(z)x$ for any $x\in\mathcal{H}$. We make the following conventions:

Convention 1.6. *In this paper, we will expand the* NMSP *functions $J^M(\mathbf{t},z)$ and $S_\tau^M(z)$ near $z=0$ when it is computed by localizations; we will expand local functions $J^{\mathrm{loc}}(\mathbf{t},z)$ and $S_\tau^{\mathrm{loc}}(z)$ near $z=\infty$.*

Remark 1.7. *Applying localization, for $\tau\in\mathcal{H}^{\mathrm{ev}}$ and $x\in H^3(Q)$ one shows that, for $\bullet$ to be "M" or "loc", $S_\tau^\bullet(z)x\in H^3(Q)$. This implies $S_\tau^\bullet(z)^{-1}$ preserves $H^3(Q)$ and thus $S_\tau^\bullet(z)$ preserves $\mathcal{H}^{\mathrm{ev}}$. For $\tau=0$ and $x\in H^3(Q)$, one has $S_{\tau=0}^M(z)x=x$. For $\tau\in\mathcal{H}^{\mathrm{ev}}$ such that $\tau|_Q=\sum_{i=0}^3\tau_iH^i$ and $x\in H^3(Q)$, one has $S_\tau^{\mathrm{loc}}(z)x=e^{\tau_0/z}x$.*

For $\bullet$ equals "Q", "Q,tw", "pt_α", or "$\mathrm{pt}_\alpha,\mathrm{tw}$" and for $\tau\in\mathbb{A}\otimes\mathcal{H}_\bullet$, $\mathbf{t}\in\mathbb{A}\otimes\mathcal{H}_\bullet[\![z]\!]$, we define

$$J^\bullet(\mathbf{t},z)\in\mathbb{A}\otimes\mathcal{H}_\bullet[\![z,z^{-1}]\!] \quad\text{and}\quad S_\tau^\bullet(z)\in\mathbb{A}\otimes\mathrm{End}\,\mathcal{H}_\bullet[\![z^{-1}]\!]$$

via the same formula (1.11) and (1.12), where as vector space we define $\mathcal{H}_X=\mathcal{H}_{X,\mathrm{tw}}:=H^*(X,\mathbb{F})$ for $X=Q$ or pt_α and the dual basis are taken under the respective pairing $(,)^\bullet$. [10]

A direct calculation shows

$$S_\tau^{Q,\mathrm{tw}}(q,z)=S_\tau^Q(q',z) \quad\text{and}\quad S_\tau^{\mathrm{pt}_\alpha,\mathrm{tw}}(z)=S_\tau^{\mathrm{pt}_\alpha}(z)=e^{\tau/z}. \tag{1.13}$$

Applying Principle ($\bigstar$), the J^M, S_τ^M are identical to the J and S function of the genus zero $\mathcal{L}_p^{\otimes 5}$-twisted G-equivariant theory of $\mathbb{P}^{4+\mathrm{N}}$. Givental's mirror theorem applies immediately.

[10] The pairing $(,)^{Q,\mathrm{tw}}$, $(,)^{\mathrm{pt}_\alpha,\mathrm{tw}}$ are defined in (1.2), (1.3) and $(,)^Q$, $(,)^{\mathrm{pt}_\alpha}$ are the ordinary Poincare pairings.

Theorem 1.1. *([Gi96, Thm 10.7]) We consider the* NMSP *J-function*

$$J^M(z) := J^M(0,z) = z\mathbf{1} + \sum_\alpha e_\alpha \left\langle \frac{e^\alpha}{z-\psi_1} \right\rangle^M_{0,1}$$

and define the NMSP *I-function*

$$I^M(q,z) = z\mathbf{1} + \sum_{d=1}^{\infty} z\, q^d \frac{\prod_{m=1}^{5d}(5p+mz)}{\prod_{m=1}^{d}(p+mz)^5 \prod_{m=1}^{d}\left((p+mz)^{\mathrm{N}} - t^{\mathrm{N}}\right)}. \tag{1.14}$$

Then the following identity (mirror theorem) holds

$$J^M(z) = I^M(q,z).$$

1.4. **Givental's Lagrangian cone.** The Lagrangian cone in GW theory was introduced in [Ba00, CG07, Gi03]. We consider the Lagrangian cone defined by the local theory

$$\mathcal{L}^{\mathrm{loc}} := \{(\mathbf{p},\mathbf{q}) := \mathbf{p}(-z) + \mathbf{q}(z) \in \mathcal{H}[\![z,z^{-1}]\!] : \mathbf{p} = d_{\mathbf{q}} F_0^{\mathrm{loc}}(\mathbf{t})\},$$

where $\mathbf{t}(z) := \mathbf{q}(z) + z \in \mathcal{H}[\![z]\!]$, and

$$F_0^{\mathrm{loc}}(\mathbf{t}) := F_0^{Q,\mathrm{tw}}(\mathbf{t}|_Q) + \textstyle\sum_\alpha F_0^{\mathrm{pt,tw}}(\mathbf{t}|_{\mathrm{pt}_\alpha})$$

is the genus zero twisted descendant potential. Applying the localization formula for the NMSP two point function, we have

Lemma 1.8. *For any $\epsilon \in \mathcal{H}^{\mathrm{ev}} \otimes \mathbb{A}$ and $\tau \in \mathcal{H}$*

$$J^M(\mathbf{t},-z) \in \mathcal{L}^{\mathrm{loc}} \quad \text{and} \quad zS^M_\epsilon(z)^{-1}\tau \in \mathcal{L}^{\mathrm{loc}} \cap T_\epsilon \mathcal{L}^{\mathrm{loc}}.$$

Proof. We compute the $J^M(\mathbf{t},-z)$ by localization. This will give us the expansion of the $J^M(\mathbf{t},-z)$ as a Laurent series at $z=0$. The reason is as follows. Suppose v is the vertex with the insertion in the localization graph. There are two possibilities:

(1) v is an unstable vertex, i.e. there are at most one more edge adjacent to v and $d_v = 0$: then the ψ class at v is invertible, the localization graph will contribute terms with only non-negative powers of z to $J^M(\mathbf{t},-z)$;
(2) v is a stable vertex, i.e. there are at least two more edges adjacent to v or $d_v > 0$: then the ψ class at v is not invertible, the localization graph will contribute terms with only negative powers of z to $J^M(\mathbf{t},-z)$.

Summarizing, the non-negative part

$$L^{\mathrm{loc}}(\mathbf{t},z) := z\mathbf{1} + [J^M(\mathbf{t},-z)]_+ \in \mathcal{H} \otimes \mathbb{A}[\![z]\!]$$

is given by the sum of the contributions of the graphs which are tails with the unstable starting vertex v; the negative part $[J^M(\mathbf{t},-z)]_-$ is the sum of the contributions of tails with stable starting vertex, which is given by

$$\sum_{k,\alpha} e^\alpha \big\langle L^{\mathrm{loc}}(\mathbf{t},\psi)^k, \frac{e_\alpha}{z-\psi} \big\rangle^{\mathrm{loc}}_{0,1+k}.$$

Hence we have

$$J^M(\mathbf{t},-z) = -z\mathbf{1} + L^{\mathrm{loc}}(\mathbf{t},z) + \sum_{k,\alpha} e^\alpha \big\langle L^{\mathrm{loc}}(\mathbf{t},-\psi)^k, \frac{e_\alpha}{-z-\psi} \big\rangle^{\mathrm{loc}}_{0,1+k} = J^{\mathrm{loc}}(L^{\mathrm{loc}},-z) \in \mathcal{L}^{\mathrm{loc}}.$$

Moreover, for $\tau \in \mathcal{H}$, $\epsilon := \epsilon(t) \in \mathcal{H}^{\mathrm{ev}}$ with some parameter t (for example t is the flat coordinate), we consider $zS^M_\epsilon(z)^{-1}\tau$. Let $\star_\epsilon$ be the quantum product of the $\mathcal{L}_p^{\otimes 5}$-twisted GW theory (Principle (★)). Then the S-function satisfies the QDE (see [CG07, LeP04])

$$zdS^M_\epsilon(z) = d\epsilon *_\epsilon S^M_\epsilon(z).$$

Suppose τ lies in the subring of the quantum coholomogy ring generated by $\epsilon' := \frac{d}{dt}\epsilon$. Then it is of the form

$$zS^M_{\epsilon(t)}(z)^{-1}\tau = \sum_k a_k(\tau) z^k \frac{d^k}{dt^k} J^M(\epsilon(t),-z).$$

Since the Lagrangian cone $\mathcal{L}^{\rm loc}$ is closed under the operation $z\frac{d}{dt}$ (c.f. [CG07], [Gi01b]), it lies in $\mathcal{L}^{\rm loc}$.

Notice that by divisor equation, when $\epsilon = tp$ the operator $D_p := p + zq\frac{d}{dq}$ plays the role of $z\frac{d}{dt}$. And it is clear in our case the hyperplane class p generates the even part of the quantum cohomology ring. At the point $\epsilon = 0$, the explicit form of the QDE is given in Lemma 1.12. □

By Givental's result on Lagrangian cone (cf. [Gi03]), the Lagrangian cone $\mathcal{L}^{\rm loc}$ consists of a family of linear spaces

$$zS^{\rm loc}_\tau(z)^{-1}\,\mathcal{H}\otimes\mathbb{A}[\![z]\!],$$

which gives us the Birkhoff factorization, defining us the desired $R^{\rm loc}$-matrix.

Definition 1.9. *There exists a unique $R^{\rm loc}_\epsilon(z)\in \operatorname{End}\mathcal{H}\otimes\mathbb{A}[\![z]\!]$ such that*

$$S^M_\epsilon(z) = R^{\rm loc}_\epsilon(z) S^{\rm loc}_{\tau^{\rm loc}}(z), \tag{1.15}$$

where $\tau^{\rm loc}=\tau^{\rm loc}(\epsilon,q)$ is defined by the Dijkgraaf-Witten map

$$\tau^{\rm loc}(\epsilon,q) := \sum_{\alpha,n}\frac{1}{n!}e^\alpha\big\langle e_\alpha,\mathbf{1},L^{\rm loc}(\epsilon,-\psi)^n\big\rangle^{\rm loc}_{0,n+2}\in\mathcal{H}\otimes\mathbb{A}. \tag{1.16}$$

By the localization formula, we have $L^{\rm loc}(0,\psi)\in\mathcal{H}\otimes q\mathbb{A}[\![\psi]\!]$ and thus $\tau^{\rm loc}(0,q)\in\mathcal{H}\otimes q\mathbb{A}$.

We denote $\tau_Q(q')$ and $\tau_\alpha(q')$ to be the components of the Dijkgraaf-Witten map:

$$\tau_Q(q') := \tau^{\rm loc}(0,q)|_Q,\qquad \tau_\alpha(q') := \tau^{\rm loc}(0,q)|_{\mathrm{pt}_\alpha}, \tag{1.17}$$

where we recall $q' := -q/t^{\rm N}$. In Lemma 5.1 and Lemma 6.1 (1), we will show that, when $\mathrm{N}>5$,

$$\tau_\alpha(q) = \zeta_{\rm N}^\alpha\, t\cdot\textstyle\int_0^q\left((1-5^5x)^{\frac{1}{\rm N}}-1\right)\frac{dx}{x},\qquad \tau_Q(q) = I_1(q)/I_0(q)H. \tag{1.18}$$

Together they give an explicit formula of $\tau^{\rm loc}(0,q)$.

Convention 1.10. *In the remainder of this paper, we will abbreviate*

$$S^M_{\epsilon=0},\quad S^{\rm loc}_{\tau^{\rm loc}(0,q)},\quad R^{\rm loc}_{\epsilon=0},\quad S^{Q,\rm tw}_{\tau_Q(q')},\quad S^Q_{\tau_Q(q)},\quad S^{\mathrm{pt}_\alpha,\rm tw}_{\tau_\alpha(q')},\quad S^{\mathrm{pt}_\alpha}_{\tau_\alpha(q)}$$

to S^M, $S^{\rm loc}$, $R^{\rm loc}$, $S^{Q,\rm tw}$, S^Q, $S^{\mathrm{pt}_\alpha,\rm tw}$, S^{pt_α} respectively.

By convention 1.10, we see

$$S^{Q,\rm tw} = S^Q|_{q\mapsto q'}\quad\text{and}\quad S^{\mathrm{pt}_\alpha,\rm tw} = S^{\mathrm{pt}_\alpha}|_{q\mapsto q'}.$$

Since both the local and global S-matrices

$$S^M\in\operatorname{End}(\mathcal{H})\otimes\mathbb{A}[\![z,z^{-1}]\!]\quad\text{and}\quad S^{\rm loc}\in\operatorname{End}(\mathcal{H})\otimes\mathbb{A}[\![z^{-1}]\!]$$

are symplectic (see e.g. [Gi01a]), we see that $R^{\rm loc}\in\operatorname{End}(\mathcal{H})\otimes\mathbb{A}[\![z]\!]$ is symplectic as well.

Remark 1.11. *By Remark 1.7, for any $x\in H^3(Q)$, $S^M(z)x=x$ and $S^{\rm loc}_{\tau^{\rm loc}(0,q)}(z)x=x$. Thus we have $R^{\rm loc}(z)x=x$. The symplectic property then implies $S^M, S^{\rm loc}_{\tau^{\rm loc}(0,q)}, R^{\rm loc}$ all preserve $\mathcal{H}^{\rm ev}$.*

1.5. **Explicit formula for** NMSP **S-function.** In last section, we have defined the localization R-matrix $R^{\rm loc}$ by using the Lagrangian cone $\mathcal{L}^{\rm loc}$ defined by the local theory. In this section, we will consider the application of the Lagrangian cone defined by NMSP-theory, which is identical to the Lagrangian cone of the $\mathcal{O}_{\mathbb{P}^{4+\rm N}}(5)$-twisted GW theory of $\mathbb{P}^{4+\rm N}$ via Principle (★). Hence we can apply the result of [CG07] and [Gi03] directly.

Lemma 1.12. *Let $D_p := p + zq\frac{d}{dq}$. Restricted to $\mathcal{H}^{\rm ev}$, the S-matrix of* NMSP *theory $S^M(z)^*$ satisfies the following quantum differential equation*

$$D_pS^M(z)^* = S^M(z)^*\cdot A^M. \tag{1.19}$$

Here $A^M \in \operatorname{End}\mathcal{H}^{\mathrm{ev}}$ has the following matrix form with respect to the basis $\{\phi_i = p^i\}_{i=0}^{\mathrm{N}+3}$:

$$A^M = \left[\begin{array}{cccccccccccc}
0 & & & & & & & 120q & & & & \\
1 & 0 & & & & & & & 770q & & & \\
 & 1 & 0 & & & & & & & 1345q & & \\
 & & 1 & 0 & & & & & & & 770q & \\
 & & & 1 & 0 & & & & & & & 120q - t^{\mathrm{N}} \\
\hline
 & & & & 1 & 0 & & & & & & \\
 & & & & & 1 & 0 & & & & & \\
 & & & & & & 1 & 0 & & & & \\
 & & & & & & & \cdots & \cdots & & & \\
 & & & & & & & & 1 & 0 & & \\
 & & & & & & & & & 1 & 0 & \\
 & & & & & & & & & & 1 & 0
\end{array}\right]. \tag{1.20}$$

Proof. The QDE matrix A^M can be computed via Birkhoff factorization, which is an algorithm starting from J-functions to get S-matrices. The $S^*(z)\phi_i$ are computed recursively as follows:

(1) By definition, for $\phi_0 = \mathbf{1}$, we have $S^M(z)^*\mathbf{1} \; = z^{-1}J^M(z)$.

(2) Suppose we have already obtained a closed formulae of
$$S^M(z)^*\mathbf{1}, S^M(z)^*\phi_1, \cdots, S^M(z)^*\phi_k.$$
Apply $D_p := zq\frac{d}{dq} + p$ to $S^M(z)^*\phi_k$. It will keep the element in the Lagrangian cone. We search for a linear combination of
$$S^M(z)^*\mathbf{1}, \quad S^M(z)^*\phi_1, \quad \cdots, \quad S^M(z)^*\phi_k, \quad D_pS^M(z)^*\phi_k$$
(with coefficients in $\mathbb{A}$) such that the combination takes the form
$$\phi_{k+1} + O(z^{-1}).$$
By Coates-Givental's result such linear combination is exactly $S^M(z)^*\phi_{k+1}$.

(3) The process stops when no new ϕ_k appears. Whenever ϕ_1 generates the even part of quantum cohomology ring, this algorithm provides us the full $S^M(z)^*$.

Applying this algorithm to the I-function $I^M(z)$, we obtain exactly (1.20). □

1.6. **The specialized S-function.** To compute the tail contributions and convert descendants to ancestors, we introduce the specialized S-functions, which will be used later.

Definition 1.13. *We define*
$$\mathcal{S}_a^\alpha := S^M(z)\mathbf{1}^\alpha\big|_{z=\frac{5t_\alpha}{a}}, \quad \text{and} \quad \mathcal{S}_{a;i}^\alpha := (\mathcal{S}_a^\alpha, p^i)^M. \tag{1.21}$$

Lemma 1.14. *Let $(a)_k := a\cdot(a-1)\cdot\dots\cdot(a-k+1)$, we have the following explicit formula*
$$\mathcal{S}_{a;0}^\alpha = 1 + \sum_{d=1}^{\lceil a/5\rceil-1} q^d \frac{(a-1)_{5d}}{\left((\frac{a}{5}-1)_d\right)^5} \frac{\left(a/5t\right)^{\mathrm{N}d}}{\prod_{m=1}^d \left((-\frac{a}{5}+m)^{\mathrm{N}} - (\frac{a}{5})^{\mathrm{N}}\right)}.$$

Proof. By the definition of $\mathcal{S}_{a;0}^\alpha$ and the relation $J^M(z) = zS^M(z)^*\mathbf{1}$, we see
$$\tfrac{5t_\alpha}{a}\cdot\mathcal{S}_{a;0}^\alpha\mathbf{1}_\alpha = \tfrac{5t_\alpha}{a}\cdot(\mathcal{S}_a^\alpha, \mathbf{1})^M \; \mathbf{1}_\alpha = J^M(\tfrac{5t_\alpha}{a})|_{\mathrm{pt}_\alpha}$$
Together with Theorem 1.1, we obtain
$$\frac{5t_\alpha}{a}\cdot\mathcal{S}_{a;0}^\alpha = I^M(q,z)\big|_{z=\frac{5t_\alpha}{a},\, p=-t_\alpha}.$$
A direct computation then proves this lemma. □

Corollary 1.15. *The following properties hold*

(1) For any α, we have $\mathcal{S}^\alpha_{1;0}=1$ and $\left\langle \frac{\mathbf{1}^\alpha}{5t_\alpha-\psi}\right\rangle^M_{0,1}=0$;

(2) For any α, a and i, we have $\mathcal{S}^\alpha_{a;i}=\zeta_{\mathrm{N}}^{i(\alpha-\beta)}\cdot\mathcal{S}^\beta_{a;i}$;

(3) For any α, a and i, $\mathcal{S}^\alpha_{a;i}\in\mathbb{Q}(\zeta_{\mathrm{N}})[q/t^{\mathrm{N}}]$ is a polynomial in q and

$$\deg_q \mathcal{S}^\alpha_{a;i}\le\begin{cases}\lceil a/5\rceil-1 & \text{if } i<\mathrm{N},\\ \lceil a/5\rceil & \text{if } i\ge\mathrm{N}.\end{cases}\tag{1.22}$$

Proof. Item (1) is a direct consequence of Lemma 1.14 and the definition of the J-function:

$$J^M\Big(\frac{5t_\alpha}{a}\Big)\Big|_{\mathrm{pt}_\alpha}=\frac{5t_\alpha}{a}\mathbf{1}_\alpha+\left\langle\frac{\mathbf{1}^\alpha}{5t_\alpha/a-\psi}\right\rangle^M_{0,1}\mathbf{1}_\alpha.$$

To prove (2) and (3), we will need the explicit QDE for the S-matrix, which is computed in Lemma 1.12. By using (1.19), we have

$$\mathcal{S}^\alpha_{a;i}=\begin{cases}\Big(-t_\alpha+\frac{5t_\alpha}{a}q\frac{d}{dq}\Big)\mathcal{S}^\alpha_{a;i-1} & \text{if } i<\mathrm{N},\\ \Big(-t_\alpha+\frac{5t_\alpha}{a}q\frac{d}{dq}\Big)\mathcal{S}^\alpha_{a;i-1}-c_i\,q\cdot\mathcal{S}^\alpha_{a;i-\mathrm{N}} & \text{if } i\ge\mathrm{N},\end{cases}\tag{1.23}$$

where $(c_j)_{j=\mathrm{N},\cdots,\mathrm{N}+3}=(120,770,1345,770)$. We see that (2) and (3) follow from inductions via (1.23), with initial conditions given by Lemma 1.14. □

2. NMSP localization as bipartite graphs

In this section we will show that the contribution of a NMSP localization graph can be obtained by combining its contributions from "$(1,\infty]$ parts" with those from its "$[0,1]$ parts". We use bipartite graphs to organize such decompositions.

A *bipartite* graph is a connected graph whose vertices are colored white or black, such that any two adjacent vertices have different colors.

Definition 2.1. *A decorated bipartite graph is a bipartite graph with decorations:*

- **vertices***: each vertex v is decorated by an integer $g_v\ge 0$, and in addition each black vertex v is decorated by $d_{\infty[v]}\in\mathbb{Z}$; the set of vertices is denoted by $V(\Lambda)$;*
- **edges***: each edge e is decorated by an integer $a_e>0$, and has its hour $\alpha_e\in[\mathrm{N}]$;*
- **legs***: all legs are incident to white vertices; the set of legs is denoted by $L(\Lambda)$;*
- **constraint***: each white vertex v with genus zero must have $|L_v|+|E_v|\ge 2$.*

For a decorated bipartite graph Λ, we denote by $V_w(\Lambda)$ (resp. $V_b(\Lambda)$) the set of its white (resp. black) vertices, and we denote by $F(\Lambda):=\{(e,v):e\in E_v\}$ the set of flags. We define its genus $g=h^1(\Lambda)+\sum_{v\in V(\Lambda)}g_v$, degree $d_\infty=\sum_{v\in V_b(\Lambda)}d_{\infty[v]}$, and n the number of its legs. For the decorated bipartite graphs we are working with, we require one additional constraint:

$$d_{\infty[v]}+\tfrac{1}{5}\big(2g_v-2-\textstyle\sum_{e\in E_v}(a_e-1)\big)\ge 0,\qquad \forall v\in V_b(\Lambda).\tag{2.1}$$

For each bipartite graph Λ, we perform the standard "stabilization" operation as described in Appendix B [11] , which induces two maps (see Example B.1)

$$cl: L(\Lambda)\to V(\Lambda)\quad\text{and}\quad c: L(\Lambda)\to F(\Lambda)\cup L(\Lambda).$$

The map cl sends each leg to its associated stable-graph-vertex in Λ (the closest survival vertex after stablization). There are two cases: (1) When the vertex v_l incident to l is a stable-graph-vertex (i.e. $cl(l)=v_l$), we set $c(l)=l$. (2) Otherwise, there is a unique subchain of Λ connecting v_l and $v_l^\circ:=cl(l)$. Let e_l° be the edge in this subchain that is incident to v_l°, we set $c(l)=(e_l^\circ,v_l^\circ)\in F_{v_l^\circ}$. In both cases $\bar\psi_{c(l)}$ is the ancestor class at the vertex $cl(l)$. For each vertex v of Λ, we define $L_v^\circ:=\{l\in L(\Lambda), cl(l)=v\}$. Then $L_v^\circ\ne\emptyset$ only if v is a stable-graph-vertex.

[11] When we stabilize a bipartite graph, we treat it as a dual graph, with only genus assignments considered.

The main theorem we will prove in this section is

Theorem 2.1. *Let $\Xi_{g,n}^{d_\infty}$ be the (finite) set of all decorated bipartite graphs of genus g, n legs and degree d_∞ satisfying* (2.1)*. For even classes insertions $\tau_1, \cdots, \tau_n$, we can express the total* NMSP *correlators via* NMSP-$[0,1]$ *correlators $\langle - \rangle_{g,n}^{[0,1]}$ and certain q-polynomials $\mathrm{Cont}_{[v]}^{\infty}$:*

$$\Big\langle \tau_1 \bar\psi_1^{k_1}, \cdots, \tau_n \bar\psi_n^{k_n} \Big\rangle_{g,n,d_\infty}^{M} = \sum_{\Lambda \in \Xi_{g,n}^{d_\infty}} \frac{1}{|\operatorname{Aut}\Lambda|} \prod_{v \in V_b(\Lambda)} \mathrm{Cont}_{[v]}^{\infty}\big(\bigotimes_{i \in L_v^\circ} \bar\psi_{c(i)}^{k_i} \big)$$
$$\prod_{v \in V_w(\Lambda)} \Big\langle \bigotimes_{i \in L_v} \tau_i \bigotimes_{i \in L_v^\circ} \bar\psi_{c(i)}^{k_i} \bigotimes_{e \in E_v, f=(e,v)} \frac{\mathbf{1}^{\alpha_e}}{\frac{5t_{\alpha_e}}{a_e} - \psi_f} \Big\rangle_{g_v,n_v}^{[0,1]}.$$

Here for each $v \in V_b(\Lambda)$, its contribution is a q-polynomial with degree bound

$$\mathrm{Cont}_{[v]}^{\infty}\Big(\bigotimes_{i \in L_v^\circ} \bar\psi_{c(i)}^{k_i} \Big) \in \mathbb{Q}(t)[q], \quad \deg_q \le d_{\infty[v]} + \frac{1}{5}\Big(2g_v - 2 - \sum_{e \in E_v} (a_e - 1) \Big),$$

and for any $x, y \in \mathcal{H}$ the unstable correlators are defined by

$$\Big\langle \frac{x}{z_1 - \psi_1}, \frac{y}{z_2 - \psi_2} \Big\rangle_{0,2}^{[0,1]} := \frac{(x,y)^M}{z_1 + z_2} + \Big\langle \frac{x}{z_1 - \psi_1}, \frac{y}{z_2 - \psi_2} \Big\rangle_{0,2}^{M};$$
$$\Big\langle \frac{x}{z_1 - \psi_1}, y \Big\rangle_{0,2}^{[0,1]} := \big(x, y\big)^M + \Big\langle \frac{x}{z_1 - \psi_1}, y \Big\rangle_{0,2}^{M}.$$

2.1. **From localization graphs to bipartite graphs.** To each regular decorated localization graph $\Theta \in G_{g,n,\mathbf{d}}^{\mathrm{reg}}$ (with $\mathbf{d} = (d_0, d_\infty)$), we associate a decorated bipartite graph as follows.

Let $v \in V_\infty(\Theta)$. A "restricted tail" of Θ rooted at v is the union of some $e \in E_{1\infty}$ incident to v, with a rational tail of Θ which is supported on $[0,1]$ and incident to e. To get the bipartite graph $\Lambda(\Theta)$, we first remove all restricted tails of Θ, resulting in a graph Θ'. We then collapse every maximal connected subgraph of Θ' supported[12] in $[0,1]$ (resp. ∞) to a white vertex (resp. a black vertex); we decorate the resulting vertices by the total genera of the subgraphs collapsed. These vertices form the vertices of the bipartite graph $\Lambda(\Theta)$ we are constructing.

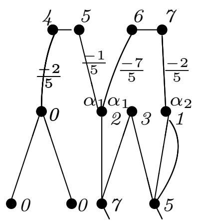
The edges and the legs of $\Lambda(\Theta)$ are the same as $E_{1\infty}(\Theta')$ and $L(\Theta)$, with the incidence relation the induced one. Because all markings are of type $(1,\rho)$, all legs in $\Lambda(\Theta)$ are incident to white vertices. By construction, each edge is incident to one white and one black vertex, giving us a bipartite graph $\Lambda(\Theta)$.

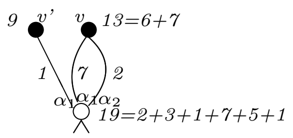
For an edge e of $\Lambda(\Theta)$, we assign its hour α_e to be the same hour of $e \in E_{1\infty}(\Theta)$; we also assign $a_e := -5d_e \in \mathbb{N}$. To a black vertex v of $\Lambda(\Theta)$, letting $\Theta_v \subset \Theta$ be the subgraph (supported in ∞) contracts to v, we assign

$$d_{\infty[v]} := d_{\infty\Theta_v} + \sum d_{\infty e}, \tag{2.2}$$

where the summation is over all edges $e \in E_{1\infty}(\Theta)$ incident to Θ_v. One checks that the inequality (2.1) holds true. This constructs the bipartite graph $\Lambda(\Theta) \in \Xi_{g,n}^{d_\infty}$ associated to Θ.

Example 2.2. *From a localization graph Θ to a decorated bipartite graph $\Lambda(\Theta)$:*

$\Theta =$ 

$\Lambda(\Theta) =$ 

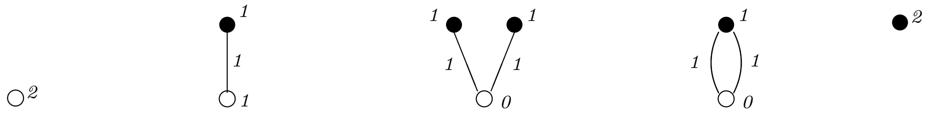
where the integer next to a vertex (an edge) is its genus (its a_e), and each vertex of Θ is indexed, including the indices $u, w \in \{1, \cdots, \mathrm{N}\}$, and we omit the decorations d_0, d_∞ here.

[12] A graph is supported in $[0,1]$ (resp. ∞) if all its vertices has level 0 or 1 (resp. ∞).

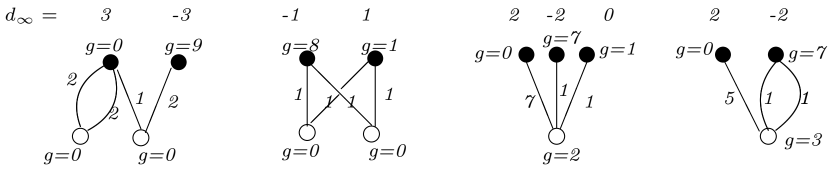
Remark 2.3. *In the construction, each white vertex of $\Lambda(\Theta)$ is collapsed from a subgraph of Θ. We call the union of all such subgraphs the $[0,1]$ part of Θ.*

Example 2.4. *We illustrate some decorated bipartite graphs of some specified (g,n,d_∞). Here in the first two Figures each vertex has zero d_∞, and each edge e is decorated by an "hour" $\alpha\in\{1,\cdots,\mathrm{N}\}$. We omit these decorations in the graphs.*

- *Figure: The list of all $(g,n,d_\infty)=(2,0,0)$ bipartite graphs. Here at each black vertex, the d_∞ must be zero by the constraint (2.1).*



- *Figure: Some bipartite graphs of $(g,n,d_\infty)=(10,0,0)$, where the numbers in the top row denotes the d_∞ of each vertex right below.*



2.2. **The contribution from a black vertex in the bipartite graph.** Next, we analysis the localization contributions of black vertices and their incident edges.

Let $\Lambda\in\Xi^{d_\infty}_{g,\ell}$ and let $v\in V_b(\Lambda)$ with $E_v=\{e_1,\cdots e_m\}$. We let $[v]$ be the bipartite graph with one black vertex v, m edges $e_1,\cdots,e_m$, m genus 0 white vertices $v_1,\cdots,v_m$, and m $(1,\rho)$-legs $l_1,\cdots,l_m$, where each v_i is incident to e_i and each l_i is incident to v_i. We set $\alpha_i:=\alpha_{e_i}$, which is the hour of e_i, the hour e_i as an edge in Λ. This way, $[v]\in\Xi^{d_{\infty[v]}}_{g_v,m}$.

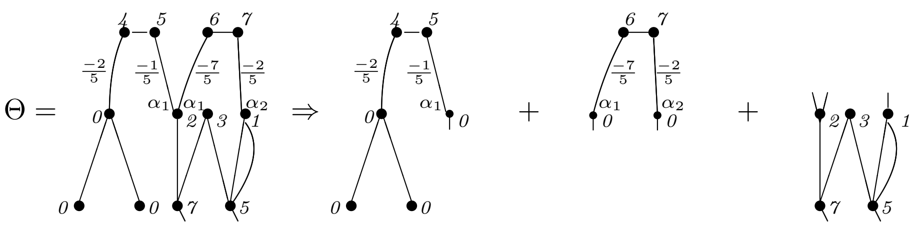
For the $[v]$ of the shape just described, we say a regular decorated localization graph Θ *strongly contracts* to $[v]$, if $\Lambda(\Theta)\cong[v]$ *and for each i the subgraph Θ_{v_i} of Θ that is collapsed to v_i (in $[v]$) has total d_0-degree zero (i.e. Θ_{v_i} is a one unstable vertex graph)* .

Definition 2.5. *Denote by $G_{[v]}$ the set of all graphs Θ strongly contracting to $[v]$. We define*

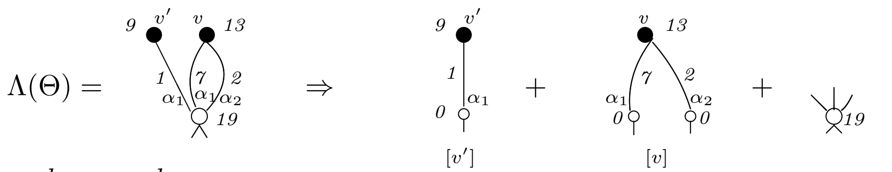
$$\mathrm{Cont}^\infty_{[v]}:=\sum_{d\geq 0}(-1)^{d+1-g}q^d\sum_{\Theta\in G_{[v]}}\int_{[F_\Theta]^{\mathrm{vir}}}\frac{1}{e(N^{\mathrm{vir}}_\Theta)}\prod_{i=1}^m\mathrm{ev}_i^*\mathbf{1}_{\alpha_i}\in\mathbb{A},\tag{2.3}$$

the sum of contributions to $\langle\mathbf{1}_{\alpha_1},\cdots,\mathbf{1}_{\alpha_m}\rangle^M_{g_v,m,d_{\infty[v]}}$ from all $\Theta\in G_{[v]}$. If a factor $\prod_{i=1}^m\bar\psi_i^{u_i}$ is added to the integrand in (2.3), we denote the result by $\mathrm{Cont}^\infty_{[v]}(\otimes_{i=1}^m\bar\psi_i^{u_i})$.

Example 2.6. *The contribution of the graph Θ in Example 2.2 can be partitioned into a product of the contributions of three subgraphs. Here the smaller black dots are unstable vertices.*



The three graphs in the above R.H.S yield the partition of $\Lambda(\Theta)$ into three parts:



Here α_1 and α_2 are hours.

To proceed, for any $\Theta \in G^{\mathrm{reg}}_{g,\gamma,\mathbf{d}}$ we define its infinity part Θ_∞ as follows: The vertices of Θ_∞ is the set $V_\infty(\Theta)$, its edges are edges in $E_{\infty\infty}(\Theta)$, and its legs are elements in $E_{1\infty}(\Theta)$, with incidence relations the induced one. For $l_e \in L(\Theta_\infty)$ associated to $e \in E_{1\infty}(\Theta)$, we assign its monodromy to be $\zeta_5^{w_e}$ satisfying $w_e - 5d_{\infty e} \equiv 0(5)$. Because Θ is regular, we can choose $w_e \in [1,4]$.

Suppose Γ is the infinity part of any regular decorated graph, and Γ has total genus g_Γ and bidegree $d_{0\Gamma}, d_{\infty\Gamma}$. Suppose $l_1, \cdots, l_m$ are the legs of Γ, of (narrow) monodromy assignments $\gamma = (\zeta_5^{w_1}, \cdots, \zeta_5^{w_m})$, with $\mathbf{w} = (w_1, \cdots, w_m)$ and $w_i \in [1,4]$. Then $\Gamma \in G^{\mathrm{reg}}_{g_\Gamma,\gamma,(d_{0\Gamma},d_{\infty\Gamma})}$, and $\Gamma_\infty = \Gamma$. We form $\mathcal{W}_{(\Gamma)}$, which is the image of

$$\mathcal{W}_\Gamma \to \mathcal{W}_{g_\Gamma,\gamma,(d_{0\Gamma},d_{\infty\Gamma})}.$$

Over $\mathcal{W}_{(\Gamma)}$ we denote the coarse-psi classes by ψ_i, and its ancestors by $\bar\psi_i$, which are pullback of the i-th psi class via the forgetful map $\mathcal{W}_{(\Gamma)} \to \overline{\mathcal{M}}_{g_\Gamma,m}$. For $(c_1, \cdots, c_m) \in (\mathbb{Z}_{\geq 0})^m$ and $\vec z = (z_1, \cdots, z_m)$, we define the extended dual twisted FJRW m-point function by

$$\mathcal{F}^\bullet_\Gamma(\vec z; \otimes_{i=1}^m \bar\psi_i^{c_i}) := q^{d_{0\Gamma}} \int_{[\mathcal{W}_{(\Gamma)}]^{\mathrm{vir}}} \frac{1}{e(N_\Gamma^{\mathrm{vir}})} \prod_{i=1}^m \frac{\bar\psi_i^{c_i}}{z_i - \psi_i/5}.$$

Let $\alpha_i = \alpha_{v_i}$ be the hour of the vertex v_i to which l_i is incident; let $\vec a = (a_1, \cdots, a_m) \in \mathbb{N}^m$ be such that $a_i \equiv w_i(5)$. We define

$$\mathcal{F}^\bullet_{\Gamma,\vec a}(\otimes_{i=1}^m \bar\psi_i^{c_i}) := (-1)^{d_{0\Gamma}+1-g_\Gamma} \cdot \mathcal{F}^\bullet_\Gamma\left(\tfrac{-t_{\alpha_1}}{a_1}, \cdots, \tfrac{-t_{\alpha_m}}{a_m}; \otimes_{i=1}^m \bar\psi_i^{c_i}\right). \tag{2.4}$$

In the case all $c_i = 0$, we denote (2.4) by $\mathcal{F}^\bullet_{\Gamma,\vec a}$.

2.3. **Proof of Theorem 2.1.** We first consider the case $k_1 = \cdots = k_n = 0$. By the localization formula (see [NMSP1, Sect. 6] for the explicit formuanmsp), we see that the decomposition of Θ into its $[0,1]$ part and $(1,\infty]$ part (c.f. Example 2.6) is consistent with their localization contributions.

Let $\Theta \in G_{[v]}$ (Definition 2.5). By definition, it has the shape given by Figure 1. Here the infinity line represents the infinity part of Θ, i.e. Θ_∞. For convenience we brief $\Gamma := \Theta_\infty$. The edges $E_{1\infty}(\Theta)$ are divided into two types: Type-I are edges $e_1, \cdots, e_m$ so that each vertex $v_i \in V_1(\Theta)$ incident to e_i is unstable and has one leg l_i incident to it; Type-II are edges $e'_1, \cdots, e'_\ell$ so that the maximal connected subgraph Γ_j of $\Theta_{[0,1]}$ attached to e'_j has total genus zero and no legs. Each Γ_j with e'_j is a restricted tail, defined in the beginning of §2.1.

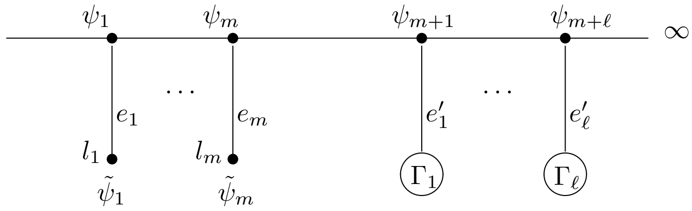


FIGURE 1.

For $s = 1, \cdots, m$ and $i = 1, \cdots, \ell$, we denote

$$d_{\infty e_s} = \frac{a_s}{5}, \quad \alpha_{e_s} = \alpha_s \quad \text{and} \quad d_{\infty e'_i} = \frac{b_i}{5}, \quad \alpha_{e'_i} = \beta_i.$$

Then $\vec a = (a_1, \cdots, a_m)$, $\vec b = (b_1, \cdots, b_\ell)$ are sequences of integers. We denote by $(\vec a, \vec b)$ the vectors $(a_1, \cdots, a_m, b_1, \cdots, b_\ell)$.

We first make the following simplification. In case $b_j = 1$, by Corollary 1.15, the summation of the contributions of all possible $[0,1]$ tails Γ_j is

$$\left\langle \frac{\mathbf{1}^{\alpha_j}}{5t_\alpha - \psi} \right\rangle^M_{0,1} = 5t_{\alpha_j}\big((\mathcal{S}^{\alpha_j}_{a=1}, \mathbf{1}^\alpha)^M - 1\big)\mathbf{1}_\alpha = 0.$$

Thus from now on we assume $b_i \geq 2$ for all $i = 1, \cdots \ell$.

We claim that

$$d_{\infty[v]} = d_{0\Theta_\infty} + \frac{1}{5}(\sum_{r=1}^{m} a_r + \sum_{i=1}^{\ell} b_i) - \frac{1}{5}(2g_\Gamma - 2 + m + \ell). \tag{2.5}$$

Indeed, for any $(\mathcal{C}, \Sigma^{\mathcal{C}}, \cdots) \in \mathcal{W}_\Theta$, we have $\mathcal{L}^{\otimes 5} \cong \omega_{\mathcal{C}}^{\log}$ when restricted to $\mathcal{C}_a$ for $a \in V(\Theta_\infty) \cup E(\Theta_\infty)$. Thus if we denote the universal line bundle of Θ_∞ by $\mathcal{L}_{\Theta_\infty}$ and $\mathcal{N}_{\Theta_\infty}$, we have

$$\frac{1}{5}(2g_\Gamma - 2 + m + \ell) = \deg \mathcal{L}_{\Theta_\infty} = \deg \mathcal{L}_{\Theta_\infty} \otimes \mathcal{N}_{\Theta_\infty} - \deg \mathcal{N}_{\Theta_\infty} = d_{0\Theta_\infty} - d_{\infty\Theta_\infty}.$$

Therefore (2.2) implies (2.5). As a consequence, (recall $\Gamma = \Theta_\infty$)

$$0 \le \deg_q \mathcal{F}^{\bullet}_{\Gamma,(\vec{a},\vec{b})} = d_{0\Gamma} = d_{\infty[v]} + \frac{2g_\Gamma - 2}{5} - \frac{\sum_{r=1}^{m}(a_r - 1)}{5} - \frac{\sum_{i=1}^{\ell}(b_i - 1)}{5}. \tag{2.6}$$

Let $\mathrm{Cont}([v], \Gamma, \vec{b})$ be the sum of contributions to (2.3) from all such Θ's with prescribed $\Gamma = \Theta_\infty$ and $\vec{b}$ in Figure 1. Then we have

$$\mathrm{Cont}^{\infty}_{[v]} = \sum_{\Gamma, \vec{b}} \mathrm{Cont}([v], \Gamma, \vec{b}), \tag{2.7}$$

where the sum is over all Γ, with the prescibed g_Γ, $\vec{\alpha}$, $\vec{a}$, $d_{\infty[v]}$, and with varying $\vec{b}$. It is a finite sum since $d_{0\Gamma}$ and all $b_i - 1$ are positive, and thus bounded using (2.5).

Theorem 2.1, in the case $k_1 = \cdots = k_n = 0$, follows from (2.7) and the following lemma:

Lemma 2.7. *The contribution*

$$\mathrm{Cont}([v], \Gamma, \vec{b}) \in A[\![q]\!]$$

is a polynomial in q whose degree is bounded by $d_{\infty[v]} + \frac{1}{5}\big(2g_v - 2 - \sum_{i=1}^{m}(a_i - 1)\big)$.

Proof. We calculate $\mathrm{Cont}([v], \Gamma, \vec{b})$ according to the following subcases.

▷ *Case 1:* $(g_\Gamma, m + \ell, d_{0\Gamma}) \neq (0, 2, 0), (0, 1, 0)$.

Recall the edge/flags contributions in [NMSP1] and [CLLL16] give $\tilde{A}_{e_s}$ and $\tilde{A}_{e''_s}$ as below. For any edge $e \in E^{\alpha}_{\infty 1}$, let $u = d_e := d_{0e} - d_{\infty e} \in \frac{-1}{5}\mathbb{N}$, and set

$$\tilde{A}_e := \frac{\prod_{j=1}^{\lceil -u \rceil - 1}(-t_\alpha - \frac{jt_\alpha}{u})^5}{\prod_{j=1}^{-5u}(-\frac{jt_\alpha}{u}) \prod_{j=1}^{\lfloor -u \rfloor}(\frac{jt_\alpha}{u})}, \qquad c_{\vec{a},\vec{b}}(\vec{\alpha}, \vec{\beta}) := \frac{1}{(\prod a_s)(\prod b_i)} \cdot (\prod_{s=1}^{m} \tilde{A}_{e_s})(\prod_{i=1}^{\ell} \tilde{A}_{e''_i}).$$

We denote $|\vec{b}| := b_1 + \cdots + b_\ell$. Applying the localization formula we have

$$\begin{aligned}
\mathrm{Cont}([v], \Gamma, \vec{b}) &= (-1)^{d_{0\Gamma} + 1 - g_\Gamma} \frac{c_{\vec{a},\vec{b}}(\vec{\alpha}, \vec{\beta})}{|\operatorname{Aut} \vec{b}\,|} \prod_{s=1}^{m}(5t_{\alpha_s}) \prod_{i=1}^{\ell} \Big((5t_{\beta_i}) J^M(0, \frac{5t_{\beta_i}}{b_i})|_{\mathrm{pt}_{\beta_i}}\Big) \\
&\quad \cdot \mathcal{F}^{\bullet}_{\Gamma}(\tfrac{-t_{\alpha_1}}{a_1}, \cdots, \tfrac{-t_{\alpha_m}}{a_m}, \tfrac{-t_{\beta_1}}{b_1}, \cdots, \tfrac{-t_{\beta_\ell}}{b_\ell}) \\
&= \frac{c_{\vec{a},\vec{b}}(\vec{\alpha}, \vec{\beta})}{|\operatorname{Aut} \vec{b}\,|} \prod_{s=1}^{m}(5t_{\alpha_s}) \Big[\prod_{i=1}^{\ell}(5t_{\beta_i})(\frac{5t_{\beta_i}}{b_i}) \mathcal{S}^{\beta_i}_{b_i;0}\Big] \mathcal{F}^{\bullet}_{\Gamma,\vec{a},\vec{b}}.
\end{aligned} \tag{2.8}$$

By Corollary 1.15, each $\mathcal{S}^{\beta_i}_{b_i;0}$ is a polynomial in q/t, independent of β_i.

▷ *Case 2:* $(g_\Gamma, m + \ell, d_{0\Gamma}) = (0, 2, 0)$ *or* $(0, 1, 0)$.

Notice the case $(m, \ell) = (0, 2)$ is negligible. Using localization formula of [CLLL16, NMSP1] we obtain their contributions as follows.

- $(m, \ell) = (2, 0)$: The only nontrivial case is when $5 \nmid a_1$, and $a_1 + a_2 \equiv 0 \bmod 5$. The contribution is calculated to be

$$\frac{\tilde{A}_{e_1}\tilde{A}_{e_2}}{a_1 a_2} \frac{1}{-\frac{t_\alpha}{a_1} - \frac{t_\alpha}{a_2}} (5t_\alpha)^2 \cdot \frac{-5t_\alpha^6}{\prod_{j \neq \alpha}(t_j - t_\alpha)}.$$

- $(m,\ell)=(1,1)$: The only nontrivial case is when $5\nmid a_1$ and $a_1+b_1\equiv 0 \mod 5$. The contribution is calculated to be

$$(5t_\alpha)\frac{\tilde{A}_{e_1}}{a_1}\frac{1}{-\frac{t_\alpha}{a_1}-\frac{t_\alpha}{b_1}}\frac{\tilde{A}_{e_1''}}{b_1}\cdot\frac{-5t_\alpha^6}{\prod_{j\neq\alpha}(t_j-t_\alpha)}\cdot\left[\frac{(5t_\alpha)^2}{b_1}\cdot\mathcal{S}^{\alpha}_{b_1;0}\right].$$

- $(m,\ell)=(1,0)$: The only nontrivial case is when $n:=\frac{a_1}{5}\in\mathbb{N}$. The contribution is

$$(5t_\alpha)\frac{5t_\alpha}{1-a_1}\frac{\tilde{A}'_{e_1}}{a_1-1}\frac{1}{\prod_{j\neq\beta}(t_j-t_\alpha)},\text{with } \tilde{A}'_{e_1}=\frac{\prod_{j=1}^{n-1}(-t_\alpha-\frac{5jt_\alpha}{-5n+1})^5}{\prod_{j=1}^{5n-1}\frac{-5jt_\alpha}{-5n+1}\prod_{j=1}^{n}\frac{5jt_\alpha}{-5n+1}}.$$

Suppose we are in Case 1. By (1.22) and (2.6) (note that $g_v=g_\Gamma$)

$$\begin{aligned}\sum_{i=1}^{\ell}\deg\mathcal{S}^{\beta_i}_{b_i;0}&\leq\sum_{i=1}^{\ell}(\lceil b_i/5\rceil-1)\leq\frac{1}{5}\sum_{i=1}^{\ell}(b_i-1)\\&=d_{\infty[v]}-d_{0\Gamma}+\frac{2g_v-2}{5}-\frac{\sum_{i=1}^{m}(a_i-1)}{5}.\end{aligned}$$

Then by (2.8) and $\deg_q\mathcal{F}^{\bullet}_{\Gamma,\vec{a},\vec{b}}=d_{0\Gamma}$ the lemma is proved. The Case 2 is easy from above calculations. □

For case with nonzero k_i, one needs to integrate each ancestor $\bar{\psi}_i^{k_i}$ over the virtual cycle at the vertex $v_l^\circ=cl(l)$, regarded as the ancestor of the leg/flag $c(i)$ on v_l°. This amounts to add $\otimes_{i\in L_v^\circ}\bar{\psi}_{c(i)}^{k_i}$ to each biparted vertex's contribution.

As the term $\mathcal{F}^{\bullet}_{\Gamma,\vec{a},\vec{b}}$ in (2.8) is substituted by $\mathcal{F}^{\bullet}_{\Gamma,\vec{a},\vec{b}}(\otimes_{cl(i)=v}\bar{\psi}_{c(i)}^{k_i})$, whose q-degree is still $d_{0\Gamma}$(c.f.(2.4)), this implies the same degree bound in the above lemma. This completes the proof of Theorem 2.1.

3. Stable graph localization formula

In this section, we first give a precise statement of Theorem 3, then we prove this theorem.

Let S^M, S^{pt_α} and S^Q be the S-matrices for the NMSP-theory, the GW theory of the point pt_α and the GW theory of the quintic at the base point 0, $\tau_\alpha(q)$ and $\tau_Q(q)$ respectively[13] (cf. Sect. 1.3). The R-matrix $R(z)\in\operatorname{End}\mathcal{H}\otimes\mathbb{A}[\![z]\!]$ is defined by the following Birkhoff factorization

$$S^M(z)\begin{pmatrix}\mathrm{diag}\{\Delta^{\mathrm{pt}_\alpha}(z)\}_{\alpha=1}^{\mathrm{N}} & \\ & 1\end{pmatrix}=R(z)\begin{pmatrix}\mathrm{diag}\{S^{\mathrm{pt}_\alpha}(z)\}_{\alpha=1}^{\mathrm{N}} & \\ & S^Q(z)\end{pmatrix}\Big|_{q\mapsto q'},\tag{3.1}$$

where $\Delta^{\mathrm{pt}_\alpha}(z)$ is given by Quantum Riemann-Roch (c.f. [CG07]) [14]

$$\Delta^{\mathrm{pt}_\alpha}(z):=\exp\Big[\sum_{k>0}\tfrac{B_{2k}}{2k(2k-1)}\Big(\tfrac{5}{(-t_\alpha)^{2k-1}}+\tfrac{1}{(5t_\alpha)^{2k-1}}+\sum_{\beta\neq\alpha}\tfrac{1}{(t_\alpha-t_\beta)^{2k-1}}\Big)z^{2k-1}\Big].\tag{3.2}$$

Definition 3.1. *We define the translated classes at fixed loci $F=Q$ or pt_α to be*

$$[-]^{F,T}_{g,n}:=\begin{cases}I_0(q')^{-(2g-2+n)}\displaystyle\sum_{d\geq 0}\frac{q^d e^{d(I_1(q)/I_0(q))}}{(-t^{\mathrm{N}})^{(d+1-g)}}\cdot\mathrm{pr}^Q_{g,n*}\big(-|_Q\cap[\overline{\mathcal{M}}_{g,n}(Q,d)]^{\mathrm{vir}}\big) & \text{if } F=Q,\\ L(q')^{\frac{\mathrm{N}+3}{2}(2g-2+n)}\dfrac{5^{1-g}}{(\mathrm{N}(-t_\alpha)^{3+\mathrm{N}})^{1-g}}\displaystyle\sum_{k\geq 0}\frac{1}{k!}\mathrm{pr}_{k*}\big(-|_{\mathrm{pt}_\alpha}\cup\tilde{T}_\alpha(\psi)^k\big) & \text{if } F=\mathrm{pt}_\alpha,\end{cases}$$

where $\mathrm{pr}^Q_{g,n}:\overline{\mathcal{M}}_{g,n}(Q,d)\to\overline{\mathcal{M}}_{g,n}$, $\mathrm{pr}_k:\overline{\mathcal{M}}_{g,n+k}\to\overline{\mathcal{M}}_{g,n}$ are the forgetful maps, and

$$\tilde{T}_\alpha(z):=z(1-L(q')^{\frac{\mathrm{N}+3}{2}}R(z)^{-1}\mathbf{1})|_{\mathrm{pt}_\alpha}\quad\textit{with}\quad L(q):=(1-5^5q)^{1/\mathrm{N}},\ q':=-q/t^{\mathrm{N}}.$$

Notice that since each term in $\tilde{T}_\alpha(z)$ is divisible by z^2, the summation of $[-]^{\mathrm{pt}_\alpha,T}_{g,n}$ is finite.

[13] The explicit formulae of τ_Q and τ_α are given in (1.18). Note they are the components of the Dijkgraaf-Witten map (1.17). This also matches with Convention 1.10.

[14] Here B_{2k} are bernoulli numbers.

Theorem 3.1 (Restatement of Theorem 3)**.** *Let $G^{\mathrm{N}}_{g,n}$ be the set of stable (dual) graphs of n legs and total genus g so that each vertex v is decorated by $F_v \in \{Q, \mathrm{pt}_1, \cdots, \mathrm{pt}_{\mathrm{N}}\}$.*

Suppose $\mathrm{N} \gg 3g-3+n$, then we have the following stable graph localization formula

$$[\tau_1(\psi_1), \cdots, \tau_n(\psi_n)]^{[0,1]}_{g,n} = \sum_{\Gamma \in G^{\mathrm{N}}_{g,n}} \frac{1}{|\operatorname{Aut}\Gamma|} \xi_{\Gamma *} \circ \mathrm{Cont}_\Gamma$$

where $\xi_\Gamma : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n}$ is the tautological map, and the contributions $\mathrm{Cont}_\Gamma \in H^*(\overline{\mathcal{M}}_\Gamma)$ are defined via the following construction:

(1) at each stable vertex v, we place a linear map

$$\bigotimes_{j=1}^{n_v} \gamma_j(z_j) \longmapsto \Big[\prod_{j=1}^{n_v} \gamma_j(\bar\psi_j)\Big]^{F_v, T}_{g_v, n_v} \quad \in H^*(\overline{\mathcal{M}}_{g_v,n_v});$$

(2) at each edge, we place a bi-vector valued series

$$\sum_\alpha \frac{e_\alpha \otimes e^\alpha - R(z)^{-1} e_\alpha \otimes R(w)^{-1} e^\alpha}{z+w} \in (\mathcal{H} \otimes \mathcal{H}) \otimes \mathbb{A}[\![z,w]\!];$$

(3) at each leg, we place a vector valued series

$$R(z)^{-1}[S^M(z^-)\tau_i(z)]_+ \in \mathcal{H} \otimes \mathbb{A}[\![z]\!].$$

Here $f(z^-)$ means that we expand $f(z)$ near $z = \infty$, and $[\ \ ]_+$ means that we take the nonnegative-power part of the series. If $\tau_i(z) = \tau_i \in \mathcal{H}$, the placement becomes $R(z)^{-1}\tau_i$.

We begin with providing a stable graph description of $[0,1]$-class in descendant form.

3.1. **Stable graphs from localization graphs.** Recall that by Definition 0.7, for $\tau_i(z) \in \mathcal{H}[z]$

$$\big[\tau_1(\psi_1), \cdots, \tau_n(\psi_n)\big]^{[0,1]}_{g,n} \tag{3.3}$$

is the sum of contributions of all NMSP localization $[0,1]$-graphs Θ of n markings and genus g

$$\sum_{d\ge 0} \sum_{\Theta \in G^{[0,1]}_{g,n,(d,0)}} (-1)^{d+1-g} q^d \big(\mathrm{pr}^W_{g,n}\big)_* \Big(\prod_{i=1}^n \mathrm{ev}_i^* \tau_i(\psi_i) \cdot \frac{[F_\Theta]^{\mathrm{vir}}}{e(N^{\mathrm{vir}}_\Theta)}\Big). \tag{3.4}$$

Given a connected localization $[0,1]$-graph Θ, we can *stabilize* it (c.f. Appendix B) to get a stable graph $\Gamma = \Theta^{\mathrm{st}}$, together with decoration g_v for $v \in V(\Theta)$, and its level $\mathrm{lev}_v \in \{0,1\}$ and $F_v \in \{Q\} \cup \{\mathrm{pt}_\alpha\}_{\alpha=1}^{\mathrm{N}}$. Here since each $v \in \Theta^{\mathrm{st}}$ corresponds to a unique stable vertex, denoted by the same v, in Θ, the level of v is the same level of v, and $F_v = \{Q\}$ when $\mathrm{lev}_v = 0$, $F_v = \{\mathrm{pt}_\alpha\}$ when $\mathrm{lev}_v = 1$ and $\alpha_v = \alpha$. Then $\Gamma \in G^{\mathrm{N}}_{g,n}$ by the definition of $G^{\mathrm{N}}_{g,n}$ in Theorem 3.1.

We now analysis the contributions from the three types of unstable objects in the localization graph: tail, end and chain. We will see that, after stabilization, the tails contract into the vertices and contribute to the translation actions; while the ends or the chains contract to the legs or edges in the stable graph, and give the leg or edge contributions respectively.

3.2. **Tail contribution via** NMSP ***J*-function.** The tail contributions naturally appear at each stable vertex by the localization formula. For example, among all decorated graphs appearing in the localization formula calculating genus g no marking NMSP correlator

$$F^M(q) := \langle\rangle^M_{g,0}\,,$$

there is a class of graphs, called "leading" graphs, each of which has a single genus g vertex v with "tails" attached to it. We will view such graph as a vertex with legs so that each leg is then replaced by a tail. By the argument in Lemma 1.8, the total contribution of all possible tails attached to a leg l is (ψ_l is the psi classes of the marking l)

$$L^{\mathrm{loc}}(\psi_l)|_{F_v} = \psi_l + [J^M(-\psi_l)]_+\big|_{F_v} \in \mathcal{H}_{F_v}[\![\psi_l]\!]. \tag{3.5}$$

3.3. **Chain contributions via two-point functions.** We introduce the following two-point function:

$$W^M(z_1, z_2) = \frac{\sum_\alpha e_\alpha \otimes e^\alpha}{-z_1 - z_2} + \sum_{\alpha,\beta} e_\alpha \otimes e_\beta \Big\langle \frac{e^\alpha}{-z_1 - \psi_1}, \frac{e^\beta}{-z_2 - \psi_2} \Big\rangle^M_{0,2}.$$

By Principle (★) it is equal to the two point function of $\mathcal{L}_p^{\otimes 5}$ twisted $\mathbb{P}^{4+\mathrm{N}}$ theory. From the string and WDVV equations, a standard argument shows

Lemma 3.2. *We have*

$$W^M(z_1, z_2) = -\sum_\alpha \frac{S^M(z_1)^{-1} e_\alpha \otimes S^M(z_2)^{-1} e^\alpha}{z_1 + z_2}. \tag{3.6}$$

The two point NMSP correlator can be computed by using NMSP localization as a graph sum, and we can see that each localization graph is a chain that connects two (localization) vertices v_1 and v_2 (which could be the same one).

At each vertex v_i ($i = 1, 2$), there are two types of graph contributions:

(1) when the vertex v_i is unstable, ψ_i is invertible and the graph will contribute to $W^M(z_1, z_2)$ of terms with non-negative power of z_i;
(2) when the vertex v_i is stable, ψ_i is not invertible and the graph will contribute to $W^M(z_1, z_2)$ of terms with negative power of z_i.

These contributions can be computed by expanding $W^M(z_1, z_2)$ as a Laurent series of z_i (at $z_i = 0$), and taking the part of non-negative or positive powers of z_i.

On the other hand, for each z_i we can expand $W^M(z_1, z_2)$ as a power series of z_i^{-1}, namely expanding at $z_i = \infty$. The coefficient of $e_\alpha z_i^{-k-1}$ corresponds to the correlator with insertion $e^\alpha \psi_i^k$ at v_i. [15]

The following two situations will be used consistently.

(a) *An edge*: Let e be an edge in the graph after stabilization, with vertices v and v' incident to it. We consider the two side truncation
$$[W^M(\psi_{(e,v)}, \psi_{(e,v')})]_{+,+} \in \mathcal{H}_v \otimes \mathcal{H}_{v'}[\![\psi_{(e,v)}, \psi_{(e,v')}]\!], \tag{3.7}$$
as a bi-vector insertion at the edge e. It gives the total contribution from all possible chains that contract to e.

(b) *A leg* : Let l be a leg incident to a vertex v in the graph after stabilization. We consider the class with one side truncation, as an insertion at the vertex v: [15]
$$\mathrm{Res}_{z=0}\Big([W^M(\psi_l, -z^-)]_+, \tau(z) \Big)^M \in \mathcal{H}_v[\![\psi_l]\!]. \tag{3.8}$$
It gives the total contributions from all possible "ends" that contract to l. Note that an end is a chain between an leg l' with insertion $\tau(\psi_{l'})$ and its closest stable-graph-vertex $v = cl(l')$ in the localization graph (c.f. Appendix B).

By using (3.6), the contribution (3.8) can be computed to be

$$\begin{aligned}&\mathrm{Res}_{z=0}\Big(-\sum_\alpha \frac{S^M(\psi_v)^{-1} e_\alpha \otimes S^M(-z^-)^{-1} e^\alpha}{\psi_v - z}\Big|_+, \tau(z) \Big)^M \\ &= [S^M(\psi_v)^{-1}[(S^M(-\psi_v^-)^{-1})^* \tau(\psi_v)]_+]_+ = [S^M(\psi_v)^{-1}[S^M(\psi_v^-)\tau(\psi_v)]_+]_+.\end{aligned} \tag{3.9}$$

Especially, when the insertion τ does not contain ψ_v-class, it is

$$[S^M(\psi_v)^{-1}\tau]_+.$$

[15] Recall that the notation $f(z^-)$ means we expand $f(z)$ near $z = \infty$.

3.4. **Stable graphs contribution in descendants.** By Section 3.2 and 3.3, the contribution from a stable $[0,1]$-graphs Γ to (3.3), which we denote by Cont_Γ, is given by the following construction:

(1) at each vertex v of Γ, we place a linear map
$$\bigotimes_{j=1}^{n}\tau_j(z_j)\longmapsto\sum_{l\geq 0}\frac{1}{l!}(\mathrm{pr}_l)_*\Big[\prod_{j=1}^{n}\tau_j(z_j),\prod_{i=1}^{l}L^{\mathrm{loc}}(\psi_{n_v+i})\Big]^{F_v,\mathrm{tw}}_{g_v,n_v+l},$$
where $L^{\mathrm{loc}}(z):=z(1-S^M(z)^{-1})\mathbf{1}|_+\in\mathcal{H}[\![z]\!]$; [16]

(2) at each edge of Γ, we place a bi-vector valued series (viewing $z=\psi_{(e,v)}, w=\psi_{(e,v')}$)
$$\sum_\alpha\frac{S^{\mathrm{loc}}(z)^{-1}e_\alpha\otimes S^{\mathrm{loc}}(w)^{-1}e^\alpha}{z+w}-\frac{S^M(z)^{-1}e_\alpha\otimes S^M(w)^{-1}e^\alpha}{z+w}\Big|_{+,+};$$
this is the contribution of chains in the localization graph that stabilize to e: the second term comes from (3.6) and (3.7), and the first term comes from the case that $v=v'$ is the chain itself, which is not allowed;

(3) at each leg l incident to v of Γ, we place a vector valued series [15]
$$[S^M(\psi_v)^{-1}[S^M(\psi_v^-)\tau_i(\psi_v)]_+]_+\in\mathcal{H}\otimes\mathbb{A}[\![\psi_v]\!];$$
this is from (3.9), the total contribution of chains connecting v and the leg l, with insertions τ_i at l in the localization graph. (Note that unlike (2), the "length" of this chain can be 0.)

This way the $\frac{1}{|\mathrm{Aut}_\Gamma|}(\xi_\Gamma)_*\,\mathrm{Cont}_\Gamma$ is the sum of the contribution in the RHS of (3.4) over all Θ with $\Theta^{\mathrm{st}}=\Gamma$. We conclude that
$$\sum_{d\geq 0}(-1)^{d+1-g}q^d(\mathrm{pr}^W_{g,n})_*\Big(\prod_{i=1}^{n}\mathrm{ev}_i^*\tau_i(\psi_i)\cap[\mathcal{W}_{g,n,(d,0)}]^{[0,1]}\Big)=\sum_\Gamma\frac{1}{|\mathrm{Aut}_\Gamma|}(\xi_\Gamma)_*\mathrm{Cont}_\Gamma.$$
Note here that the automorphisms of the tails are parts of the definition of Cont_Θ, hence only $|\mathrm{Aut}\,\Gamma|$ is in the identity.

Remark 3.3. *The argument here is essentially the idea of Givental* [Gi01b]*, and is close to the treatment in* [CGT15].

3.5. **Stable graph contribution in ancestors.** To prove Theorem 3.1, we need to convert descendant classes to an ancestors classes $\bar\psi_i\in H^*(\overline{\mathcal{M}}_{g_v,n_v})$, at stable-graph-vertex v.

Let $\{\tau_i(\psi)\}_{i=1}^{n_v}$ be the insertions to v from the two types of chains (3.7) and (3.9). Here n_v is the valence of v in the graph after stabilization. Let $u=\tau^{\mathrm{loc}}(0,q)$ (c.f. Definition 1.9), by applying Kontsevich-Manin's descendant-ancestor formula [KM94],
$$\begin{aligned}&\sum_{s\geq 0}\frac{1}{s!}(\mathrm{pr}_s)_*\Big[\bigotimes_{i=1}^{n_v}\tau_i(\psi_i),L^{\mathrm{loc}}(\psi)^s\Big]^{F_v,\mathrm{tw}}_{g_v,n_v+s}\\&=\sum_{k,l\geq 0}\frac{1}{k!\,l!}(\mathrm{pr}_{k+l})_*\Big[\bigotimes_{i=1}^{n_v}\tau_i(\psi_i),(L^{\mathrm{loc}}(\psi)-u)^k,u^l\Big]^{F_v,\mathrm{tw}}_{g_v,n_v+k+l},\\&=\sum_{k,l\geq 0}\frac{1}{k!\,l!}(\mathrm{pr}_{k+l})_*\Big[\bigotimes_{i=1}^{n_v}S^{\mathrm{loc}}(\bar\psi_i)\tau_i(\bar\psi_i)_+,T^{\mathrm{loc}}(\bar\psi)^k,u^l\Big]^{F_v,\mathrm{tw}}_{g_v,n_v+k+l},\end{aligned}\tag{3.10}$$
where ψ_i's are the psi-class of $\overline{\mathcal{M}}_{g_v,n_v+k+l}(F_v,d)$; $\bar\psi_i,\bar\psi$ are the ancestor classes pullback from $\overline{\mathcal{M}}_{g_v,n_v+k}$. Here $T^{\mathrm{loc}}(z)$ is defined by
$$\begin{aligned}T^{\mathrm{loc}}(z):=&S^{\mathrm{loc}}(z)(L^{\mathrm{loc}}(z)-u)_+\\=&S^{\mathrm{loc}}(z)\Big(z(1-S^M(z)^{-1})\mathbf{1}\big|_+-u\Big)_+=z(1-R^{\mathrm{loc}}(z)^{-1})\mathbf{1},\end{aligned}\tag{3.11}$$

[16] The infinite summation is convergent in the q-adic topology. See remark 3.4.

and by Convention 1.10 we omit the subscript in S^{loc},R^{loc} and T^{loc} with this choice of u.

Since $T^{\mathrm{loc}}(z)$ has no $z^{n\le 0}$ terms, namely $T^{\mathrm{loc}}(z)=O(z)$, (3.10) still holds if we replace the $\{\bar\psi_i\}_{i=1}^{n_v}$ by the ancestor classes $\{\bar{\bar\psi}_i\}_{i=1}^{n_v}$ from $\overline{\mathcal{M}}_{g_v,n_v}$. In this case, (3.10) is equal to

$$\sum_{k,l\ge 0}\frac{1}{k!\,l!}(\mathrm{pr}_{k+l})_*\Big[\bigotimes_{i=1}^{n_v} S^{\mathrm{loc}}(\bar{\bar\psi}_i)\tau_i(\bar{\bar\psi}_i)_+, T^{\mathrm{loc}}(\bar\psi)^k, u^l\Big]^{F_v,\mathrm{tw}}_{g_v,n_v+k+l}.$$

Remark 3.4. *Note that by the localization formula, each term in L^{loc} has positive q-degree. Since $u=\tau^{\mathrm{loc}}(0,q)$ also has this property (c.f. Definition 1.9), T^{loc} only has positive q-degree. Hence for fixed d, there are only finitely many terms that contribute to the coefficient of q^d in the infinite sum. This implies that the infinite sum is well-defined.*

By using the Birkhoff factorization (cf. (1.15))

$$S^M(z)=R^{\mathrm{loc}}(z)S^{\mathrm{loc}}(z),$$

we see the contribution of a stable graph Γ is given by the following construction which can be realized as an R-matrix action:

(1) at each vertex v, we place a linear map

$$\otimes_{j=1}^{n_v}\tau_j(z_j)\longmapsto \sum_{k,l\ge 0}\frac{1}{k!l!}(\mathrm{pr}_{k+l})_*\Big[\bigotimes_{j=1}^{n_v}\tau_j(\bar{\bar\psi}_i), T^{\mathrm{loc}}(\bar\psi)^k, u^l\Big]^{F_v,\mathrm{tw}}_{g_v,n_v+k+l} \tag{3.12}$$

(2) at each edge e with $E_v=\{v_1,v_2\}$, we place a bi-vector valued series

$$V^{\mathrm{loc}}(z,w):=\sum_\alpha \frac{e_\alpha\otimes e^\alpha - R^{\mathrm{loc}}(z)^{-1}e_\alpha\otimes R^{\mathrm{loc}}(w)^{-1}e^\alpha}{z+w}\in(\mathcal{H}\otimes\mathcal{H})\otimes\mathbb{A}[\![z,w]\!]$$

(3) at each leg l incident to v, we place a vector valued series

$$R^{\mathrm{loc}}(z_v)^{-1}\big[S^M(z_v^-)\tau_i(z_v)\big]_+\in\mathcal{H}\otimes\mathbb{A}[\![z_v]\!].$$

Recall that Grothendieck-Riemann-Roch theorem is used by Mumford to express the twisted class $[-]^{\mathrm{pt}_\alpha,\mathrm{tw}}$ in terms of its degree 0 component $[-]^{\mathrm{pt}_\alpha,\mathrm{top}}$ (c.f. (1.6)). Further, the Mumford-Faber-Pandharipande's formula (c.f. [Gi01b, Sect. 2.2]) can be applied to conclude that the above graph sum remains the same if we do the following:
(i) at edge and leg, replace all R^{loc} by [17]

$$R(z)=\ R^{\mathrm{loc}}(z)\cdot\Big(\mathrm{I}_{\mathcal{H}_Q}\oplus\mathrm{diag}\{\Delta^{\mathrm{pt}_\alpha}(z)\}_{\alpha=1}^{\mathrm{N}}\Big); \tag{3.13}$$

(ii) at each vertex replace $[-]^{F_v,\mathrm{tw}}$ by $[-]^{F_v,\mathrm{top}}$, (cf. (3.12))

- if $F_v=Q$, replace $[-]^{Q,\mathrm{tw}}$ by $[-]^{Q,\mathrm{top}}$, notice $[-]^{Q,\mathrm{tw}}=[-]^{Q,\mathrm{top}}$;
- if $F_v=\mathrm{pt}_\alpha$, replacing $[-]^{\mathrm{pt}_\alpha,\mathrm{tw}}$ by $[-]^{\mathrm{pt}_\alpha,\mathrm{top}}$,

and replace T^{loc} in (3.12) by

$$T(z):=\ z(1-R(z)^{-1})\mathbf{1}. \tag{3.14}$$

Remark 3.5. *The formula $T(z)$ comes from, for $\Delta(z):=\Big(\mathrm{I}_{\mathcal{H}_Q}\oplus\mathrm{diag}\{\Delta^{\mathrm{pt}_\alpha}(z)\}_{\alpha=1}^{\mathrm{N}}\Big)$, the tail formula of the composed R matrices (c.f.* [PPZ15, NMSP3]*)*

$$\Delta(z)^{-1}T^{\mathrm{loc}}(z)+z[\mathbf{1}-\Delta(z)^{-1}\mathbf{1}]=z\mathbf{1}-\Delta(z)^{-1}R^{\mathrm{loc}}(z)^{-1}\mathbf{1}=z(1-R(z)^{-1})\mathbf{1}. \tag{3.15}$$

---

[17] Here $\Delta^{\mathrm{pt}_\alpha}$ are defined in (3.2); they can be viewed as the R matrices of Grothendieck-Riemann-Roch formula reducing $[-]^{\mathrm{pt}_\alpha,\mathrm{tw}}$ to its topological part $[-]^{\mathrm{pt},\mathrm{top}}$.

The equation (3.13) can also be understood as a composition of R-matrix actions on CohFTs. By a theorem of [PPZ15], the R-matrix action on CohFT is a left group action. Hence by first applying the constant R-matrix action $\Delta^{\mathrm{pt}_\alpha}(z)$ at the fixed loci pt_α, and then applying the R^{loc} which is from the localization, the composition gives us the R-matrix $R(z)$. For more details of this point of view, see Section 2.5.1 and Theorem 2.7 in the sequent paper [NMSP3].

Remark 3.6. *The $R(z)$ defined in* (3.13) *satisfies the (defining) identity* (3.1) *because*

$$\begin{aligned} S^M \begin{pmatrix} \Delta^{\mathrm{Npt}} & \\ & \mathrm{I} \end{pmatrix} &= R^{\mathrm{loc}} \begin{pmatrix} S^{\mathrm{Npt,tw}} & \\ & S^{Q,\mathrm{tw}} \end{pmatrix} \begin{pmatrix} \Delta^{\mathrm{Npt}} & \\ & \mathrm{I} \end{pmatrix} \\ &= R^{\mathrm{loc}} \begin{pmatrix} \Delta^{\mathrm{Npt}} & \\ & \mathrm{I} \end{pmatrix} \begin{pmatrix} S^{\mathrm{Npt,tw}} & \\ & S^{Q,\mathrm{tw}} \end{pmatrix} = R \begin{pmatrix} S^{\mathrm{Npt,tw}} & \\ & S^{Q,\mathrm{tw}} \end{pmatrix}. \end{aligned}$$

Here we have used the abbreviations $S^{\mathrm{Npt,tw}} = \mathrm{diag}\{S^{\mathrm{pt}_\alpha,\mathrm{tw}}\}$, $S^{\mathrm{Npt}} = \mathrm{diag}\{S^{\mathrm{pt}_\alpha}\}$, and $\Delta^{\mathrm{Npt}} = \mathrm{diag}\{\Delta^{\mathrm{pt}_\alpha}\}_{\alpha=1}^{\mathrm{N}}$. In the first and second identity, we have used (1.15), *and that $\Delta^{\mathrm{pt}_\alpha}$ commutes with $S^{\mathrm{pt}_\alpha,\mathrm{tw}} = e^{\tau_\alpha(q')/z}$.*

Complete the proof of Theorem 3.1. We will finish the proof by applying the Divisor and Dilaton equations. Notice that we need to evaluate the vertex contribution

$$\sum_{k,l\geq 0} \frac{1}{k!l!} (\mathrm{pr}_{k+l})_* \big[\bigotimes_{j=1}^{n_v} \tau_j(\bar{\bar{\psi}}_i), T(\bar{\psi})^k, u^l \big]^{F_v,\mathrm{tw}}_{g_v,n_v+k+l}. \tag{3.16}$$

At $F_v = Q$, the Divisor equation implies that for $f(q) \in F[\![q]\!]$, $u = f(q)H$, and $\tau_i \in \mathcal{H}_Q$, we have

$$\sum_{l\geq 0} \frac{1}{l!} (\mathrm{pr}_l)_* \big[\tau_1, \cdots, \tau_{n_v}, u^l\big]^{Q,\mathrm{top}}_{g_v,n_v+l} = \big[\tau_1, \cdots, \tau_{n_v}\big]^{Q,\mathrm{top}}_{g_v,n_v}\Big|_{q\mapsto q\cdot e^{f(q)}}.$$

For our case, this gives the mirror map $q \mapsto q \cdot e^{I_1/I_0}$.

At $F_v = \mathrm{pt}_\alpha$, we have by dimension reason

$$(\mathrm{pr}_l)_* \big[\tau_1, \cdots, \tau_n, u^l\big]^{\mathrm{pt}_\alpha,\mathrm{top}}_{g_v,n_v+l} = 0, \quad \text{when } l > 0.$$

Let $T_v(z) := T(z)|_{F_v}$. We set

$$\tilde{T}_v(z) := z + \delta_{F_v} \cdot (T_v(z) - z 1_{F_v}) \tag{3.17}$$

where $q' := -q/t^{\mathrm{N}}$ and

$$\delta_F := \begin{cases} I_0^{-1}(q') & \text{if } F = Q; \\ L^{\frac{\mathrm{N}+3}{2}}(q') & \text{if } F = \mathrm{pt}_\alpha. \end{cases}$$

Then $\tilde{T}_v(z) = \tilde{T}_\alpha(z)$ when $F_v = \mathrm{pt}_\alpha$ (cf. $\tilde{T}_\alpha(z)$ is defined in Definition 3.1). Further

$$T_v(z) = (1 - \delta_{F_v}^{-1}) z + \delta_{F_v}^{-1} \tilde{T}_v(z).$$

Using the class-version of Dilaton equation,[18]

$$\begin{aligned} T_v \big[\otimes_i^{n_v} \tau_i \big]^{F_v,\mathrm{top}}_{g_v,n_v} &:= \sum_{s\geq 0} \frac{1}{s!} (\mathrm{pr}_s)_* \big[\otimes_i^{n_v} \tau_i, T_v(\bar{\psi})^s \big]^{F_v,\mathrm{top}}_{g_v,n_v+s} \\ &= \sum_{\ell,m\geq 0} \frac{1}{\ell! m!} (\mathrm{pr}_{\ell+m})_* \big[\otimes_i^{n_v} \tau_i, [(1-\delta_{F_v}^{-1})\bar{\psi} 1]^\ell, [\delta_{F_v}^{-1} \tilde{T}_v(\bar{\psi})]^m \big]^{F_v,\mathrm{top}}_{g_v,n_v+\ell+m} \\ &= \sum_{\ell,m\geq 0} \frac{(1-\delta_{F_v}^{-1})^\ell}{m!} \binom{2g_v - 2 + n_v + m + \ell - 1}{\ell} (\mathrm{pr}_m)_* \big[\otimes_i^{n_v} \tau_i, [\delta_{F_v}^{-1} \tilde{T}_v(\bar{\psi})]^m \big]^{F_v,\mathrm{top}}_{g_v,n_v+m} \\ &= \sum_{m\geq 0} \frac{\delta_{F_v}^{2g-2+n_v}}{m!} (\mathrm{pr}_m)_* \big[\otimes_i^{n_v} \tau_i, [\tilde{T}_v(\bar{\psi})]^m \big]^{F_v,\mathrm{top}}_{g_v,n_v+m}. \end{aligned} \tag{3.18}$$

Together with the explicit formula [19]

$$\begin{aligned} T(z)|_Q = T^{\mathrm{loc}}(z)|_Q &= \big(1 - I_0(q')\big) \cdot \mathbf{1}_Q z + O(z^{\mathrm{N}-2}), \\ T(z)|_{\mathrm{pt}_\alpha} &= \big(1 - L(q')^{\frac{-\mathrm{N}-3}{2}} + O(z)\big) \cdot \mathbf{1}_\alpha z, \end{aligned} \tag{3.19}$$

[18] Let $\pi_1 : \overline{\mathcal{M}}_{g,n+1}(X,d) \to \overline{\mathcal{M}}_{g,n}(X,d)$ be the forgetful map, let η be a homology class of $\overline{\mathcal{M}}_{g,n}(X,d)$. Then $\pi_{1*}(\pi_1^*\eta \cap \psi_{n+1}) = (2g-2+n)\eta$.

[19] $T(z)|_Q = T^{\mathrm{loc}}(z)$ because $\Delta^*|_Q$ is identity and Δ^* preserves $H^*(\mathrm{Npt})$ in (3.15).

(c.f. Lemma 5.1 and Lemma 6.1) we obtain (using (3.17) for definition of $\tilde{T}_v$)

$$\tilde{T}_v(z) = \begin{cases} O(z^{\mathrm{N}-2}) & \text{if } F_v = Q, \\ O(z^2) & \text{if } F_v = \mathrm{pt}_\alpha. \end{cases}$$

Applying the discussions to (3.16), we see that when

$$\mathrm{N} \gg 3g - 3 + n \geq 3g_v - 3 + n_v,$$

(3.16) matches the vertex contribution in Thm. 3.1. This completes the proof of Thm. 3.1. □

Note that in the above proof, the condition $\mathrm{N} \gg 3g-3+n$ is only used for (3.19). This allows us to define the translated classes at Q as in Definition 3.1, which has no tails contributions. For general N the proof still works, with the translated classes defined by the translation action on the local CohFT (see [PPZ15, NMSP3] for the translation action). Indeed, similar R-matrix formula as in Theorem 3 should hold for general localization formula.

4. The polynomiality of $[0,1]$-theory

In this section, we will prove Theorem 4. The idea is to use Theorem 2.1 and the polynomiality of whole NMSP-theory. First, we convert the descendants correlators

$$\Big[\bigotimes_{i\in L_v} \tau_i, \bigotimes_{e\in E_v} \frac{\mathbf{1}^{\alpha_e}}{\frac{5t}{a_e} - \psi_{(e,f)}}\Big]^{[0,1]}_{g_v,n_v}$$

which is used in Theorem 2.1 into the ancestors. There are two cases to consider: the unstable ones and the stable ones.

4.1. Unstable cases.

Let $v \in V_w(\Lambda)$ be unstable, which is when $(g_v, n_v) = (0,2)$.

Lemma 4.1. *For $f(z) \in \mathcal{H}[\![z]\!]$, we introduce $\mathcal{S}f(z) := S^M(z^-)f(z) \in \mathcal{H}[\![z, z^{-1}]\!]$* [15]*. Then*

$$\big\langle \frac{\tau_1}{z-\psi_1}, \tau_2(\psi_2)\big\rangle^{[0,1]}_{0,2} = \big(S^M(z)\tau_1, \mathcal{S}\tau_2(-z)_+\big)^M.$$

Proof. Using the definition of W and (3.6), we have

$$\begin{aligned}\big\langle \frac{\tau_1}{z-\psi_1}, \tau_2(\psi_2)\big\rangle^{[0,1]}_{0,2} &= \mathrm{Res}_{w=0}\Big(\tau_1 \otimes \tau_2(w), \frac{S^M(z)^* e_\alpha \otimes S^M(w)^* e^\alpha}{w+z}\Big)^M \\ &= \mathrm{Res}_{w=0}\Big(S^M(z)\tau_1 \otimes \mathcal{S}\tau_2(w), \frac{e_\alpha \otimes e^\alpha}{w+z}\Big)^M = (S^M(z)\tau_1, \mathcal{S}\tau_2(-z)_+)^M,\end{aligned}$$

where we always expand $\frac{1}{w+z}$ as $\sum_{k\geq 0} w^{-k-1}z^k$. And in the last step, we have used that for any formal series $f(z)$ one has $\mathrm{Res}_{w=0}\frac{f(w)}{w-z} = f(z)_+$. □

Example 4.2. *For every genus zero white vertex with two edges e_1 and e_2, of hours α_1 and α_2, respectively, the unstable contribution is (by using* (3.6)*)*

$$\Big\langle \frac{1^{\alpha_1}}{\frac{5t_{\alpha_1}}{a_{e_1}} - \psi_1}, \frac{1^{\alpha_2}}{\frac{5t_{\alpha_2}}{a_{e_2}} - \psi_2}\Big\rangle^{[0,1]}_{0,2} = \frac{(\mathcal{S}^{\alpha_1}_{a_{e_1}}, \mathcal{S}^{\alpha_2}_{a_{e_2}})^M}{\frac{5t_{\alpha_1}}{a_{e_1}} + \frac{5t_{\alpha_2}}{a_{e_2}}} =: \Big\langle \frac{\mathcal{S}^{\alpha_1}}{\frac{5t_{\alpha_1}}{a_{e_1}} - \psi_1}, \frac{\mathcal{S}^{\alpha_2}}{\frac{5t_{\alpha_2}}{a_{e_2}} - \psi_2}\Big\rangle^{[0,1]}_{0,2}. \tag{4.1}$$

For genus zero white vertex with one edge of hour α and one insertion τ, we have

$$\Big\langle \tau, \frac{1^\alpha}{\frac{5t_\alpha}{a_e} - \psi_2}\Big\rangle^{[0,1]}_{0,2} = (\tau, \mathcal{S}^\alpha_{a_e})^M =: \Big\langle \tau, \frac{\mathcal{S}^\alpha_{a_e}}{\frac{5t_\alpha}{a_e} - \bar{\psi}_2}\Big\rangle^{[0,1]}_{0,2}. \tag{4.2}$$

In both examples, the degrees are bounded via (1.22)*.*

4.2. **Stable cases.** This can be done by the Kontsevich-Manin (KM) type formula

Lemma 4.3 (Cycle version of KM formula for the $[0,1]$ classes)**.** *For $2g-2+n>0$,*

$$\big[\tau_1(\psi_1),\cdots,\tau_n(\psi_n)\big]^{[0,1]}_{g,n}=\big[\mathcal{S}\tau_1(\bar\psi_1)_+,\cdots,\mathcal{S}\tau_n(\bar\psi_n)_+\big]^{[0,1]}_{g,n}.$$

Proof. Notice that by equation (3.9), for an insertion $\tau_j(\psi_j)\in\mathcal{H}[\psi_j]$, the corresponding contribution in the stable graph localization formula at the fixed locus is given by

$$[S^M(\psi)^{-1}[\mathcal{S}\tau_j(\psi)]_+]_+.$$

Via the KM descendant-ancestor relation for the fixed locus, as an ancestor insertion it becomes

$$R^{\mathrm{loc}}(\bar\psi)^{-1}[\mathcal{S}\tau_j(\bar\psi)]_+=[S^{\mathrm{loc}}(\bar\psi)S^M(\bar\psi)^{-1}[\mathcal{S}\tau_j(\bar\psi)]_+]_+.$$

On the other hand, if we consider an ancestor insertion $\tau_j'(\bar\psi)$ in the master space, the corresponding contribution at the fixed locus is

$$R^{\mathrm{loc}}(\bar\psi)^{-1}\tau_j'(\bar\psi).$$

By setting $\tau_j'(\bar\psi)=\mathcal{S}\tau_j(\bar\psi_j)_+$, we finish the proof. □

By this lemma and the definition of $\mathcal{S}^\alpha_a$ (cf. (1.21)), we see that the ancestor version of the white vertex contribution in Theorem 2.1, becomes

$$\Big\langle\bigotimes_{i\in L_v}\tau_i\bigotimes_{i\in L_v^\circ}\bar\psi_i^{k_i}\bigotimes_{e\in E_v,f=(e,v)}\frac{\mathcal{S}^{\alpha_e}_{a_e}}{\frac{5t_{\alpha_e}}{a_e}-\bar\psi_f}\Big\rangle^{[0,1]}_{g_v,n_v}. \tag{4.3}$$

Here we have used that, for $\frac{x}{u-z}=\frac{x}{u}(1+\frac{z}{u}+\cdots)$, one has $\big[S^M(z^-)\frac{x}{u-z}\big]_+=\Big(\frac{S^M(w)x}{u-z}\Big)|_{w=u}$.

4.3. **The proof of Theorem 4.** By dimension reasons, the $[0,1]$-correlator in Theorem 4 lies in $\mathbb{Q}(\zeta_{\mathrm{N}})[\![q']\!]$. Observe that the localization (0.13) used to define the correlator is symmetric under the permutations group S_{N}, which acts on the last N coordinate of $\mathbb{P}^{4+\mathrm{N}}$. This implies that the $[0,1]$-correlator lies in $\mathbb{Q}[\![q']\!]$. Therefore, it is sufficient to show that $[0,1]$-correlator lies in $\mathbb{F}[q]$ with the desired q-degree bound.

In this proof, we define the degree of $\sum_{i=0}^{\mathrm{N}+3}c_ip^i$, $c_i\in\mathbb{F}$, to be $\max\{i\mid c_i\neq0\}$. Then, $\forall\ i$,

$$\deg\phi^i\le\mathrm{N}+3-i. \tag{4.4}$$

Recall $\deg[\mathcal{W}_{g,n,(d,0)}]^{\mathrm{vir}}=\mathrm{N}(d+1-g)+n$ ([NMSP1]). Thus for pure degree insertions $\{\tau_i\}$,

$$\big\langle\tau_1\bar\psi^{k_1},\cdots,\tau_n\bar\psi^{k_n}\big\rangle^M_{g,n} \tag{4.5}$$

vanishes when $\sum_i\deg\tau_i+k_i<\mathrm{N}(d+1-g)+n$. Because $\bar\psi$ are ancestor classes, (4.5) also vanishes when $3g-3+n<\sum_ik_i$. Adding the two inequalities, the vanishing holds when

$$\textstyle\sum_i\deg\tau_i<\mathrm{N}(d+1-g)-(3g-3).$$

This proves that (4.5) is a polynomial in $\mathbb{F}[q]$ of degree bounded by

$$g-1+\frac{3g-3+\sum_i\deg\tau_i}{\mathrm{N}}. \tag{4.6}$$

We now apply the bipartite graph sum formula (Theorem 2.1) to the NMSP correlator (4.5), and prove the desired q-degree bound for the $[0,1]$-correlator by induction on g. First, since

$$\big\langle\tau_1\bar\psi^{k_1},\cdots,\tau_n\bar\psi^{k_n}\big\rangle^{[0,1]}_{0,n}=\big\langle\tau_1\bar\psi^{k_1},\cdots,\tau_n\bar\psi^{k_n}\big\rangle^M_{0,n},$$

when $g=0$ the stated bound holds.

We now suppose that the stated bound holds for all genus $h<g$ with arbitrary many insertions, then we apply Theorem 2.1 to the genus g, n insertions case. Note that the set of bipartite graphs $\Xi_{g,n}$ contains a "leading" one, the graph Λ_g with a single genus g white vertex and n markings. We let $(\Xi_{g,n})^\circ=\Xi_{g,n}-\{\Lambda_g\}$. Theorem 2.1 gives us

$$\big\langle\tau_1\bar\psi^{k_1},\cdots,\tau_n\bar\psi^{k_n}\big\rangle^M_{g,n}=\big\langle\tau_1\bar\psi^{k_1},\cdots,\tau_n\bar\psi^{k_n}\big\rangle^{[0,1]}_{g,n}+\sum_{\Lambda\in(\Xi_{g,n})^\circ}(*).$$

Here the first term on the R.H.S. of the identity is the contribution from Λ_g. As was argued at the beginning, the L.H.S. of the identity is a q-polynomial of degree bound by (4.6), thus to prove the theorem, we only need to show that each term in the summation is a q-polynomial of degree bounded from above by the same quantity.

Let $\Lambda \in (\Xi_{g,n})^\circ$. By equation 4.1, (4.2) and (4.3), the contribution from Λ is,

$$\prod_{v\in V_b(\Lambda)} \mathrm{Cont}^\infty_{[v]}\Big(\bigotimes_{i\in L_v^\circ}\bar\psi_i^{k_i}\Big)\prod_{v\in V_w(\Lambda)}\Big\langle \bigotimes_{i\in L_v}\tau_i\bigotimes_{i\in L_v^\circ}\bar\psi_i^{k_i}\bigotimes_{e\in E_v}\frac{\mathcal{S}^{\alpha_e}_{a_e}}{\frac{5t_\alpha}{a_e}-\bar\psi_{(e,v)}}\Big\rangle^{[0,1]}_{g_v,n_v}. \tag{4.7}$$

Assume (4.7) is not zero, we substitute $\mathcal{S}^{\alpha_e}_{a_e}=\sum_{i=0}^{\mathrm{N}+3}\mathcal{S}^{\alpha_e}_{a;i}\phi^i$ and apply inductions.

- At each stable vertex $v\in V_w(\Lambda)$, an easy argument shows that the total genus of all black vertices of Λ is at least one, thus we have $g_v<g$. Consequently, by induction hypothesis the term $\langle - \rangle^{[0,1]}_{g_v,n_v}$ is a polynomial of degree no more than
$$g_v-1+|E_v|+\sum_{e\in E_v}\frac{a_e-1}{5}+\frac{3g_v-3+3|E_v|+\sum_i\deg\tau_i}{\mathrm{N}}$$
where we have used that (1.22) and (4.4) imply that each insertion from edge contributes degree
$$\le \deg_q \mathcal{S}^{\alpha_e}_{a_e;i}+\frac{\mathrm{N}+3-i}{\mathrm{N}}\le\frac{a_e-1}{5}+1+\frac{3}{\mathrm{N}}.$$
- At each unstable white vertex $v\in V_w(\Lambda)$, by Example 4.2:
(1) if there are one edge e and one insertion $\tau(\bar\psi)$, the $[0,1]$-correlator is a polynomial of degree
$$\lceil\frac{a_e}{5}\rceil-1+\frac{\deg\tau}{\mathrm{N}}\le\frac{a_e-1}{5}+\frac{\deg\tau}{\mathrm{N}};$$
(2) if there are two edges e_1,e_2, the $[0,1]$-correlator is a polynomial of degree
$$\lceil\frac{a_{e_1}}{5}\rceil+\lceil\frac{a_{e_2}}{5}\rceil-1\le\frac{a_{e_1}-1}{5}+\frac{a_{e_2}-1}{5}+1+\frac{3}{\mathrm{N}}$$
where we have used (1.22) and that $(\phi^i,\phi^j)^M=0$ whenever $i,j\ge\mathrm{N}$.
- At each black vertex $v\in V_b(\Lambda)$
$$\deg\mathrm{Cont}^\infty_{[v]}\le d_{\infty[v]}+\frac{2}{5}(g_v-1)-\sum_{e\in E_v}\frac15(a_e-1).$$

Using $\sum_{v\in V_b(\Lambda)}d_{\infty[v]}=0$ and $G_{[v]}\ne\emptyset$, we have $\sum_{v\in V_b(\Lambda)}(g_v-1)\ge0$, after applying $d_{0\Theta_\infty}\ge0$ in (2.5), and summing over v. Hence
$$\sum_{v\in V_b(\Lambda)}\frac25(g_v-1)\le\sum_{v\in V_b(\Lambda)}(g_v-1).$$
This proves that the total q-degree is no more than
$$\begin{aligned}
&\sum_{v\in V_w(\Lambda)}(g_v-1+|E_v|)+\sum_{v\in V_b(\Lambda)}\frac25(g_v-1)+\frac1{\mathrm{N}}\Big(\sum_{v\in V_w(\Lambda)}3(g_v-1+|E_v|)+\sum_i\deg\tau_i\Big)\\
\le&\sum_{v\in V_w(\Lambda)}(g_v-1+|E_v|)+\sum_{v\in V_b(\Lambda)}(g_v-1)+\frac1{\mathrm{N}}\Big(\sum_v 3(g_v-1)+3|E(\Lambda)|+\sum_i\deg\tau_i\Big)\\
=&\quad g-1+\frac{3(g-1)+\sum_i\deg\tau_i}{\mathrm{N}},
\end{aligned}$$
where in the last step we have used
$$|E(\Lambda)|+\sum_v(g_v-1)=\sum_{v\in V(\Lambda)}g_v+|E(\Lambda)|-|V(\Lambda)|=\sum_v g_v+g(\Lambda)-1=g-1.$$

This proves the theorem.

5. Proof of Theorem 1 and 2

5.1. Computing tail contributions, DW map and R-matrix at $F_v = Q$ via large N. By using (3.5) in Sect. 3.2, the tail contribution at the fixed locus Q is given by

$$\begin{aligned} L^{\mathrm{loc}}(z)|_Q &= z\mathbf{1} + J^M(0,-z)|_{Q,+} \qquad (5.1)\\ &= -\sum_{d=1}^{\infty} z\, q^d \frac{\prod_{m=1}^{5d}(5H-mz)}{\prod_{m=1}^{d}(H-mz)^5 \prod_{m=1}^{d}\big((H-mz)^{\mathrm{N}}+t^{\mathrm{N}}\big)}\Big|_+ . \end{aligned}$$

By $H^4=0$ and the explicit formula of the quintic I-functions (0.1), we have

$$\begin{aligned} L^{\mathrm{loc}}(z)|_Q =& -\sum_{d=1}^{\infty} z\,(q')^d \frac{\prod_{m=1}^{5d}(5H-mz)}{\prod_{m=1}^{d}(H-mz)^5}\cdot\Big(1+O(z^{\mathrm{N}}) \\ &\qquad + H\cdot O(z^{\mathrm{N}-1}) + H^2\cdot O(z^{\mathrm{N}-2}) + H^3\cdot O(z^{\mathrm{N}-3})\Big)\Big|_+ \\ =&\ \ z + I^Q(q',-z)\big|_+ + O(z^{\mathrm{N}-2}) \\ =&\ \ z\cdot\big(1-I_0(q')\big) + I_1(q')H + O(z^{\mathrm{N}-2}). \qquad (5.2) \end{aligned}$$

Lemma 5.1. *Suppose* $\mathrm{N}>5$.[20] *We have the following formula for* $\tau^{\mathrm{loc}}(0,q)|_Q$ *and* $T^{\mathrm{loc}}(z)|_Q$ *(see* (1.16) *and* (3.11) *for their definitions)*

$$\tau^{\mathrm{loc}}(0,q)|_Q = \tau_Q(q') := \frac{I_1(q')}{I_0(q')}H, \quad \textit{and} \quad T^{\mathrm{loc}}(z)|_Q = \big(1-I_0(q')\big)\cdot \mathbf{1}_Q z + O(z^{\mathrm{N}-2}).$$

Proof. By applying (5.2) to the definition (1.16) and the method deriving (3.18), we have

$$\begin{aligned} \tau^{\mathrm{loc}}(0,q)|_Q =& \sum_{\alpha,n}\frac{1}{n!}e^\alpha \Big\langle e_\alpha, \mathbf{1}, L^{\mathrm{loc}}(-\psi)|_Q^n \Big\rangle_{0,n+2}^{Q,\mathrm{tw}} \\ =& \sum_{\alpha,n}\frac{1}{n!}e^\alpha \big\langle e_\alpha, \mathbf{1}, \big((1-I_0(q'))\psi + I_1(q')H\big)^n\big\rangle_{0,n+2}^{Q,\mathrm{tw}} \\ =& \sum_{\alpha,n}\frac{1}{n!}e^\alpha \big\langle e_\alpha, \mathbf{1}, \big(\tau_Q(q')\big)^n\big\rangle_{0,n+2}^{Q,\mathrm{tw}} = \tau_Q(q'). \end{aligned}$$

For the second equality, we use (5.2), (A.4), and (1.13) to obtain

$$\begin{aligned} T^{\mathrm{loc}}(z)|_Q =& S_{\tau_Q}^{Q,\mathrm{tw}}(z)\big(L^{\mathrm{loc}}(z)-\tau_Q\big)|_{Q,+} + O(z^{\mathrm{N}-2}) \\ =& \big(1-I_0(q')\big)\cdot\mathbf{1}_Q z + O(z^{\mathrm{N}-2}). \end{aligned}$$

This proves the lemma. □

The above results, together with the QDE (1.19) gives us an effective way to compute the S^M- and R-matrix when restricting to the fixed loci Q. We illustrate the algorithm as follows:

Example 5.2. *By using equation* (5.1) *and* (5.2)*, we have*

$$S^M(z)^*\mathbf{1}|_{Q,+} = z^{-1}J^M(z)|_{Q,+} = I_0(q') + O(z^{\mathrm{N}-3}).$$

Applying (1.19) *to* $\phi_0 = p^0 = \mathbf{1}$*, we get*

$$S^M(z)^* p|_{Q,+} = \big(D_p S^M(z)^*\mathbf{1}\big)|_{Q,+} = HI_0(q') + zD(I_0(q')) + D(I_1(q'))H + O(z^{\mathrm{N}-2}).$$

Recall (3.1) *and* (1.13) *give formally*

$$\big(R(z)^{-1}x\big)|_Q = S^Q(q',z)\big(S^M(z)^{-1}x\big)|_Q, \qquad \forall x\in\mathcal{H}. \qquad (5.3)$$

---

[20] Indeed, a more careful analysis shows $\mathrm{N}>2$ in enough. Further, for the cases $\mathrm{N}=1,2$ one can still compute the Dijkgraaf-Witten map and prove certain polynomiality of entries of the R-matrix, however the computations will be much more involved. For example, for the original MSP (i.e. $\mathrm{N}=1$) we have

$$\tau^{\mathrm{loc}}(0,q)|_Q = 5!q + \tau_Q + \tau_2 H^2 + \tau_3 H^3$$

with $\tau_2 = -\frac{1}{2}I_{11}I_{22}$ and $\tau_3 = I_{11}^2 I_{22}\big(-\frac{5}{12}+\frac{Y}{4}-\frac{B}{2}-\frac{A}{4}\big)$ (here I_{22} is defined in (A.3)).

Together with the explicit formula for S^Q (A.4), they lead to the first two R-matrix entries:

$$R(z)^*\mathbf{1}|_Q = I_0(q') + O(z^{\mathrm{N}-3}),$$
$$R(z)^*p|_Q = zD(I_0(q')) + H\, I_0(q')\, I_{11}(q') + O(z^{\mathrm{N}-2}),$$

where in the last equality we have used $I_{11} = 1 + D(I_1/I_0)$.

5.2. **Polynomiality of the entries of the R-matrix.** We define the normalized basis for $\mathcal{H}_Q$ and its dual by [21]

$$\varphi_b := I_0(q')I_{11}(q')\cdots I_{bb}(q')H^b\,, \qquad \varphi^b := \frac{-t^{\mathrm{N}}}{5\, I_0(q')I_{11}(q')\cdots I_{bb}(q')}H^{3-b};$$

for $b = 0,1,2,3$. Here I_{22} is defined in (A.3) and we let $I_{33} := I_{11}$.

Recall $L(q) := (1-5^5q)^{1/\mathrm{N}}$. We define the normalized basis for $H^*(\mathrm{pt}_\alpha)$ and its dual by

$$\bar{\mathbf{1}}_\alpha := L(q')^{-\frac{\mathrm{N}+3}{2}}\mathbf{1}_\alpha, \qquad \bar{\mathbf{1}}^\alpha := L(q')^{\frac{\mathrm{N}+3}{2}}\mathbf{1}^\alpha.$$

Convention 5.3. *In this and the next section, we will regard the symbols*

$$I_0,\ I_1,\ I_2,\ I_3,\ A_1,\ B_1,\ B_2,\ B_3,\ Y$$

as their original definition with q substituted by $q' = -q/t^{\mathrm{N}}$. For example,

$$I_1 = I_1(q')\ ,\ Y = Y(q') = (1+5^5q/t^{\mathrm{N}})^{-1}, \quad e.t.c.\ .$$

We set $L_\alpha := \zeta_{\mathrm{N}}^\alpha\, t\cdot L(q') = \zeta_{\mathrm{N}}^\alpha(t^{\mathrm{N}}+5^5q)^{1/\mathrm{N}}$ and introduce (for $j = 0,\cdots,\mathrm{N}+3$)

$$(R_k)_j^b := (R_k\varphi^b, p^j)^M, \qquad (R_k)_j^\alpha := L_\alpha^{-(j-k)}\cdot(R_k\bar{1}^\alpha, p^j)^M.$$

Lemma 5.4. *Suppose $k < \mathrm{N}-3$. At the fixed loci Q, we have that*

$$(R_k)_j^b = 0, \quad \text{if } j \not\equiv b+k \mod \mathrm{N}\,, \tag{5.4}$$

and for $b = 0,1,2,3$

$$(R_k)_{b+k}^b \ \text{ and }\ \tfrac{Y}{t^{\mathrm{N}}}\cdot(R_k)_{b+\mathrm{N}+k}^b \ \in\ \mathbb{Q}[A,B,B_2,B_3,Y]. \tag{5.5}$$

At the fixed loci pt_α, we have that the $(R_k)_j^\alpha$ is independent of α, and

$$(R_k)_j^\alpha \in \mathbb{Q}[Y]_{k+\lfloor\frac{j}{\mathrm{N}}\rfloor}. \tag{5.6}$$

Proof. First we prove the vanishing property (5.4) and polynomiality (5.5) of the R-matrix when restricted to the fixed loci Q. By Example 5.2, we have

$$(R_k)_0^b = \delta_{b,0}\delta_{k,0} \quad \text{for} \quad k < \mathrm{N}-3. \tag{5.7}$$

To compute the other columns, recall that the QDE for S^M and S^Q are (c.f. (1.19), (A.1))[22]

$$\big(p + z\,q\frac{d}{dq}\big)S^M(z)^* = S^M(z)^*\cdot A^M,$$
$$\big(H + z\,q\frac{d}{dq}\big)S^Q(z)^* = S^Q(z)^*\cdot A^Q.$$

Together with the Birkhoff factorization (5.3), we obtain the QDE for $R(z)^*|_Q$:

$$(zD + A^Q)\,(R(z)^*x)\big|_Q = (R(z)^*\cdot A^Mx)\big|_Q, \qquad \forall x\in\mathcal{H}. \tag{5.8}$$

This implies, under the basis $\{\varphi_i\}$, for $j = 1,\cdots,\mathrm{N}+3$

$$(R_k)_j^b = (D+C_b)(R_{k-1})_{j-1}^b + (R_k)_{j-1}^{b-1} - c_jq\,(R_k)_{j-\mathrm{N}}^b, \tag{5.9}$$

$$C_b := D\log(I_0I_{11}\cdots I_{bb}) \in \mathcal{R}, \qquad b = 0,1,2,3,^{23} \tag{5.10}$$

[21] The same normalized basis has been used in [GR17, Sect. 6.2], which naturally appears in the computation of the canonical basis for the twisted theory (see also [GR16, Sect. 7.3]).

[22] By Convention 5.3, to entries of S^Q, A^Q((A.2), (A.4)), we apply $q\mapsto q'$, and still denoted them as S^Q, A^Q in QDE.

[23] To be explicit, by definition (0.2) and by using $I_0^2I_{11}^2I_{22} = Y$ (see Appendix A and [ZZ08]), we have $C_0 = B_1, C_1 = A_1+B_1, C_2 = 1-Y-A_1-B_1, C_3 = 1-Y-A_1$.

$$(c_j)_{j=1,\cdots,\mathrm{N}+3} := (0,\cdots,0,120,770,1345,770). \tag{5.11}$$

Recall that the ring $\mathcal{R}$ is closed under D, hence (5.5) follows from (5.9) by recursion.

For (5.4), just notice that in the inductive formula (5.9), the difference of the index $j-b-k$ mod N is preserved. Hence (5.4) is implied by (5.7) recursively.

Next, we prove the property (5.6). The $j=0$ case will be proved in Lemma 6.1 (2). We now prove $j>0$ case. Applying (1.19) to (3.1):

$$(\mathbf{1}^\alpha, R(z)^*\phi_j) = e^{\tau_\alpha/z}\Delta^{\mathrm{pt}_\alpha}(z)^*(\mathbf{1}^\alpha, S^M(z)^*\phi_j).$$

we see that for $j>1$, the entries $(\mathbf{1}^\alpha, R(z)^*\phi_j)$ can be recursively computed by the QDE

$$(\mathbf{1}^\alpha, R(z)^*\phi_j) = D_{L_\alpha}(\mathbf{1}^\alpha, R(z)^*\phi_{j-1}) - c_j\, q\,(\mathbf{1}^\alpha, R(z)^*\phi_{j-\mathrm{N}}) \tag{5.12}$$

where $D_{L_\alpha} := zD + L_\alpha$ and $(c_j)_{j=1,\cdots,\mathrm{N}+3}$ is defined as in (5.11). Namely,[24]

$$(R_k)^\alpha_j = \Big(D - \tfrac{1}{\mathrm{N}}\big(\tfrac{\mathrm{N}+3}{2} - j + k\big)(1-Y)\Big)(R_{k-1})^\alpha_{j-1} + (R_k)^\alpha_{j-1} + \frac{c_j}{5^5}\,(1-Y)\,(R_k)^\alpha_{j-\mathrm{N}}.$$

By induction on j, using $D(Y) = Y(Y-1)$ and the initial result ($j=0$), we conclude that $(R_k)^\alpha_j$ does not depend on α and that $(R_k)^\alpha_j \in \mathbb{Q}[Y]_{k+\lfloor \frac{j}{\mathrm{N}}\rfloor}$. □

Lemma 5.5. *Let V-matrix be the edge contribution in the stable graph localization formula (Theorem 3.1), which is given by*

$$V(z,w) = \sum_{k,l\geq 0} V_{kl} z^k w^l := \sum_\alpha \frac{e_\alpha \otimes e^\alpha - R(z)^{-1}e_\alpha \otimes R(w)^{-1}e^\alpha}{z+w}.$$

Then the coefficients V_{kl} can be written in the following form

$$\begin{aligned} V_{kl} =& \textstyle\sum_{a,b=0}^3 (V_{kl})^{ab}\varphi_a\otimes\varphi_b + \sum_{b=0}^3\sum_{\alpha=1}^{\mathrm{N}} L_\alpha^{2-b-k-l}\cdot (V_{kl})^{\alpha b}\bar{1}_\alpha \otimes \varphi_b + \\ &+ \textstyle\sum_{\alpha,\beta=1}^{\mathrm{N}}\sum_j L_\alpha^{j-k}L_\beta^{2-j-l}\cdot (V_{kl})^{\alpha\beta;j}\,\bar{1}_\alpha\otimes\bar{1}_\beta \end{aligned} \tag{5.13}$$

such that (see Appendix C for a more detailed discussion)

$$\tfrac{Y}{t^{\mathrm{N}}}\cdot (V_{kl})^{\alpha\beta;j} \in \mathbb{Q}[Y], \qquad \tfrac{Y}{t^{\mathrm{N}}}\cdot (V_{kl})^{\alpha b} \quad \text{and} \quad \tfrac{Y}{t^{\mathrm{N}}}\cdot (V_{kl})^{ab} \in \mathbb{Q}[A,B,B_2,B_3,Y]$$

are indepedent of α,β. Furthermore, the edge contribution is homogeneous of degree 2.

Proof. For any basis $\{e_a\}$'s of $\mathcal{H}$ (with dual basis $\{e^a\}$), we have

$$V(z,w) = \sum_a \sum_{\substack{k\leq i-1\\ i-1-k=l-j\geq 0}} (-1)^{k+j} z^k R_i^* e_a \otimes w^l R_j^* e^a \tag{5.14}$$

which implies, by the definition of V_{kl}

$$V_{kl} = \sum_a \sum_{\substack{i\geq k+1\\ i+j=k+l+1}} (-1)^{k+j} R_i^* e_a \otimes R_j^* e^a \tag{5.15}$$

Choose $\{e_a\} = \{\mathbf{1}_\alpha\}_{\alpha\in[\mathrm{N}]} \cup \{\varphi_j\}_{j=0,1,2,3}$, we obtain (5.13). Then the last statement follows from Lemma 5.4. □

5.3. **Proof of Theorem 1.** To simplify the computation, we pick t such that $t^{\mathrm{N}} = -1$. This makes $q' = q$, By definition, it is clear that for $2g-2+n>0$

$$P_{g,n+1} = \big(q\tfrac{d}{dq} + (g-1)(2B+1-Y) - n\,A\big)P_{g,n}. \tag{5.16}$$

Hence if $P_{g,n}\in\mathcal{R}$, so does $P_{g,n+1}$.

By Theorem 3.1 , any correlator of $[0,1]$-theory is equal to a summation over stable graphs. We define the leading graph of the stable graphs in $G^{\mathrm{N}}_{g,0}$ to be the one with single genus g vertex labeled by Q (quintic fixed loci). We now prove the theorem by induction.

By (0.4) both $P_{0,3}$ and $P_{1,1}$ lie in $\mathcal{R}$. Thus by (5.16)

$$P_{0,n}(n\geq 3),\ \text{and}\ \ P_{1,n}(n\geq 1)\ \in\ \mathcal{R}.$$

---

[24] Note we have used $L^{-\mathrm{N}}q = (1-Y)/5^5$, $L_\alpha = \zeta_{\mathrm{N}}^\alpha t\,L$ and $DL = \frac{1}{\mathrm{N}}L(1-Y)$.

We next assume $g \geq 2$. Assume for any genus $h < g$, and any $2h-2+n>0$ one has $P_{h,n} \in \mathcal{R}$. We consider the normalized NMSP-$[0,1]$ potential $Y^{g-1} \cdot F_g^{[0,1]}$. By Theorem 4,

$$Y^{g-1} \cdot F_g^{[0,1]} \in \mathbb{Q}[Y]_{g-1}.$$

On the other hand, we have the graph sum formula by Theorem 3.1. The leading stable graph of the leading bipartite graph is a single genus g vertex labelled by 0, with contribution $\frac{Y^{g-1}}{I_0^{2g-2}} F_g = P_g$. For the remainder graphs, via the relation

$$\textstyle\sum_v (g_v - 1) + E = g - 1$$

we put the factor Y^{g-1} into vertices and edges. Together with Lemma 5.4 and 5.5, the contribution of each non-leading graph is given by the followings:

- At each edge, the contribution is of the form (by (5.13))

$$Y \cdot (V_{kl})^{ab}, \quad Y \cdot (V_{kl})^{\alpha b} \quad \text{or} \quad Y \cdot (V_{kl})^{\alpha\beta;j} \tag{5.17}$$

which lies in the ring $\mathcal{R}$. We pick (any) one of them from (5.13), and vary the hour $\alpha = 1, \cdots, \mathrm{N}$ of each level 1 vertex. The variation provides a multiplicative factor below, since (5.17) are independent of hours (c.f. Lemma 5.5).

- At each quintic vertex v, the contribution is

$$P_{g_v, \vec{a}, \vec{k}} := Y^{g_v - 1} \int_{\overline{\mathcal{M}}_{g_v, n_v}} \left[\varphi_{a_1} \bar{\psi}_1^{k_1}, \cdots, \varphi_{a_{n_v}} \bar{\psi}_{n_v}^{k_{n_v}} \right]_{g_v, n_v}^{Q,T}.$$

Here we recall the translated correlators defined by the equation in Definition 3.1. It vanishes unless $\sum a_i + k_i = n_v$. By using string and dilation equations once and again, this correlator will reduce to $P_{g_v,m}$ multiplied by a constant[25] . Since $g_v < g$, by induction hypothesis we have $P_{g_v, \vec{a}, \vec{k}} \in \mathcal{R}$.

- At each pt_α vertex, the contribution is

$$\begin{aligned} & Y^{g_v-1} \int_{\overline{\mathcal{M}}_{g_v, n_v}} \left[L_\alpha^{j_1 - k_1} \bar{\psi}_1^{k_1}, \cdots, L_\alpha^{j_{n_v} - k_{n_v}} \bar{\psi}_{n_v}^{k_{n_v}} \right]_{g_v, n_v}^{\mathrm{pt}_\alpha, T} \\ = & \sum_m \frac{L^{\frac{3}{2}(2g_v - 2)}}{m!} \left\langle L_\alpha^{j_1 - k_1} \bar{\psi}_1^{k_1}, \cdots, L_\alpha^{j_{n_v} - k_{n_v}} \bar{\psi}_{n_v}^{k_{n_v}}, \prod_{s=1}^m \tilde{T}_\alpha(\bar{\psi}_{n_v+s}) \right\rangle_{g_v, n_v+m}. \end{aligned}$$

We claim that, after summing over $\alpha = 1, \cdots, \mathrm{N}$, the contribution lies in $\mathcal{R}$.

(1) For $s = 1, \cdots, m$, if each monomial in

$$\tilde{T}_\alpha(\bar{\psi}_{n_v+s}) = \textstyle\sum_{l_s} (\tilde{T}_\alpha)_{l_s} \bar{\psi}_{n_v+s}^{l_s+1}$$

contribute $(\tilde{T}_\alpha)_{l_s} \bar{\psi}^{l_s+1}$, the correlator is non-zero only if

$$\textstyle\sum_{i=1}^{n_v} k_i + \sum_{s=1}^m (l_s + 1) = 3g_v - 3 + n_v + m.$$

(2) Together with the fact $L_\alpha^l \cdot (\tilde{T}_\alpha)_l = (R_l)_{j\alpha} \in \mathbb{Q}[Y]$ (for all l), we see that the total factor involving L_α is $L_\alpha^{(\sum_i j_i) - n_v}$. This makes $\sum_{\alpha=1}^{\mathrm{N}} L_\alpha^{(\sum_i j_i) - n_v}$ a multiplicative factor of the contribution. Since N may be chosen to be a prime, we do such assumption in the beginning. Then this multiplicative factor is non-zero only if

$$\textstyle\sum_{i=1}^{n_v} j_i = n_v \mod \mathrm{N}.$$

The total factors in the graph Γ becomes

$$\textstyle\prod_{v \in \Gamma} L_{\alpha(v)}^{\sum_{i=1}^{n_v} j_i - n_v} = \prod_{v \in \Gamma} Y^{(\sum_{i=1}^{n_v} j_i - n_v)/\mathrm{N}} = 1.$$

Here we have used that for each edge, if at one end it contributes L_α^j then in the other end it contributes L_β^{2-j}.

[25] For $g = 1, \vec{a} = 0^{n_v}$ case, the correlator will reduce to

$$\langle \bar{\psi} \rangle_{1,1}^Q = \frac{\chi}{24} \quad \text{with} \quad \chi = -200.$$

This shows that the summation of the contributions from the remainder graphs lies in $\mathcal{R}$. Therefore the leading graph contribution

$$P_g \in \mathcal{R}.$$

Finally, by (5.16) we have $P_{g,n} \in \mathcal{R}$ for any $n > 0$. This proves Theorem 1.

5.4. **Proof of Theorem 2.** Notice that the quintic I-functions $\{I_i\}_{i=0}^3$ (see (0.1)) are analytic functions in the disk $\{|q| < \frac{1}{5^5}\}$ (c.f. [CdGP91, (3,14)]). So are the mirror map I_1/I_0 and the generators A_k and B_k. Furthermore, the map

$$q \to \mathrm{Q}(q) = q \cdot e^{I_1(q)/I_0(q)}$$

is an analytic homeomorphism between neighborhoods of zeros. Hence, any element in $\mathcal{R}$ is an analytic function near $\mathrm{Q} = 0$ as a function of Q.

6. Proof of a key property of the R-matrix

In this section, we will prove the following key Lemma:

Lemma 6.1. *(1) Let $\tau^{\mathrm{loc}}(0, q)$ be the DW map (see (1.15)) and $L := (1 + 5^5 q/t^{\mathrm{N}})^{1/\mathrm{N}}$, then*

$$\tau^{\mathrm{loc}}(0, q)|_{\mathrm{pt}_\alpha} = \tau_\alpha(q') := -t_\alpha \int_0^{q'} (L(x) - 1)\frac{dx}{x}.$$

(2) Let $R(z) = \sum_{k\geq 0} R_k z^k$ be the R-matrix defined via the Birkhoff factorization (3.1) (see also (3.13) or Remark 3.6). We consider the $R(z)^*\mathbf{1}$ restricted at the fixed loci pt_α. We claim

$$(R_k)_0^\alpha := L^{\frac{\mathrm{N}+3}{2}} L_\alpha^k \cdot (R_k \mathbf{1}^\alpha, \mathbf{1})^M \quad \text{is independent of } \alpha, \quad \text{and } \in \mathbb{Q}[Y]_k. \tag{6.1}$$

(3) The tail contribution (3.14) at pt_α is given by

$$T(z)|_{\mathrm{pt}_\alpha} = \mathbf{1}_\alpha z - L^{\frac{-\mathrm{N}-3}{2}} \mathbf{1}_\alpha z \cdot \Big[1 - \Big(\Big(\frac{\mathrm{N}}{24} + \frac{43}{120}\Big) + \frac{Y-1}{\mathrm{N}}\Big(\frac{47}{24} + \frac{23\mathrm{N}}{24} + \frac{\mathrm{N}^2}{12}\Big)\Big)\frac{z}{L_\alpha} + O(z^2)\Big]. \tag{6.2}$$

We will give the proof in the following subsections. The idea is to use the Picard-Fuchs (PF) equation to solve R_k recursively and to use Givental's oscillator integral to determine their initial values.

6.1. **Applying Picard-Fuchs equation.** The first columns $(R_k)_0^\alpha$ can be solved from the PF equation for NMSP I-function. It is clear I^M (see (1.14)) satisfies the PF equation:

$$\Big(D_p^5 \prod_{\alpha=1}^{\mathrm{N}} (D_p + t_\alpha) - q \prod_{k=1}^{5} (5D_p + kz)\Big) I^M(q, z) = 0, \tag{6.3}$$

where $D := q\frac{d}{dq}$ and $D_p := zD + p$. By (3.1), (1.13) and (1.17) we also have[26]

$$\Delta^{\mathrm{pt}_\alpha}(z)^* J^M(0, z)|_{\mathrm{pt}_\alpha} = z\Delta^{\mathrm{pt}_\alpha}(z)^* S^M(z)^*\mathbf{1}|_{\mathrm{pt}_\alpha} = ze^{\tau_\alpha/z} R^*(z)\mathbf{1}|_{\mathrm{pt}_\alpha}.$$

Via the mirror theorem (Theorem 1.1) we obtain

$$\Delta^{\mathrm{pt}_\alpha}(z)^* I^M(q, z)|_{\mathrm{pt}_\alpha} = e^{\tau_\alpha/z} z R^*(z)\mathbf{1}|_{\mathrm{pt}_\alpha}. \tag{6.4}$$

The LHS of equation (6.4) satisfies the PF equation (6.3) as well since $\Delta^{\mathrm{pt}_\alpha}(z)$ is a constant in q . Hence we see $R^*(z)\mathbf{1}|_{\mathrm{pt}_\alpha}$ satisfies

$$\Big(D_{L_\alpha}^5 (D_{L_\alpha}^{\mathrm{N}} - t^{\mathrm{N}}) - q \prod_{k=1}^{5} (5D_{L_\alpha} + kz)\Big) R^*(z)\mathbf{1}|_{\mathrm{pt}_\alpha} = 0, \tag{6.5}$$

[26] in this section we always abbreviate $\tau_\alpha = \tau_\alpha(q')$;

where $D_{L_\alpha} := zD + L_\alpha$ and $L_\alpha := -t_\alpha + q\frac{d}{dq}\tau_\alpha$. Note that for any $k > 0$

$$D^k_{L_\alpha} = L^k_\alpha \cdot \Big[1 + \frac{z}{L_\alpha} \cdot \Big(k\,D + \frac{k(k-1)}{2}\frac{DL_\alpha}{L_\alpha}\Big) + \cdots\Big] \tag{6.6}$$

By solving this equation we obtain both L_α and $R^*(z)\mathbf{1}|_{\mathrm{pt}}$.

We solve them recursively as follows: First we look at the coefficient of z^0 of (6.5), which gives us

$$L^5_\alpha(L^{\mathrm{N}}_\alpha - t^{\mathrm{N}}) - q\cdot 5^5 L^5_\alpha = 0.$$

This equation has two types of solutions: $L_\alpha = 0$ with multiplicity 5 and $L_\alpha = \zeta^\alpha_{\mathrm{N}}\, t(1 + 5^5 q/t^{\mathrm{N}})^{1/\mathrm{N}}$ with multiplicity 1. The first solution corresponds to the matrix S^Q which is not diagonalizable, while the second solution corresponds to the diagonalizable part $S^{\mathrm{pt}_\alpha} = e^{\tau_\alpha/z}$, which is what we need. We then obtain

$$\tau_\alpha = \tau_\alpha(q') = \int_0^{q'} (L_\alpha + t_\alpha)\frac{dq}{q}$$

where we have used the initial condition $\tau_\alpha|_{q=0} = 0$. This proves Lemma 6.1 (1).

Next we look at the coefficient of z^1 of (6.5), and use the value of L_α we compute

$$L_\alpha = \zeta^\alpha_{\mathrm{N}}\, t(1 + 5^5 q/t^{\mathrm{N}})^{1/\mathrm{N}}, \quad \text{and} \quad DL_\alpha = (L^{\mathrm{N}}_\alpha - t^{\mathrm{N}})/\mathrm{N}L^{\mathrm{N}-1}_\alpha.$$

Recall $R(z)$ is a symplectic transformation such that

$$R^*(z) := R^*_0 + R^*_1 z + R^*_2 z^2 + \cdots \in \mathrm{Hom}(\mathcal{H}, \mathcal{H}_Q \oplus \mathcal{H}_{\mathrm{Npt}})[\![z, z^{-1}]\!].$$

The coefficient of z^1 of the equation (6.5) becomes

$$\frac{3+\mathrm{N}}{2} 5^5 q\, R^*_0\mathbf{1}|_{\mathrm{pt}_\alpha} + \mathrm{N}(5^5 q + t^{\mathrm{N}})\, D(R^*_0\mathbf{1}|_{\mathrm{pt}_\alpha}) = 0.$$

Solving it we obtain

$$R^*_0\mathbf{1}|_{\mathrm{pt}_\alpha} = L^{-\frac{\mathrm{N}+3}{2}}.$$

Here we have used the initial condition $R^*_0\mathbf{1}|_{\mathrm{pt}_\alpha, q=0} = [z^0]\Delta^{\mathrm{pt}_\alpha}(z) = 1$.

Then we look at the coefficient of z^2 of this equation, and solve $R_1\mathbf{1}|_{\mathrm{pt}_\alpha}$ up to a constant. Repeating the steps, we can solve $R^*_k\mathbf{1}|_{\mathrm{pt}_\alpha}$ for any k. Note that at each step we have one constant to fix. The constant can be fixed by the $\Delta^{\mathrm{pt}_\alpha}$-matrix.

By using the above idea, we now prove (6.1) and (6.2):

Proof of Lemma 6.1 (2),(3). We write

$$R(z)^*\mathbf{1}|_{\mathrm{pt}_\alpha} = L^{-\frac{\mathrm{N}+3}{2}} \cdot \big(1 + \tfrac{r_1}{L_\alpha} z + \tfrac{r_2}{L^2_\alpha} z^2 + \cdots\big). \tag{6.7}$$

The equation (6.5) becomes of the form

$$L^{\mathrm{N}+5}_\alpha \cdot \mathcal{PF}\Big(L^{-\frac{\mathrm{N}+3}{2}} \cdot \big(1 + \frac{r_1}{L_\alpha} z + \frac{r_2}{L^2_\alpha} z^2 + \cdots\big)\Big) = 0, \tag{6.8}$$

where we denote the operator on LHS of (6.5) by $L^{\mathrm{N}+5}_\alpha \cdot \mathcal{PF}$. Recall $q' = -q/t^{\mathrm{N}}$ and let

$$X := 1 - Y = (-5^5 q')/(1 - 5^5 q') = 1 - L^{-\mathrm{N}}.$$

By the explicit formula of (6.5), we see the operator

$$\mathcal{PF} = \tfrac{1}{L^5_\alpha}\big(\tfrac{1}{L^{\mathrm{N}}_\alpha} D^{\mathrm{N}}_{L_\alpha} + 1 - X\big) D^5_{L_\alpha} + \tfrac{X}{5^5 L^5_\alpha} \textstyle\prod_{k=1}^5 (5D_{L_\alpha} + kz)$$

is of the form $\mathcal{PF} = \sum_{k=1}^{\mathrm{N}+5} \frac{z^k}{L^k_\alpha} \cdot \mathcal{PF}_k$, with

$$\mathcal{PF}_k = \textstyle\sum_{j=0}^k f_{k,j}(X) D^{k-j} \quad \text{and} \quad f_{k,j} \in \mathbb{Q}[X]_j \quad \forall k, j. \tag{6.9}$$

Here we have used the following observations:

(1) acting on L_α or X, the operator D increases the X-degree by 1:

$$DL_\alpha = L_\alpha \cdot X/\mathrm{N} \quad \text{and} \quad DX = X(1 - X); \tag{6.10}$$

(2) in each monomial of $\mathcal{PF}_n$, the operator D_{L_α} appear at most n-times;

(3) for any n, by using (6.10) the operator $L_\alpha^{-n} D^n_{L_\alpha}$ has the following form

$$L_\alpha^{-n} D^n_{L_\alpha} = \sum_{k=0}^{n} z^k L_\alpha^{-k} \cdot \sum_{j=0}^{k} c_{k,j}(X) D^{k-j} \tag{6.11}$$

where $c_{k,0} = \binom{n}{k}$ and for $j > 0$, $c_{k,j} \in X\mathbb{Q}[X]_{j-1}$ (See (6.6) for the leading terms).

For example, we have

$$\mathcal{PF}_1 = \mathrm{N}\, D + \frac{(\mathrm{N}+3)X}{2}, \qquad \mathcal{PF}_2 = \frac{\mathrm{N}(\mathrm{N}+9)}{2} D^2 + \frac{(\mathrm{N}^2+12\mathrm{N}+23)X}{2} D + \cdots .$$

Now let $\widetilde{\mathcal{PF}} := L^{\frac{\mathrm{N}+3}{2}}\, \mathcal{PF}(L^{-\frac{\mathrm{N}+3}{2}}) = \sum_{k=1}^{\mathrm{N}+5} z^k \cdot \widetilde{\mathcal{PF}}_i$, with $\widetilde{\mathcal{PF}}_1 = \mathrm{N}D$. Then (6.8) becomes

$$\begin{aligned}
&\widetilde{\mathcal{PF}}_1\big(\tfrac{r_1}{L_\alpha}\big) + \tfrac{1}{L_\alpha} \cdot \widetilde{\mathcal{PF}}_2 1 = 0,\\
&\widetilde{\mathcal{PF}}_1\big(\tfrac{r_2}{L_\alpha^2}\big) + \tfrac{1}{L_\alpha} \cdot \widetilde{\mathcal{PF}}_2\big(\tfrac{r_1}{L_\alpha}\big) + \tfrac{1}{L_\alpha^2} \cdot \widetilde{\mathcal{PF}}_3 1 = 0,\\
&\cdots\ \cdots .
\end{aligned}$$

By the shape of (6.9), we know that any transformation

$$D \mapsto D + a \cdot X, \qquad a \in \mathbb{Q}$$

will not change the degree estimate of the coefficients $f_{k,j}$. Hence the operator $\widetilde{\mathcal{PF}}_1$ has exactly the same properties as (6.9). We observe that the coefficient of z^{k+1} in (6.8) can be always written in the form

$$\mathrm{N} \cdot D\big(\tfrac{r_k}{L_\alpha^k}\big) = \tfrac{1}{L_\alpha^k} \mathcal{P}_k(r_0, r_1, \cdots, r_{k-1})$$

where $r_0 = 1$ and $\mathcal{P}_k$ are differential polynomials in r_i and X: [27]

$$\mathcal{P}_k(r_0, \cdots, r_{k-1}) = -\sum_{j=1}^{k-1} \widetilde{\mathcal{PF}}_{k+1-j}\big|_{D \mapsto D - \frac{j}{\mathrm{N}} X}(r_j).$$

Lemma 6.1 (2) can be proved by induction: Suppose "for any $i < k$, r_i is a polynomial in X of degree i, and does not depend on α" holds. Then $\mathcal{P}_k(r_0, \cdots, r_{k-1})$ is a polynomial in X of degree $k+1$ [27], and is divisible by X[28]. By using $DL_\alpha = \frac{1}{\mathrm{N}} L_\alpha \cdot X$ again one has

$$\frac{r_k}{L_\alpha^k} = \int \frac{\mathrm{N}}{L_\alpha^k} \mathcal{P}_k(r_1, \cdots, r_{k-1})\, \frac{dL_\alpha}{X \cdot L_\alpha}. \tag{6.12}$$

Notice $X^{-1}\mathcal{P}_k \in \mathbb{Q}[X]_k = \mathbb{Q}[Y]_k$ and $X = 1 + L_\alpha^{-\mathrm{N}}$. Thus the integrand is a polynomial of L_α^{-1}. Further, we claim that *the integration constant is zero*. Hence we obtain $r_k \in Q[X]_k$ and independent on α. Together with the initial data $r_0 = 1$, this complete the inductive proof of (6.1). Further, an explicit computation via (6.12) gives (6.2). □

The claim that the integration constant vanishes is equivalent to the following lemma:

Lemma 6.2. *The coefficient of z^k in the series $L^{\frac{\mathrm{N}+3}{2}} \cdot R(z)^* \mathbf{1}|_{\mathrm{pt}_\alpha}$, which is exactly the LHS of the equation* (6.12)*, approaches zero when $L_\alpha^{-1} \to 0$.*

We will prove Lemma 6.2 in the following two sections, as a special case of Proposition 6.1.

6.2. **Asymptotic expansion.** Following [Gi96, CG07], we introduce the Landau-Ginzburg potential $W : (\mathbb{C}^*)^{n+1} \to \mathbb{C}$ for the equivariant GW theory of Fano hypersurface $X_m \subset \mathbb{P}^n$, or equivalently, $\mathcal{O}(m)$-twisted GW theory of $\mathbb{P}^n$ with $m < n+1$:

$$W(x_0, \cdots, x_n) := \sum_{i=0}^{n} (x_i - \lambda_i \ln x_i) - y, \quad y^m q = \prod_{i=0}^{n} x_i.$$

We consider the critical points of W

$$\Big\{ \mathbf{x}_\alpha : \frac{\partial W}{\partial x_i}(\mathbf{x}_\alpha) = 0 \Big\}$$

There are $n+1$-critical points for generic equivariant parameters $\{\lambda_i\}$:

$$(x_i)_\alpha = L_\alpha + \lambda_i, \qquad y_\alpha = m L_\alpha \tag{6.13}$$

[27] We can see if $\deg X = 1, \quad \deg r_i = i, \quad \deg D = 1$, then $\deg \mathcal{P}_k = \deg \widetilde{\mathcal{PF}}_{k+1} = \deg \mathcal{PF}_{k+1} = k+1$.

[28] We have used the property of $c_{k,j}$ in (6.11), and that $Df(X)$ is divisible by X for any polynomial $f(X)$.

with critical value $u_\alpha = W(\mathbf{x}_\alpha)$, where $\{L_\alpha\}$ are $n+1$-solutions[29] of $\prod_i(L+\lambda_i) = (mL)^m q$. Near each critical point $\mathbf{x}_\alpha$ we have a *Lefschetz thimble* γ_α, which is a real n-dimensional cycle in $\mathbb{C}^n$ such that restricted to the Milnor fibre $W^{-1}(u)$ it is the vanishing cycle.

We consider the oscillatory integral

$$\mathcal{I}_\alpha(q,z) := \int_{\gamma_\alpha \subset (\mathbb{C}^*)^n} e^{W/z} \frac{dx_0 \wedge \cdots \wedge dx_n}{x_0 \cdots x_n}.$$

We assume $z<0$ and study its asymptotic expansion as $z \to 0^-$, from the negative real axis. By the result of [Gi96, CCIT16], the asymptotic expansion of $\mathcal{I}_\alpha$ coincides with[30]

$$(-mzD)\mathcal{I}_\alpha(q,z) \asymp e^{u_\alpha/z}(-2\pi z)^{\frac{n+1}{2}} R^\lambda(z)^* \mathbf{1}|_{\mathrm{pt}_\alpha}. \tag{6.14}$$

Here let $S^\lambda(z)$ be the S-matrix of the equivalently $\mathcal{O}(m)$-twisted GW theory of $\mathbb{P}^n$, and $R^\lambda(z)$ is defined via the Birkhoff factorization

$$\Delta^\lambda(z) S^\lambda(z) = R^\lambda(z) e^{u/z}$$

with $\Delta^\lambda(z) = \mathrm{diag}\big\{ \exp \sum_{k>0} \frac{B_{2k}}{2k(2k-1)} \big(\frac{1}{(-m\lambda_\alpha)^{2k-1}} + \sum_{\beta\neq\alpha} \frac{1}{(\lambda_\alpha-\lambda_\beta)^{2k-1}} \big) z^{2k-1} \big\}_{\alpha=0}^n$.

We now use the saddle point method to compute the asymptotic expansion. We consider the Taylor expansion of W near critical point $\mathbf{x}_\alpha$:

$$W = u_\alpha + \frac{1}{2}\mathcal{Q}(\xi) + \sum_{k\geq 3} \frac{1}{k!} \sum_{i_1,\cdots,i_k} \partial_{x_{i_1}} \cdots \partial_{x_{i_k}} W(\mathbf{x}_\alpha) \xi_{i_1} \cdots \xi_{i_k}$$

where $\xi = \mathbf{x} - \mathbf{x}_\alpha$ is the local coordinate and $\mathcal{Q}(\xi) = \sum_{i,j} \partial_{x_i}\partial_{x_j} W(\mathbf{x}_\alpha)\xi_i\xi_j$ is a non-degenerate quadratic form.

For $I = (i_1, \cdots, i_k)$, we denote by

$$\partial_{x_I} := \partial_{x_{i_1}} \cdots \partial_{x_{i_k}}, \qquad \xi_I := \xi_{i_1} \cdots \xi_{i_k}.$$

Then the oscillatory integral has the following form (note $z = -s < 0$ is real)

$$\begin{aligned} \mathcal{I}_\alpha(q,z) &= e^{u_\alpha/z} \int_{\gamma_\alpha} e^{\sum_{k\geq 3}\sum_{I=(i_1,\cdots,i_k)} \frac{1}{z}\partial_I W(\mathbf{x}_\alpha)\xi_I} e^{\frac{1}{2}\mathcal{Q}(\xi)\frac{1}{z}} d\omega \\ &\asymp e^{u_\alpha/z}(-z)^{\frac{n+1}{2}} \Psi_\alpha \cdot \Big(1 + \sum_{l>0} f_l \cdot (-z)^l\Big) \end{aligned}$$

where $d\omega := (\xi_0 + (x_0)_\alpha)^{-1} \cdots (\xi_n + (x_n)_\alpha)^{-1} d\xi_0 \cdots d\xi_n$ and

$$\Psi_{-1\bar{\alpha}} := \textstyle\prod_i (x_i)_\alpha^{-1} \int_{\mathbb{R}^n} e^{-\frac{1}{2}\mathcal{Q}(\xi)} d\xi_1 \cdots d\xi_n. \tag{6.15}$$

Proposition 6.1. *For $l>0$, f_l are rational functions of L_α. Further*

$$\lim_{L_\alpha^{-1}\to 0} f_l(L_\alpha) = 0. \tag{6.16}$$

Remark 6.3. *For CY case,* (6.16) *is no longer true, since the estimate of* $\det B$ *is changed to $O(L_\alpha^{-n-2})$. Indeed, for CY case with $\sum_i \lambda_i \neq 0$, we have $\lim_{L_\alpha^{-1}\to 0} f_l(L_\alpha) = constant$.*

We first prove two lemmas, then give the proof of Proposition 6.1.

Lemma 6.4. *Write $k = k_0 + \cdots + k_n$. When k is even, we have*

$$L_\alpha^{-\frac{n+1+k}{2}} \int_{\mathbb{R}^n} \xi_0^{k_0} \cdots \xi_n^{k_n} e^{\frac{1}{2z}\mathcal{Q}(\xi)} d\xi_0 \cdots d\xi_n = (-z)^{\frac{n+1+k}{2}} O(1) \quad \text{as } L_\alpha^{-1} \to 0\ ; \tag{6.17}$$

when k is odd, (6.17) *vanishes.*

---

[29] At the critical point, the condition $x_i\partial_{x_i}W = 0$ implies $x_i - \lambda_i = -y/m$ for all i. Hence by using $\prod_i(-\frac{y}{m} + \lambda_i) = y^m q$ we solve y_α and then $(x_i)_\alpha$.

[30] Here the critical value $u_\alpha = \int L_\alpha \frac{dq}{q}$ differs from the coordinate τ_α defined in (1.18) by a logarithmic term $t_\alpha \log q$ plus some constant.

Proof. At each critical point (6.13) we have

$$\frac{\partial^2 W}{\partial x_i^2} = \frac{1}{(L_\alpha+\lambda_i)^2}\Big(\lambda_i + \frac{(m-1)}{m}L_\alpha\Big) \quad \text{and} \quad \frac{\partial^2 W}{\partial x_i \partial x_j} = -\frac{L_\alpha}{m(L_\alpha+\lambda_i)(L_\alpha+\lambda_j)}$$

Denote $B_{ij} = \sum_{i,j} \partial_{x_i}\partial_{x_j} W(\mathbf{x}_\alpha)$, we have $B_{ij} = B_{ji}$. Further, we compute

$$\det B = \frac{\sum_{j=0}^{n+1}(m-j)\, e_{n+1-j} L_\alpha^j}{m\prod_i (L_\alpha+\lambda_i)^2} = m^{n+1} L_\alpha^{-(n+1)}\Big(1 - \frac{n+1}{m} + O(L_\alpha^{-1})\Big) \tag{6.18}$$

where e_k are the symmetric polynomials of $\{\lambda_i\}$. In our case, which is Fano, we have $1-\frac{n+1}{m} \neq 0$. This implies

$$(B^{-1})_{ij} = O(L_\alpha) \qquad L_\alpha^{-1} \to 0.$$

Gaussian integral says (6.17) vanishes if $k := \sum k_i$ is not even. Assume $k = 2\ell$ with $\ell \in \mathbb{N}$. By Wick's theorem, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} \xi_{d_0}\cdots\xi_{d_k} e^{\frac{1}{2z}\mathcal{Q}(\xi)} d\xi_0\cdots d\xi_n &= (-z)^{\frac{n+1+k}{2}} \frac{(2\pi)^{n/2}}{\sqrt{\det B}} \frac{1}{2^\ell \ell!} \sum_{\sigma\in\mathfrak{S}_{2\ell}} \prod_{i=1}^{l} (B^{-1})_{d_{\sigma(2i-1)}d_{\sigma(2i)}} \\ &= (-z)^{\frac{n+1+k}{2}} L_\alpha^{\frac{n+1}{2}} L_\alpha^\ell \cdot O(1) \qquad \text{as } L_\alpha^{-1}\to 0.\end{aligned}$$

Here $\mathfrak{S}_{2\ell}$ is the set of all permutations of $\{1,2,\cdots,2\ell\}$ which are products of ℓ disjoint transpositions. ☐

Lemma 6.5. *The following explicit formula for the Gaussian integral holds*

$$\Psi_{-1\bar\alpha} := \textstyle\prod_i (x_i)_\alpha^{-1} \int_{\mathbb{R}^n} e^{-\frac{1}{2}\mathcal{Q}(\xi)} d\xi_1\cdots d\xi_n = \Big(m^{-1}\sum_{j=0}^{n+1}(m-j)\, e_{n+1-j}L_\alpha^j\Big)^{-1/2}$$

Proof. By using (6.18), we have LHS $= \prod_i (x_i)_\alpha^{-1}\cdot(\det\mathcal{Q})^{-1/2} =$ RHS. ☐

Proof of Proposition 6.1. Note that $(x_i)_\alpha^{-1} = O(L_\alpha^{-1})$. First by Lemma 6.4, we have for any $\sum_i k_i = k > 0$

$$\prod_i (x_i)_\alpha^{-k_i-1}\int_{\mathbb{R}^n} \xi_0^{k_0}\cdots\xi_n^{k_n} e^{\frac{1}{2z}\mathcal{Q}(\xi)} d\xi_0\cdots d\xi_n = (-z)^{\frac{n+k+1}{2}} O(L_\alpha^{-1}) \tag{6.19}$$

Next, by using $x_i\partial_{x_i} y = m^{-1}y$, we have

$$\frac{\partial^k W}{\partial x_{i_1}\cdots\partial x_{i_k}} \in \frac{1}{x_{i_1}\cdots x_{i_k}}\mathbb{Q}[y]_1$$

for $k\geq 3$. At the critical point, by (6.13) we see $\partial_{x_I} W(\mathbf{x}_\alpha)$ are rational function of L_α with the following property

$$\partial_{x_{i_1}}\cdots\partial_{x_{i_k}} W(\mathbf{x}_\alpha) = O(L_\alpha^{-k+1}) \qquad \text{when } L_\alpha^{-1}\to 0$$

By combining the estimate proved by Lemma 6.4, one obtains that for any $|I_j|\geq 3$

$$\partial_{x_{I_1}} W(\mathbf{x}_\alpha)\cdots\partial_{x_{I_r}} W(\mathbf{x}_\alpha)\cdot \int_{\mathbb{R}^n} \xi_{I_1}\cdots\xi_{I_r} e^{\frac{1}{2z}\mathcal{Q}(\xi)} \frac{d\xi_0\cdots d\xi_n}{(\xi_0+(x_0)_\alpha)\cdots(\xi_n+(x_n)_\alpha)} \tag{6.20}$$

has leading term (lowest order of L_α^{-1} expansion)

$$(-z)^{\frac{n+1+\sum_j |I_j|}{2}} L_\alpha^{\frac{n+1+\sum_j|I_j|}{2} - \frac{n+1}{2} + \sum_j(1-|I_j|)} = (-z)^{\frac{n+r+1}{2}+l} L_\alpha^{-l} \tag{6.21}$$

where $l := -\sum_j (1-\frac{|I_j|}{2}) > 0$. The Gaussian integral vanishes if l is not an integer. We see equation (6.19) and (6.21) together prove the proposition. ☐

6.3. **Specialization.** In our case, we have $m = 5$ and $n = \mathrm{N} + 4$. Further the equivariant parameters are given by the following special values

$$(t_i) = (0, 0, 0, 0, 0, -\xi_{\mathrm{N}} t, -\xi_{\mathrm{N}}^2 t, \cdots, -\xi_{\mathrm{N}}^{\mathrm{N}} t).$$

Then the equation for L_α becomes $y^5(y^{\mathrm{N}}/5^{\mathrm{N}} - t^{\mathrm{N}}) = -5^5 q \cdot y^5$. We consider

$$\left\{y_\alpha = 5\zeta_{\mathrm{N}}^\alpha (t^{\mathrm{N}} + 5^5 q)^{1/\mathrm{N}} = 5L_\alpha, \quad (x_i)_\alpha = \lambda_i + L_\alpha\right\}, \qquad \alpha = 1, \cdots, \mathrm{N}.$$

At these N critical points, the Hessian $(\partial_{x_i}\partial_{x_j} W)$ is non-degenerate. Hence all the argument in Section 6.2 can be applied. In particular, $\Psi_{-1\bar\alpha} = \sqrt{5} L_\alpha^{-(\mathrm{N}+5)/2}$ and we obtain

$$\begin{aligned} R(z)^* \mathbf{1}|_{\mathrm{pt}_\alpha} &= (zD + L_\alpha)\Big(5^{-\frac12}\Psi_{-1\bar\alpha}(1 + \textstyle\sum_l f_l(-z)^l)\Big) \\ &= L_\alpha^{-(\mathrm{N}+3)/2}\big(1 - \tfrac{\mathrm{N}+5}{2\mathrm{N}} z L_\alpha^{-1} X + z L_\alpha^{-1} D\big)\Big(1 + \textstyle\sum_l f_l(-z)^l\Big) \end{aligned}$$

where we have used $D\tau_\alpha = L_\alpha$ and $DL_\alpha = N^{-1} L_\alpha X$ (Recall $X = 1 + L_\alpha^{-\mathrm{N}}$). Together with the initial condition (6.16), we obtain Lemma 6.2.

Appendix A. Quintic's QDE and the Yukawa coupling

Let $S^Q(\tau, z) = \sum_{k\geq 0} S_k^Q(\tau) z^{-k}$ be the S-function for the quintic. Under the flat basis, it satisfies the following quantum differential equation (QDE)

$$z\, dS^Q(\tau, z) = d\tau *_\tau S^Q(\tau, z).$$

When τ is the mirror map $\tau_Q(q) = I_1(q)H/I_0(q)$, by using divisor equation the QDE becomes

$$(H + zD)\, S^Q(z)^* = S^Q(z)^* * \dot\tau_Q. \tag{A.1}$$

where $D = q\frac{d}{dq}$ and $\dot\tau_Q := H + D\tau_Q = I_{11} H$.

Suppose the quantum product $*\dot\tau_Q$ is given by a matrix A^Q under the flat basis, namely $H^k * \dot\tau_Q = H^j (A^Q)_j^k$. Then we have

$$A^Q = \begin{pmatrix} 0 & & & \\ I_{11} & 0 & & \\ & I_{22} & 0 & \\ & & I_{11} & 0 \end{pmatrix}, \tag{A.2}$$

by the definition of the quantum product and the formulae for the three point functions:

$$\big\langle 1, H, H^2\big\rangle_{0,3}^Q = 1, \quad \langle H, H, H\rangle_{0,3}^Q = I_{22}/I_{11}, \quad \text{with } I_{22} := 1 + D\Big(\frac{DJ_2 + J_1}{I_{11}}\Big). \tag{A.3}$$

Further, by solving the QDE we obtain the explicit formula for the S-matrix at $\tau = \tau_Q$:

$$S^Q(z)^* = \mathrm{I} + \frac{1}{z}\begin{pmatrix} 0 & & & \\ J_1 & 0 & & \\ & \frac{J_2'}{J_1'} & 0 & \\ & & J_1 & 0 \end{pmatrix} + \frac{1}{z^2}\begin{pmatrix} 0 & & & \\ & 0 & & \\ J_2 & & 0 & \\ & \frac{J_2'}{J_1'} J_1 - J_2 & & 0 \end{pmatrix} + \frac{1}{z^3}\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ J_3 & & & 0 \end{pmatrix}. \tag{A.4}$$

where $J_1' := 1 + DJ_1 = I_{11}$ and $J_2' := J_1 + DJ_2$.

The I-functions I_0, I_{11}, I_{22} defined above satisfies the following Zagier-Zinger relation

$$I_0^2 I_{11}^2 I_{22} = (1 - 5^5 q)^{-1}. \tag{A.5}$$

Indeed, (A.5) is equivalent to the original physicists' mirror conjecture proposed in [COGP91]:

Theorem A.1. *The normalized Yukawa coupling*

$$Y := I_0^2 \cdot (I_{1,1})^3 \cdot \langle H, H, H\rangle_{0,3}^Q\,(\mathrm{Q})$$

is a rational function of q. To be precise,

$$Y = (1 - 5^5 q)^{-1}. \tag{A.6}$$

Appendix B. Staiblization of grahps

We consider a genus g connected graph with n legs, and with each vertex v labeled by its genus $g_v \in \mathbb{Z}_{\geq 0}$. A vertex v with valence n_v is called stable, if $2g_v - 2 + n_v > 0$. A graph is called stable if all its vertices are stable.

Given such a graph Θ with $2g-2+n > 0$, we can apply "stabilization" to it to get a stable graph Θ^{st}, by removing all (maximal rational) tails; replacing all (maximal rational) chains between two stable vertices by single edges; and by contracting all (maximal rational) chains between one stable vertex and one leg(and with no other legs in the chain). In this paper, we will use "tail", "chain", and "end" to describe such three types of subgraphs just mentioned. (Here we follow [CGT15] for the terminologies of "tail" and "end".)

Any vertex of Θ that remains a vertex after stabilization is called a "stable-graph-vertex". For any leg l of Θ, the stabilization associates to it a unique(closest) stable-graph-vertex $cl(\ell) \in V(\Theta)$ that is the vertex of Θ so the leg l will be attached to after stabilization.

In this paper we apply stabilization to two kinds of decorated graphs: NMSP localization graphs and bipartite graphs. The following is an example of the second kind:

Example B.1 (From a decorated bipartite graph Λ to its stablization)**.** *In the following graph Λ, the integer next to each vertex/edge is its genus/a_e. For example $g_{v_1} = 3, g_{v_2} = 5, g_{v_3} = 9, a_{e_1} = 1, a_{e_2} = 3$. Each edge is decorated by some hour in* $[\mathrm{N}]$ *which we omit here. And d_∞ of each black vertex is in the first row. There are three legs in Λ, labeled by l_1, l_2, l_3.*

After stablization, only vertices v_1, v_2, v_3 survives and are relabeled $\bar{v}_1, \bar{v}_2, \bar{v}_3$. Thus v_1, v_2, v_3 are the stable-graph-vertices of Λ. Hence we have $cl(l_1) = v_1, cl(l_2) = v_2, cl(l_3) = v_3$. By definiton $c(l_1) = (e_1, v_1), c(l_2) = (e_2, v_2), c(l_3) = l_3$.

Appendix C. Refinement of a key property

Recall the R-matrix was defined in (3.1). Let $R(z) = \sum_k R_k z^k$ and

$$V(z,w) = \sum_{k,l} V_{kl} z^k w^l := \sum_j \tfrac{1}{z+w}(\phi_j \otimes \phi^j - R(z)^{-1}\phi_j \otimes R(w)^{-1}\phi^j).$$

The following is a refinement of Lemma 5.4 and 5.5, which will be used in [NMSP3].

Lemma C.1. *For $k, l \geq 0$, $a = 0, 1, \cdots, \mathrm{N}{+}3$ and $\alpha, \beta \in [\mathrm{N}]$, we have the following properties.*

(a) *The entries*

$$(R_k)^\alpha_a := L^{\frac{\mathrm{N}+3}{2}} \cdot L_\alpha^{-a+k} (\mathbf{1}^\alpha, R_k^* \phi_a)^M \in \mathbb{Q}[X]_{k+\lfloor \frac{a}{\mathrm{N}} \rfloor}, \tag{C.1}$$

and are independent of α. Here we recall $X := 1 - Y$ and $L_\alpha := \zeta_{\mathrm{N}}^\alpha \cdot t \cdot L$.

(b) *The matrix $V_{kl}|_{\mathrm{Npt}\times\mathrm{Npt}}$ is of the following form*

$$V_{kl}|_{\mathrm{Npt}\times\mathrm{Npt}} = L^{-3} t^{\mathrm{N}} \sum_{\alpha,\beta} \sum_s L_\alpha^s L_\beta^{2-s-l-k} \cdot (V_{kl})^{\alpha\beta;s+k}\, \mathbf{1}_\alpha \otimes \mathbf{1}_\beta, \tag{C.2}$$

such that the entries $(V_{kl})^{\alpha\beta;s}$ are independent of α, β and satisfy

$$(V_{kl})^{\alpha\beta;s+k} := \sum \Big((R_i)^\alpha_a (R_j)^\beta_{\mathrm{N}+3-a} + (R_i)^\alpha_{\mathrm{N}+a} (R_j)^\beta_{3-a} - Y \cdot (R_i)^\alpha_a (R_j)^\beta_{3-a} \Big) \in \mathbb{Q}[X]_{k+l+1} \tag{C.3}$$

Here the summation is over $\{(i,j,k) \mid i+j = k+l+1,\ i \geq k+1,\ a = i+s \in [0, \mathrm{N})\}$, and we set $(R_j)_a = 0$ for $a < 0$.

Proof. The statements involving R_k are proved in Lemma 5.4. For the statements involving V_{kl}, by using (5.15) and (C.1) we compute $(\mathbf{1}^\alpha \otimes \mathbf{1}^\beta, V_{kl})$

$$= L^{-(\mathrm{N}+3)} \sum_{i+j=k+l+1,\ i\geq k+1} \frac{(-1)^{k+j}}{5} \Big(\sum_{a=0}^{\mathrm{N}+3} L_\alpha^{a-i} L_\beta^{(\mathrm{N}+3-a)-j} (R_i)_a^\alpha (R_j)_{\mathrm{N}+3-a}^\beta - t^{\mathrm{N}} \sum_{a=0}^{3} L_\alpha^{a-i} L_\beta^{(3-a)-j} (R_i)_a^\alpha (R_j)_{3-a}^\beta \Big)$$

$$= L^{-3} t^{\mathrm{N}} \sum_{i+j=k+l+1,\ i\geq k+1} \frac{(-1)^{k+j}}{5} \sum_{a=0}^{\mathrm{N}-1} L_\alpha^{a-i} L_\beta^{(3-a)-j} \Big((R_i)_a^\alpha (R_j)_{\mathrm{N}+3-a}^\beta + (R_i)_{\mathrm{N}+a}^\alpha (R_j)_{3-a}^\beta - Y (R_i)_a^\alpha (R_j)_{3-a}^\beta \Big)$$

where we used $L_\alpha^{\mathrm{N}} = L_\beta^{\mathrm{N}} = t^{\mathrm{N}} L^{\mathrm{N}} = t^{\mathrm{N}}/Y$. This proves (C.2) and (C.3).

Now we prove the degree estimate of $(V_{kl})^{\alpha\beta;s+k}$. Since $(R_k)_a^\alpha$ does not depend on α, we will simply denote it by $(R_k)_a$. We first claim that for $j = \mathrm{N}, \cdots, \mathrm{N}+3$,

$$\mathrm{Coef}_{X^{k+1}} (R_k)_j = \frac{c_j'}{5^5} \mathrm{Coef}_{X^k} (R_k)_{j-\mathrm{N}}, \tag{C.4}$$

where $(c_j')_{j=\mathrm{N},\cdots,\mathrm{N}+3} = (-120, -890, -2235, -3005)$.

We prove this claim as follows. By using QDE for R-matrix (5.12): for $j = 1, \cdots, \mathrm{N}+3$,

$$R_k^* \phi_j|_{\mathrm{pt}_\alpha} = D(R_{k-1}^* \phi_{j-1}|_{\mathrm{pt}_\alpha}) + L_\alpha \cdot (R_k^* \phi_{j-1})|_{\mathrm{pt}_\alpha} - c_j\, q\, R_k^* \phi_{j-\mathrm{N}}|_{\mathrm{pt}_\alpha}, \tag{C.5}$$

where $(c_j)_{j=1,\cdots,\mathrm{N}+3} = (0, \cdots, 0, 120, 770, 1345, 770)$. Note here $\phi_i = 0$ whenever $i < 0$. As an example, for $j = 1$, (C.5) gives

$$R_k^* \phi_1|_{\mathrm{pt}_\alpha} = D(R_{k-1}^* \phi_0|_{\mathrm{pt}_\alpha}) + L_\alpha \cdot (R_k^* \phi_0)|_{\mathrm{pt}_\alpha}. \tag{C.6}$$

Apply $L^{\frac{\mathrm{N}+3}{2}} \cdot L_\alpha^{k-\mathrm{N}} (\mathbf{1}^\alpha, \cdot)$ to $j = \mathrm{N}$ case of the equation (C.5), we obtain

$$(R_k)_{\mathrm{N}} = L^{\frac{\mathrm{N}+3}{2}} L_\alpha^{k-\mathrm{N}} \Big((\mathbf{1}^\alpha, D(R_{k-1}^* \phi_{N-1}|_{\mathrm{pt}_\alpha})\Big) + (R_k)_{\mathrm{N}-1} - (Y/t^{\mathrm{N}}) c_N\, q\, (R_k)_0. \tag{C.7}$$

We make two observations: (i) by (C.1), $\mathrm{Coef}_{X^{k+1}} (R_k)_j = 0$ for all $j < \mathrm{N}$; (ii) the operator D increase the X-degree by at most 1 because $DL_\alpha = L_\alpha \cdot X/\mathrm{N}, DL = L \cdot X/\mathrm{N}$ and $DX = X(1-X)$. By (i), (ii), (C.1) and that D is a derivation, we have

$$\mathrm{Coef}_{X^{k+1}} (R_k)_{\mathrm{N}-1} = 0 \quad \text{and} \quad \mathrm{Coef}_{X^{k+1}} L^{\frac{\mathrm{N}+3}{2}} L_\alpha^{k-\mathrm{N}} \Big(\mathbf{1}^\alpha, D(R_{k-1}^* \phi_{N-1}|_{\mathrm{pt}_\alpha}) \Big) = 0.$$

This implies

$$\mathrm{Coef}_{X^{k+1}} (R_k)_{\mathrm{N}} = \mathrm{Coef}_{X^{k+1}} \big(Y/t^{\mathrm{N}} \, (-120q)(R_k)_0 \big).$$

Then the case $j = \mathrm{N}$ in (C.4) follows from $5^5\, Y q = t^{\mathrm{N}} X$. Similarly, by the QDE (C.5) for case $R_k^* \phi_{\mathrm{N}+1}$ and $R_{k-1}^* \phi_{\mathrm{N}}$, one calcuates

$$\begin{aligned} R_k^* \phi_{\mathrm{N}+1}|_{\mathrm{pt}_\alpha} = &\, D\Big(D(R_{k-2}^* \phi_{\mathrm{N}-1}|_{\mathrm{pt}_\alpha}) + L_\alpha \cdot (R_{k-1}^* \phi_{\mathrm{N}-1})|_{\mathrm{pt}_\alpha} \Big) - c_{\mathrm{N}}\, q\, R_{k-1}^* \phi_0|_{\mathrm{pt}_\alpha} \\ &+ L_\alpha \cdot \Big(D(R_{k-1}^* \phi_{N-1}|_{\mathrm{pt}_\alpha}) + L_\alpha \cdot (R_k^* \phi_{N-1})|_{\mathrm{pt}_\alpha} \Big) - c_{\mathrm{N}}\, q\, R_k^* \phi_1|_{\mathrm{pt}_\alpha} - c_{\mathrm{N}+1}\, q\, R_k^* \phi_1|_{\mathrm{pt}_\alpha}, \end{aligned}$$

where we have used equation (C.6). We see the only contribution to $\mathrm{Coef}_{X^{k+1}}$ is from the last two terms. Similar arguments applied to the cases $R_k^* \phi_{\mathrm{N}+2}$ and $R_k^* \phi_{\mathrm{N}+3}$, we obtain

$$\begin{aligned} \mathrm{Coef}_{X^{k+1}} (R_k)_{\mathrm{N}+1} &= \mathrm{Coef}_{X^{k+1}} \big(Y/t^{\mathrm{N}} \, (-120q - 770q)(R_k)_1 \big) \\ \mathrm{Coef}_{X^{k+1}} (R_k)_{\mathrm{N}+2} &= \mathrm{Coef}_{X^{k+1}} \big(Y/t^{\mathrm{N}} \, (-120q - 770q - 1345q)(R_k)_2 \big) \\ \mathrm{Coef}_{X^{k+1}} (R_k)_{\mathrm{N}+3} &= \mathrm{Coef}_{X^{k+1}} \big(Y/t^{\mathrm{N}} \, (-120q - 770q - 1345q - 120q)(R_k)_3 \big). \end{aligned}$$

Now we are able to prove a vanishing result for V. Applying (C.1) to each summand of (C.3), we see $(V_{kl})^{\alpha\beta;s+k} \in \mathbb{Q}[X]_{k+l+2}$ is independent of α, β. Each summand in (C.3) is

$$(R_i)_a (R_j)_{\mathrm{N}+3-a} + (R_i)_{\mathrm{N}+a} (R_j)_{3-a} - Y \cdot (R_i)_a (R_j)_{3-a}.$$

We look at its coefficient of X^{k+l+2}(recall $i+j=k+l+1$ by (C.3)). In case $a\in(3,\mathrm{N})$, such coefficient vanishes by (C.1) again. Furthermore by (C.4) we have, for any $a=0,1,2,3$,

$$\begin{aligned}&\mathrm{Coef}_{X^{k+l+2}}\Big((R_i)_a(R_j)_{\mathrm{N}+3-a}+(R_i)_{\mathrm{N}+a}(R_j)_{3-a}-Y\,(R_i)_a(R_j)_{3-a}\Big) \qquad (C.8)\\ &=\big(\frac{c'_{\mathrm{N}+a}}{5^5}+\frac{c'_{\mathrm{N}+3-a}}{5^5}-1\big)\cdot\mathrm{Coef}_{X^{k+l+1}}\Big((R_i)_a(R_j)_{3-a}\Big)=0\end{aligned}$$

Here we have used $Y=1-X$. This finished the proof. □

List of Symbols

N	a large prime integer
p	the equivariant hyperplane class $c_1(\mathcal{O}_{\mathbb{P}^{4+\mathrm{N}}}(1))$
H	the hyperplane class of quintic
Q	the quintic 3-fold
$\mathbb{F}$	the base field $\mathbb{F}=\mathbb{Q}(t)$
$\mathbb{A}$	coefficient ring $\mathbb{A}=\mathbb{Q}(t)[\![q]\!]$ of all cohomologies
Npt	the union of N points $\mathrm{Npt}:=\cup_{\alpha=1}^{\mathrm{N}}\{\mathrm{pt}_\alpha\}$
$\aleph$	the union of the quintic and N points: $\aleph=Q\cup\mathrm{Npt}$
$\mathcal{H}$	the NMSP state space $H^*(\aleph,\mathbb{F})$ with pairing $(\,,)^M$ and unit $\mathbf{1}$ (Def. 1.1)
$\mathcal{H}_Q$	the quintic state space with twisted pairing[9] $(\,,)^{Q,\mathrm{tw}}$ and unit $1_Q\in H^0(Q)$
$\mathcal{H}_{\mathrm{pt}_\alpha}$	the point state space with twisted pairing[9] $(\,,)^{\mathrm{pt}_\alpha,\mathrm{tw}}$ and unit $1_\alpha\in H^0(\mathrm{pt})$
I_i,J_i	the I-functions (0.1) and the J-functions for the quintics, by mirror theorem they are related by $J_i:=I_i/I_0$ for $i=0,\cdots,3$
I_{ii}	Zagier-Zinger's I-functions: $I_{11}=I_{33}:=1+DJ_1$, $I_{22}:=1+D\big(\frac{DJ_2+J_1}{I_{11}}\big)$
Y,X	the normalized Yukawa coupling $Y=(1-5^5q)^{-1}$ (see Thm A.1), $X:=1-Y$
A_k,B_k	generators of Yamaguchi-Yau's ring: $A_k:=I_{11}^{-1}D^kI_{11},\ B_k:=I_0^{-1}D^kI_0$, and we brief $A:=A_1,\ B:=B_1$
q',ζ_{N}	we define $q':=-q/t^{\mathrm{N}}$ and $\zeta_{\mathrm{N}}=e^{2\pi\sqrt{-1}/\mathrm{N}}$
L,L_α	we define $L(q):=(1-5^5q)^{1/\mathrm{N}}$ and $L_\alpha:=\zeta_{\mathrm{N}}^\alpha\,t\cdot L(q')$
$\mathcal{R}$	the Yamaguchi-Yau's ring of five generators $\mathcal{R}:=\mathbb{Q}[A,B,B_2,B_3,Y]$
$\mathbb{Q}[X]_d$	the set of polynomials in X of degree $\le d$, by definition $\mathbb{Q}[X]_d=\mathbb{Q}[Y]_d$
ϕ_i,ϕ^i	the basis $\{\phi_i:=p^i\}_{i=0}^{\mathrm{N}+3}$ of $\mathcal{H}$ with dual basis $\{\phi^i\}_{i=0}^{\mathrm{N}+3}$
φ_i,φ^i	the normalized basis $\{\varphi_i:=I_0I_{11}\cdots I_{ii}H^i\}_{i=0}^3$ of $\mathcal{H}_Q$ with dual basis $\{\varphi^i\}_{i=0}^3$
$\mathcal{W}_{g,n,\mathbf{d}}$	the moduli space of stable NMSP fields of numerical data $(g,n,\mathbf{d})$
$\mathcal{W}^-_{g,n,\mathbf{d}}$	the degeneracy locus of the cosection, which is used to defined the cosection localized virtual cycle $[\mathcal{W}_{g,n,\mathbf{d}}]^{\mathrm{vir}}$
$\langle-\rangle^M_{g,n,d_\infty}$	NMSP correlator. (defined in (0.12))
$\langle-\rangle^M_{g,n}$	same as $\langle-\rangle^M_{g,n,0}$
$\langle-\rangle^{[0,1]}_{g,n}$	NMSP [0,1] correlator. (defined in (0.13))
pr_k	the map $\overline{\mathcal{M}}_{g,n+k}\to\overline{\mathcal{M}}_{g,n}$ defined by forgetting last k markings
$\mathrm{pr}^W_{g,n}$	the natural projection $\mathcal{W}_{g,n,\mathbf{d}}\to\overline{\mathcal{M}}_{g,n}$
$\mathrm{pr}^Q_{g,n}$	the natural projection $\overline{\mathcal{M}}_{g,n}(Q,d)\to\overline{\mathcal{M}}_{g,n}$
$J^\bullet(z),S^\bullet(z)$	for $\bullet=$ "M", "loc", "Q", "Q,tw", "pt_α" and "$\mathrm{pt}_\alpha,\mathrm{tw}$", they are the J-functions and S-functions of the total NMSP, local, quintic, twisted quintic, point and twisted point theory, respectively (defined in Sect. 1.3)

$f(z^-)$	for a rational function $f(z)$, we define $f(z^-)$ to be its expansion near $z=\infty$
$\mathcal{S}f(z)$	for any $f(z)\in\mathcal{H}[\![z]\!]$, we define $\mathcal{S}f(z):=S^M(z^-)f(z)\in\mathcal{H}[\![z,z^{-1}]\!]$
$\mathcal{S}^{\alpha}_{a;i}$	the specialized S-matrix $\mathcal{S}^{\alpha}_{a}:=S^M(z)\mathbf{1}^{\alpha}\big\vert_{z=\frac{5t_\alpha}{a}}$, and $\mathcal{S}^{\alpha}_{a;i}:=(\mathcal{S}^{\alpha}_{a},p^i)^M$
R^{loc}	the R-matrix $R^{\mathrm{loc}}(z)\in\operatorname{End}\mathcal{H}\otimes\mathbb{A}[\![z]\!]$ defined by NMSP localization in (1.15)
R	the R-matrix defined by NMSP localization composited with the GRR formula (at pt_α) in (3.13), an equivalent definition is (3.1) (see Remark 3.6)
Θ	a regular[7] localization graph or a $[0,1]$ localization graph [12]
Λ	a decorated bipartite graph (Def. 2.1)
Γ	a stable graph (Appendix B)
$G^{\mathrm{fl}}_{g,n,\mathbf{d}}$	the set of flat localization graphs (c.f. [NMSP1, Sect. 4.2])
$G^{\mathrm{reg}}_{g,n,\mathbf{d}}$	the set of regular[7] localization graphs
$\Xi^{d_\infty}_{g,n}$	the set of decorated bipartite graphs satisfying (2.1)
$G^{\mathrm{N}}_{g,n}$	the set of stable graphs with vertices decorated by Q or pt_α

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