Numerical algorithms for R&D stochastic control models

Chi Man Leung
Department of Mathematics,
Hong Kong Baptist University, Hong Kong

Yue Kuen Kwok[1]
Department of Mathematics,
Hong Kong University of Science and Technology,
Hong Kong

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Abstract
We consider the optimal strategy of R&D expenditure adopted by a firm that engages in R&D to develop an innovative product to be launched in the market. The firm faces with technological uncertainty associated with the success of the R&D effort and market uncertainty of the stochastic revenue flow generated by the new product. Our model departs from most R&D models by assuming that the firm's knowledge accumulation has impact on the R&D progress, so the hazard rate of arrival of R&D success is no longer memoryless. Also, we assume a finite life span of the technologies that the product resides on. In this paper, we propose efficient finite difference schemes that solve the Hamilton-Jacobi-Bellman formulation of the resulting finite time R&D stochastic control models with an optimal control on R&D expenditure and an optimal stopping rule on the abandonment of R&D effort. The optimal strategies of R&D expenditure with varying sets of model parameters are analyzed. In particular, we observe that R&D expenditure decreases with firm's knowledge stock and may even drop to zero when the accumulation level is sufficiently high.

Keywords: stochastic control, R&D model, knowledge accumulation, finite difference schemes

1 Introduction

R&D efforts that lead to technological innovations provide the engine for economic growth. R&D process can be visualized as a sequence of iterated cycles of generation of knowledge that enhance problem solving skill. Quite often, the successful completion of a R&D project is cumulative. That means it is sequential and building on intermediate results. West and Iansiti (2003) examine the impact on firm performance in the semiconductor industry arising

[1]Correspondence author; email: maykwok@ust.hk; fax number: 852-2358-1643

from generation of knowledge through experimentation and retention of knowledge through experience. They show that projects with high levels of complexity gain from organizational processes that integrate experience of the specific context of the project. Also, experience with the technology can be used to narrow the option set to be searched. Several recent papers have considered the modeling of learning and knowledge accumulation in the R&D process. Doraszelski (2003) models the hazard rate of arrival of discovery as the sum of current R&D effort and knowledge stock. The knowledge stock is assumed to grow with R&D effort and decreases with organizational forgetting (say, due to turnover of experienced staff). Knowledge accumulation adds path dependence to the dynamic R&D process. Hussain (2007) considers R&D races with innovation uncertainty and timing uncertainty, where the hazard rate is modeled as the product of the knowledge stock level and a technical feasibility parameter. The feasibility parameter is modeled as a Bernoulli random variable, which assumes value zero when the innovation is technically infeasible. In his R&D race model, Steinmetz (2009) shows that the level of knowledge stock may exhibit strong impact on the leader-follower equilibrium. The knowledge effect may outweigh the competition effect, for example, the leader's R&D effort may decrease with the lead in knowledge stock. In these R&D race models with knowledge accumulation, they consider R&D effort as a stochastic control variable. However, they do not include a stochastic fundamental state variable that considers market uncertainty. In our model, we allow for the uncertain profit flow rate generated from launching of the innovative product.

In this paper, we consider R&D process conducted by single firm within a finite time horizon under both market and technological uncertainty. We also allow for uncertainty in the life span of the technology. Suppose a firm would like to engage in R&D to develop a new product using an existing technology, like the development of new iPhones using 3G wireless technologies. One would expect that the existing 3G technologies may be superseded by future generation of mobile telecommunications technologies. The mobile phones manufactured using 3G technologies will then become obsolete. The exact date for the emergence of a new type of technology may not be certain, the terminal date T that the product becomes obsolete is modeled by an exponential distribution in our R&D model. Distinctive from most R&D race models with stochastic control on the R&D effort, we include market uncertainty of the innovative product in our model where the revenue flow rate is assumed to follow a stochastic process. The firm is allowed to control its level of R&D expenditure at any time during R&D (including the right to abandon R&D completely) so as to maximize the expected value of the net profit from the R&D process. The technological uncertainty is quantified by the hazard rate of arrival of discovery, which depends on the current level of R&D effort and the accumulation of knowledge throughout the R&D process. As part of the solution to the model, we would like to determine the optimal control of R&D expenditure and the value function of R&D. More specifically, our model aims to address the following issues:

1. How would the firm's R&D expenditure evolve under different market conditions, say, in response to the current level and volatility of the profit flow rate, and the remaining life span of the relevant technology?

2. How would the firm's R&D expenditure change under varying levels of knowledge stock? Would the firm put off R&D effort when its knowledge stock reaches certain threshold level?

Our R&D stochastic control model is formulated using the Hamilton-Jacobi-Bellman (HJB) optimal control approach. However, the resulting non-linear HJB formulation does not lead to analytical tractability, so one must resort to numerical method for its solution. There have been several papers that address the construction of efficient, reliable and accurate numerical schemes for solving HJB equations in finance. Forsyth and Labahn (2007) provide a general procedure for numerical solution of single factor optimal control problems in finance. They show that the discretized scheme is guaranteed to converge to the viscosity solution of the control problem (relevant to our R&D problem) provided that the discretized schemes observe consistency, monotonicity and l_∞-stability. Wang and Forsyth (2008) show how to adopt the maximal use of central differencing in the discretization of the HJB equations in order to achieve higher order of convergence rate while convergence of the numerical solution to the viscosity solution is ensured at the same time. Huang *et al.* (2010) propose a combined fixed point and policy iteration approach for solving the nonlinear equations resulting from a full implicit discretization of the HJB equations. At each time step, the policy iteration proceeds by solving a linear equation and finding the control that gives the best local solution. They also derive sufficient conditions that ensure convergence of the policy iteration scheme.

Due to path dependence associated with knowledge accumulation in our stochastic R&D model, the resulting HJB equations contain two state variables: stochastic profit flow rate and knowledge stock. Also, an optimal stopping feature is included since we allow the firm to abandon the R&D process optimally when it becomes non-profitable to proceed. We adopt some of the useful and relevant techniques in the above numerical schemes papers in the construction of discretized schemes coupled with policy iteration technique to solve the HJB equations. We also adopt the penalty approximation (Forsyth and Vetzal, 2002) to approximate the optimality condition associated with the optimal stopping feature in our model.

The paper is organized as follows. In the next section, we present our finite time R&D stochastic control model. In particular, we give the details on the modeling of knowledge accumulation, revenue generated from the delivery of the product, and abandonment right. We also extend our model to the case where the life span of the technology is uncertain, the randomness of which is modeled by an exponential distribution. In Section 3, the HJB formulation of our stochastic control problem is derived, with full attention paid to the prescription of the auxiliary conditions in the formulation. Following a similar technique proposed by Windcliff *et al.* (2004), we deduce the far field boundary condition by adopting the linear asymptotic boundary condition on the value function. In Section 4, we solve the HJB equations using the finite difference approach coupled with the policy iteration technique. In the discretization procedure, special attention is taken to ensure convergence of the numerical scheme to the viscosity solution of the HJB formulation. Theoretical studies on the convergence properties of the numerical schemes are also presented. We performed numerical tests to verify the theoretical results on convergence of the scheme and examine the impact of the far field boundary condition on accuracy of the numerical calculations. In Section 5, we present our numerical studies on the sensitivity analysis of the optimal R&D expenditure with respect to different model parameters, like the level of knowledge stock and market conditions. Conclusive remarks are presented in the last section.

2 R&D stochastic control model

We develop a single firm R&D model with finite time horizon subject to both technological uncertainty and market uncertainty. Unlike most real option models where perpetuity is assumed, the innovative product to be developed by the R&D process has a finite life. The modeling assumption is more realistic since technologies associated with commercialized products always have finite life span. Indeed, as seen in the wireless telecommunications technologies, the technology life span tends to become much shortened in recent decades. In this section, we assume that the technology adopted by the innovative commercialized product has a deterministic finite duration $T > 0$. Later, we show how to extend our model to allow for exponentially distributed terminal date of the product.

R&D effort as stochastic control

The technological uncertainty is commonly modeled by the hazard rate of arrival of discovery of the innovative product. Earlier R&D models assume constant hazard rate, independent of the knowledge accumulation in the R&D process. The resulting memorylessness neglects history dependence of R&D effort. In our model, we assume that the hazard rate increases with the firm's current R&D effort (expenditure) and knowledge stock. Also, the R&D effort is considered as a control variable and the firm adopts an optimal control strategy in order to maximize the expected value of the R&D process. This leads to a stochastic control R&D model.

Let $u(t)$ denote the control variable for the rate of expending R&D effort and $z(t)$ be the path dependent variable of knowledge stock. Similar to Doraszelski's knowledge accumulation model (2003), the hazard rate at time t is modeled by

$$h(t) = au(t) + bz(t), \qquad 0 \leq t \leq T, \tag{2.1}$$

where $a > 0$ and $b \geq 0$ are constant parameters. The first term may be interpreted as the contribution of experimentation to R&D success while the second term may be associated with experience accumulation (West and Iansiti, 2003). The special case $b = 0$ corresponds to memoryless R&D process where knowledge stock plays no role in facilitating R&D success.

We assume that the firm's knowledge stock $z(t)$ grows with R&D effort $u(t)$ as depicted by the following differential equation:

$$\frac{dz}{dt} = u(t), \qquad 0 \leq t \leq T, \tag{2.2}$$

where $z(0) = z_0 \geq 0$ is the initial knowledge stock of the firm. Here, $u(t)$ is chosen from a compact set Q where $Q \subset [0, \infty)$ for $t \in [0, T]$. Furthermore, we assume the rate of cost $c(u)$

incurred in R&D to be a continuous and strictly increasing function. In the literature, the cost rate function is commonly taken to be a power function in u (Doraszelski, 2003).

Profit flow rate and expected profit
To model market uncertainty associated with the innovative product, we assume that the profit flow rate x_t generated from the developed product in the market follows the geometric Brownian motion

$$dx_t = \mu x_t \, dt + \sigma x_t \, dZ_t, \tag{2.3}$$

where μ is the constant drift rate, σ is the constant volatility parameter, and Z_t is the standard Brownian motion. Assuming the usual no-bubble condition, μ is taken to be less than the riskfree interest rate r.

Suppose at the current time t, the product has been launched in the market. Given that the life of the product ends at T, where $T > t$, and conditional on $x_t = x$, the expected profit earned by the firm from the product is given by

$$\begin{aligned} W(x,t) = & \; E\left[\int_t^T e^{-r(s-t)} x_s \, ds \mid x_t = x\right] \\ = & \; \frac{x}{r-\mu}\left[1 - e^{-(r-\mu)(T-t)}\right], \qquad t < T. \end{aligned} \tag{2.4}$$

At time $t = T$, the profit flow terminates due to expiration of the technology, so $W(x,T) = 0$.

Abandonment right
In our model, we allow the firm to adopt the irreversible decision of abandonment of R&D. Note that even at $u = 0$, the cost $c(0)$ may remain to be positive (say, maintenance of research facilities), so the abandonment decision helps save the cost of maintaining the R&D process. The firm may choose to abandon R&D optimally when x_t falls to a sufficiently low level. The analysis of the abandonment right requires the determination of the optimal stopping rule in our R&D stochastic control problem.

Remarks

1. *Calibration of model parameters and specification of cost function*
   To apply our R&D model to real life applications, one major challenge is the calibration of the model parameter values. While the parameter values for r, μ and σ can be calibrated from the observable price process of the product, the calibration of the parameters a and b in the hazard function and the specification of the cost function may not be straightforward. By performing various sensitivity analyzes of the optimal control policy and abandonment decision with respect to different model parameter values and various functional forms of the cost functions, our model may serve the role of examining how different rates of knowledge stock and R&D costs may affect R&D policy decision.

2. *Random date of termination of existing technology*
   So far, we have assumed the date of termination of the existing technology to be deterministic. However, in reality, the arrival of a new technology that makes the product obsolete may be uncertain. We would like to show how to model this uncertainty in the life span of the product by assuming the arrival time T of the new technology to be exponentially distributed with parameter $\lambda = \frac{1}{\bar{T}}$, where $\bar{T}$ is the mean of T. In other words, the arrival time is modeled as the first jump time of a Poisson process with intensity rate λ and this Poisson process is independent of the profit flow process x_t. The corresponding reward function $W(x)$ then becomes

$$\begin{aligned} W(x) &= \int_0^\infty \lambda e^{-\lambda u} E\left[\int_0^T e^{-rs} x_s \, ds | T = u\right] du \\ &= \int_0^\infty \lambda e^{-\lambda u} \left\{\frac{x}{r-\mu}\left[1 - e^{-(r-\mu)u}\right]\right\} du \\ &= \frac{x}{r+\lambda-\mu}. \end{aligned} \tag{2.5}$$

   The inclusion of the feature of random date of termination of the existing technology amounts to the modification of the reward function, which can be easily incorporated under the current model formulation.

3 Hamilton-Jacobi-Bellman formulation

Let $V(x,z,t)$ denote the time-t value function of the R&D project conditional on $x_t = x$ and $z_t = z$. Using the Bellman optimality condition, the HJB equation that governs the value function is given by

$$\begin{aligned} V(x,z,t) = \lim_{dt\to 0} \max(0, \sup_{u\in Q}\{&-c(u)dt + h(t)W(x,t)dt \\ &+ [1 - h(t)dt]e^{-rdt}E\left[V(x_{t+dt}, z_{t+dt}, t+dt)|x_t = x, z_t = z\right]\}). \end{aligned} \tag{3.1}$$

The above expression indicates that the optimal stopping rule is applied when the firm either chooses to abandon the project (with zero value being resulted) or continues the R&D process. When continuation of the R&D process is optimally chosen, the corresponding optimal control $u^*(t)$ is determined so that the continuation value is maximized. The continuation value consists of 3 terms: (i) cost of operating R&D (negative value), (ii) with probability $h(t)dt$, R&D succeeds within $(t, t+dt)$ and the expected profit derived from the product is $W(x,t)$, (iii) with probability $1 - h(t)dt$, R&D continues at $t + dt$ and the discounted expected value of the project is given by $e^{-rdt}E\left[V(x_{t+dt}, z_{t+dt}, t+dt)|x_t = x, z_t = z\right]$. By applying Ito's lemma, the last term can be expressed as

$$\begin{aligned} & E\left[V(x_{t+dt}, z_{t+dt}, t+dt)|x_t = x, z_t = z\right] \\ = \; & V + \frac{\partial V}{\partial t}dt + \mu x\frac{\partial V}{\partial x}dt + \frac{\sigma^2 x^2}{2}\frac{\partial^2 V}{\partial x^2}dt + u\frac{\partial V}{\partial z}dt + O((dt)^{\frac{3}{2}}), \end{aligned}$$

where $V = V(x, z, t)$. Substituting the above relation into eq. (3.1), we obtain the following combined HJB formulation of optimal control on R&D effort and linear complementarity formulation of the optimal stopping rule at abandonment:

$$\max(-V, \sup_{u \in Q}\{h(t)W(x,t) - [h(t)+r]V + \frac{\partial V}{\partial t} + \mu x \frac{\partial V}{\partial x} + \frac{\sigma^2}{2}x^2 \frac{\partial^2 V}{\partial x^2}$$

$$+ u\frac{\partial V}{\partial z} - c(u)\}) = 0, \qquad x > 0, z \geq 0, t \in [0, T). \tag{3.2}$$

Auxiliary conditions

Since the R&D process is sure to terminate at T, we have the obvious terminal condition

$$V(x, z, T) = 0 \tag{3.3a}$$

for all values of x and z. In the stopping region, we have abandonment of the project, so the value function becomes zero for $x \leq x^*(t)$. Here, $x^*(t)$ is the stopping boundary at time t. It becomes more tricky to consider the far field boundary conditions at $z \to \infty$ and $x \to \infty$.

(i) At exceedingly high value of z, the hazard rate tends to infinite value. The innovative product is almost surely to be delivered at the next instant, so

$$V(x, z, t) \to W(x, t), \qquad \text{as } z \to \infty. \tag{3.3b}$$

(ii) At $x \to \infty$, Forsyth and Labahn (2007) provide the justification of adopting the linear asymptotic boundary condition on V, where $\frac{\partial^2 V}{\partial x^2} \to 0$. We then have

$$V(x, z, t) \to C_1(z, t)x + C_2(z, t), \qquad \text{as } x \to \infty. \tag{3.3c}$$

As $x \to \infty$, one may deduce that the firm increases its R&D effort to the maximum level. In this case, the optimal control u^* is independent of x, z and t (Forsyth and Labahn, 2007). Under these assumptions, we may derive the corresponding partial differential equations for both coefficient functions $C_1(z, t)$ and $C_2(z, t)$. The closed form analytic formulas for them are obtained as follows:

$$C_1(z, t) = \frac{1 - e^{-(r-\mu)(T-t)}}{r - \mu} - e^{\frac{(a\hat{u}^* + bz + r - \mu)^2}{2b\hat{u}^*}} \sqrt{\frac{2\pi}{b\hat{u}^*}}[N(d_{11}) - N(d_{12})] \tag{3.4a}$$

and

$$C_2(z, t) = -c(\hat{u}^*)e^{\frac{(a\hat{u}^* + bz + r)^2}{2b\hat{u}^*}} \sqrt{\frac{2\pi}{b\hat{u}^*}}[N(d_{21}) - N(d_{22})], \tag{3.4b}$$

where $\hat{u}^*$ is the supremum value of u within the admissible set of controls, and

$$d_{11} = -\frac{a\hat{u}^* + bz + r - \mu}{\sqrt{b\hat{u}^*}}, \qquad d_{12} = d_{11} - \sqrt{b\hat{u}^*}(T - t),$$

$$d_{21} = -\frac{a\hat{u}^* + bz + r}{\sqrt{b\hat{u}^*}}, \qquad d_{22} = d_{21} - \sqrt{b\hat{u}^*}(T - t).$$

The derivation of these closed form expressions for $C_1(z, t)$ and $C_2(z, t)$ is presented in Appendix A.

4 Construction of numerical schemes and theoretic studies on performance

In this section, we first discuss the discretization of the HJB equations using the finite difference approach. We then proceed to show that the proposed numerical scheme satisfies the relevant properties of consistency, monotonicity and stability. Thus, according to Barles (1997), solution to the numerical scheme converges to the viscosity solution of the HJB equations. Next, we describe the solution of the non-linear discretized scheme using the policy iteration method (Huang *et al.*, 2010). We then present the numerical tests that demonstrate the convergence of the numerical solution.

We start with the transformation of the linear complimentarity formulation (3.2) into the penalty approximation formulation (Forsyth and Labahn, 2002). This is done by appending a penalty term $-\frac{AV}{\varepsilon}$, where $A \in \{0,1\}$ and ε is a sufficiently small parameter, such that the appended term becomes dominant when the state variables lie in the stopping region in which the R&D process should be optimally abandoned. The corresponding penalized form can be expressed as

$$\sup_{A\in\{0,1\},u\in Q}\left\{-\frac{\partial V}{\partial \tau}+\mu x\frac{\partial V}{\partial x}+\frac{\sigma^2}{2}x^2\frac{\partial^2 V}{\partial x^2}+u\frac{\partial V}{\partial z}-c(u)\right.$$
$$\left.-[h(\tau)+r]V+h(\tau)W(x,\tau)-\frac{AV}{\varepsilon}\right\}=0, \tag{4.1}$$

where $\tau = T - t$. The corresponding auxiliary conditions are prescribed as follows:

$$\begin{aligned}
&V(x,z,0)=0, && (x,z)\in(0,\infty)\times[0,\infty),\\
&V(0,z,\tau)=0, && (z,\tau)\in[0,\infty)\times[0,T],\\
&V(x,z,\tau)\to W(x,\tau) \text{ as } z\to\infty, && (x,\tau)\in[0,\infty)\times[0,T],\\
&V(x,z,\tau)=C_1(z,\tau)x+C_2(z,\tau) \text{ as } x\to\infty, && (z,\tau)\in[0,\infty)\times[0,T].
\end{aligned}$$

For notational convenience, we write the governing equation (4.1) together with the auxiliary conditions as

$$F(V)=0$$

where $V = V(x,z,\tau)$ and $(x,z,\tau) \in [0,\infty)\times[0,\infty)\times[0,T]$.

The discretized domain is restricted to a finite domain: $[0,x_{max}]\times[0,z_{\max}]\times[0,T]$, where $x_{\max}$ and $z_{\max}$ are chosen to be sufficiently large. The $(j,k,n)^{th}$ node in the discretized domain corresponds to $x_j = j\Delta x$, $z_k = k\Delta z$ and $\tau_n = n\Delta\tau$, where $j = 1,2,...,j_{\max}$, $k = 1,2,...,k_{\max}$, and $n = 0,1,...,N$. Let $V^n_{j,k}$ and W^n_j denote the numerical approximation to $V(x_j,z_k,\tau_n)$ and $W(x_j,\tau_n)$, respectively. We also let $u^n_{j,k}$ and $A^n_{j,k}$ denote the respective control strategy at the nodal point (x_j,z_k,τ_n). In our discretization procedure, we follow the techniques developed in Forsyth and Labahn (2007) where fully implicit discretization is adopted and approximate forward / central / backward differencing is applied to various spatial differential operators so that the condition of positive coefficients is enforced. Using these discretization procedures, it can be shown that convergence of the numerical scheme to the viscosity solution is guaranteed (see Proposition 1). The resulting discretized scheme is obtained as follows:

$$\frac{V_{j,k}^{n+1} - V_{j,k}^{n}}{\Delta\tau}$$
$$= \sup_{A_{j,k}^{n+1}\in\{0,1\},u_{j,k}^{n+1}\in Q} \left\{ a_j V_{j+1,k}^{n+1} - (a_j + b_j + c_{j,k}^{n+1})V_{j,k}^{n+1} + b_j V_{j-1,k}^{n+1} + d_{j,k}^{n+1} V_{j,k+1}^{n+1} + e_{j,k}^{n+1} \right\},$$

where

$$a_j = \frac{\mu x_j}{\Delta x} + \frac{\sigma^2}{2}\frac{x_j^2}{\Delta x^2}, \qquad b_j = \frac{\sigma^2}{2}\frac{x_j^2}{\Delta x^2},$$
$$c_{j,k}^{n+1} = a u_{j,k}^{n+1} + b z_k + r + \frac{u_{j,k}^{n+1}}{\Delta z} + \frac{A_{j,k}^{n+1}}{\varepsilon},$$
$$d_k^{n+1} = \frac{u_{j,k}^{n+1}}{\Delta z}, \qquad e_{j,k}^{n+1} = (a u_{j,k}^{n+1} + b z_k) W_j^{n+1} - c(u_{j,k}^{n+1}). \tag{4.2}$$

The coefficients a_j, b_j, $c_{j,k}^{n+1}$ and $d_{j,k}^{n+1}$ are all non-negative. Similarly, for notational convenience, we write the discretized scheme together with the appropriate numerical boundary conditions as

$$G_{j,k}^{n+1}(V_{j,k}^{n+1}, V_{j+1,k}^{n+1}, V_{j-1,k}^{n+1}, V_{j,k+1}^{n+1}, V_{j,k}^{n}) = 0,$$

where $V_{j,k}^n$ is the grid value function defined at (x_j, z_k, τ_{n+1}), $j = 0, 1, ..., j_{\max}$, $k = 0, 1, ..., k_{\max}$, $n = 0, 1, ..., N$.

Recall that $\Delta x = \frac{x_{\max}}{j_{\max}}$, $\Delta z = \frac{z_{\max}}{k_{\max}}$ and $\Delta\tau = \frac{T}{N}$. The stepwidth parameters and time step are chosen such that

$$\Delta x = \beta_1 \delta, \qquad \Delta z = \beta_2 \delta \quad \text{and} \quad \Delta\tau = \beta_3 \delta, \tag{4.3}$$

where β_1, β_2 and β_3 are positive constants independent of the small parameter δ. These assumptions on Δx, Δz and $\Delta\tau$ are necessary in order to establish pointwise consistency of the numerical scheme. In addition, the l_∞-stability property and monotonicity property of the numerical scheme (4.2) can be established. The formal definitions of these properties of the discretized schemes are stated in Definitions 5.1-5.3 in Forsyth and Labahn (2007) (also see Appendix B).

Proposition 1 *The numerical scheme (4.2) observes the properties of consistency (pointwise), monotonicity and l_∞-stability. Provided that the strong comparison property holds, the numerical solution to scheme (4.2) converges to the viscosity solution of the HJB formulation (4.1).*

The proof of Proposition 1 relies on some technical results in Barles (1997), the details of which are relegated to Appendix B.

Numerical implementation of policy iteration

The numerical scheme (4.2) is non-linear and one has to search for the optimal control variables $u_{j,k}^{n+1*}$ and $A_{j,k}^{n+1*}$ as part of solution procedure. Following Huang *et al.* (2010), we derive the appropriate policy iteration scheme that solves scheme (4.2) through an iterative search for $u_{j,k}^{n+1*}$ and $A_{j,k}^{n+1*}$. Noting that the boundary values $V_{j,k_{\max}}^{n+1}$ at $k = k_{\max}$ are known, we proceed to solve for $V_{j,k}^{n+1}$ through marching backward in k, $k = k_{\max} - 1, ..., 1$. For a fixed value of k, we solve recursively for the optimal control variables, where each iteration requires numerical solution of a system of $j_{\max} - 1$ algebraic equations.

First, we rewrite scheme (4.2) as follows:

$$\begin{aligned}\sup_{u_{j,k}^{n+1}, A_{j,k}^{n+1}} &\{a_j \Delta\tau V_{j+1,k}^{n+1} - [1 + (a_j + b_j + c_{j,k}^{n+1})\Delta\tau] V_{j,k}^{n+1} \\ &+ b_j \Delta\tau V_{j-1,k}^{n+1} + h_{j,k}^{n+1}\} = 0, \qquad j = 1, 2, ..., j_{\max} - 1,\end{aligned} \tag{4.4}$$

where the known quantities are lumped into $h_{j,k}^{n+1}$ defined as

$$h_{j,k}^{n+1} = V_{j,k}^n + \Delta\tau(d_{j,k}^{n+1} V_{j,k+1}^{n+1} + e_{j,k}^{n+1}).$$

The solution to the formulation in eq. (4.4) is given by solving the following system of algebraic equations:

$$-a_j \Delta\tau V_{j,k}^{n+1} + [1 + (a_j + b_j + c_{j,k}^{n+1})\Delta\tau] V_{j,k}^{n+1} - b_j \Delta\tau V_{j,k}^{n+1} = h_{j,k}^{n+1}, j = 1, 2, ..., j_{\max} - 1. \tag{4.5}$$

Here, the coefficients a_j, b_j, $c_{j,k}^{n+1}$, $h_{j,k}^{n+1}$ and the grid value function $V_{j,k}^{n+1}$ are evaluated at the optimal control variables $u_{j,k}^{n+1*}$ and $A_{j,k}^{n+1*}$, which are determined by

$$\begin{aligned}&(u_{j,k}^{n+1*}, A_{j,k}^{n+1*}) \\ &= \operatorname*{argmax}_{u \in Q, A \in \{0,1\}} \{a_j \Delta\tau V_{j+1,k}^{n+1} - [1 + (a_j + b_j + c_{j,k}^{n+1})\Delta\tau] V_{j,k}^{n+1} + b_j \Delta\tau V_{j-1,k}^{n+1} + h_{j,k}^{n+1}\}.\end{aligned} \tag{4.6}$$

We define the vector of grid value function values as follows:

$$\mathbf{V}_k^{n+1} = (V_{0,k}^{n+1} \;\; V_{1,k}^{n+1} \;\; ... \;\; V_{j_{\max},k}^{n+1})^T, \;\; k = 0, 1, ..., k_{\max} - 1.$$

Also, we define the tridiagonal matrix B_k of dimension $(j_{\max}+1)\times(j_{\max}+1)$, $k = 0, 1, ..., k_{\max} - 1$, whose entries are given by

$$[B_k]_{l,m} = \begin{cases} 1 & (l, m) = (j_{\max}, j_{\max}) \\ -b_l \Delta\tau & m = l - 1, \;\; l = 1, ..., j_{\max} - 1 \\ -a_l \Delta\tau & m = l + 1, \;\; l = 0, 1, ..., j_{\max} - 1 \\ 1 + (a_l + b_l + c_{l,k}^{n+1})\Delta\tau & m = l, l = 0, 1, ..., j_{\max} - 1 \\ 0 & \text{otherwise} \end{cases}$$

Lastly, we define the column vector $\mathbf{h}_{j,k}^{n+1}$, $k = 0, 1, ..., k_{\max} - 1$, by

$$\mathbf{h}_k^{n+1} = (h_{0,k}^{n+1} \quad h_{1,k}^{n+1} \quad ... \quad h_{j_{\max}-1,k}^{n+1} \quad C_1(z_k, \tau_{n+1}) x_{\max} + C_2(z_k, \tau_{n+1}))^T.$$

In terms of $\mathbf{V}_k^{n+1}$, B_k, $\mathbf{h}_k^{n+1}$, scheme (4.4) can be expressed into the following matrix form:

$$\sup_{u_{j,k}^{n+1}, A_{j,k}^{n+1}} \{-B_k \mathbf{V}_k^{n+1} + \mathbf{h}_k^{n+1}\} = \mathbf{0}, \qquad k = 0, 1, ..., k_{\max} - 1. \tag{4.7}$$

The tridiagonal matrix B_k can be easily checked to be a M-matrix. In the policy iteration scheme presented below, the M-matrix property provides a sufficient condition for the convergence of the policy iteration procedure (Huang *et al.*, 2010).

Let $(\mathbf{V}_k^{n+1})^i$ denote the i^{th} iterate of the vector $\mathbf{V}_k^{n+1}$. The sequence of steps in the policy iteration are outlined as follows:

1. Set the initial guess of $\mathbf{V}_k^{n+1}$ to be $(\mathbf{V}_k^{n+1})^0 = \mathbf{V}_k^n$.

2. Assuming that the value of $(\mathbf{V}_k^{n+1})^i$ is known, the i^{th} iterate of the pair of optimal control variables $(u_{j,k}^{n+1}, A_{j,k}^{n+1})^i$ is determined by

$$(u_{j,k}^{n+1}, A_{j,k}^{n+1})^i = \underset{u \in Q, A \in \{0,1\}}{\operatorname{argmax}} \{(-B_k (\mathbf{V}_k^{n+1})^i + \mathbf{h}_k^{n+1})_j\},$$

   where $(-B_k(\mathbf{V}_k^{n+1})^i + \mathbf{h}_k^{n+1})_j$ denotes the j^{th}-component of the vector, $j = 1, 2, ..., j_{\max} - 1$.

3. Solve the following linear system of equations:

$$-(B_k)^i (\mathbf{V}_k^{n+1})^{i+1} + (\mathbf{h}_k^{n+1})^i = \mathbf{0},$$

   where

$$(B_k)^i = B_k|_{(u_{j,k}^{n+1}, A_{j,k}^{n+1})^i} \text{ and } (\mathbf{h}_k^{n+1})^i = \mathbf{h}_k^{n+1}|_{(u_{j,k}^{n+1}, A_{j,k}^{n+1})^i}.$$

   A well defined iteration termination criterion should be imposed. For example, the policy iteration is terminated when

$$\max_j \frac{(V_{j,k}^{n+1})^{i+1} - (V_{j,k}^{n+1})^i}{(V_{j,k}^{n+1})^{i+1}} < \text{tolerance value.}$$

Convergence of the policy iteration

It is important to ensure that the iterates of the value function under the policy iteration procedure converge to the solution of the discretized scheme (4.4). As the discretization observes the positive coefficients condition, one can establish the following proposition.

Proposition 2 *The iterates* $(\mathbf{V}_k^{n+1})^i$, $i = 1, 2, ...,$ *of the iteration algorithm converge to the unique solution of eq. (4.7) for any initial guess* $(\mathbf{V}_k^{n+1})^0$.

The proof of Proposition 2 is presented in Appendix C. By following similar theoretical arguments in Huang *et al.* (2010), the iterates of the policy iteration algorithm converge to the discretized scheme (4.2).

Numerical tests on convergence of the numerical algorithm
There are two major sources of errors in the numerical algorithm, one is the discretization error arising from the discretization of the differential terms and the other is the error arising from approximating the auxiliary conditions via numerical boundary conditions. First, we performed numerical calculations to test for first order temporal accuracy of the algorithm. In Table 1, we list the numerical solution values of the value function evaluated at $x = 5$, $z = 5$ and $T = 1$ with varying values of number of time steps. The parameter values used in our calculations were chosen to be: $r = 0.05$, $\mu = 0.01$, $\sigma = 0.3$, $a = b = 1$, $\varepsilon = 10^{-8}$, $T = 1$, $x_{\max} = 10$ and $z_{\max} = 10$ and $c(u) = 0.01 + \frac{u^2}{2}$. It is seen from Table 1 that the ratio of difference in numerical solutions is close to 2 when the number of time steps is doubled, indicating that first order termporal accuracy of the numerical algorithm is achieved. Next, we examine the impact of the choices of the upper boundaries, $x_{\max}$ and $z_{\max}$, in the discretized domain on the accuracy of the numerical solution. In Table 2, we list the numerical solution of $V(50, 50, 1)$ with various choices of $x_{\max}$ and $z_{\max}$. In our numerical calculations, we use the same set of model parameters as those for Table 1. It is seen from Table 2 that the percentage difference in numerical solutions is less than 0.005% when the values for $x_{\max}$ and $z_{\max}$ have been chosen to be about 200, indicating that the numerical errors caused by finite truncation of the computational domain is insignificant with these choices of values for $x_{\max}$ and $z_{\max}$.

5 Analysis of the optimal R&D effort

In this section, we would like to explore how the firm adopts its optimal R&D effort in response to various market conditions and economic scenarios by performing the sensitivity analysis of the optimal control u^* with respect to different model parameters, like time to expiry, knowledge stock level, volatility of the stochastic state variable, etc. Also, we explore the optimal abandonment policy adopted by the firm by computing the optimal abandonment boundary that separates the continuation region and abandonment region. Unless otherwise stated, the following set of parameter values are used in generating the numerical plots: $r = 0.05$, $\mu = 0.01$, $\sigma = 0.3$, $a = b = 1$, $\varepsilon = 10^{-8}$, $T = 1$, $c(u) = 10 + \frac{u^2}{2}$, $Q = [0, 10]$.

First, we explore the dependence of u^* on time to expiry τ. In Figure 1, we show the numerical plot of u^* against τ at the fixed value of $x = 10$. It is shown in the figure that when $\tau < \tau^*$ (our calculations give $\tau^* = 0.1$), u^* assumes zero value indicating that the firm chooses to abandon the project (zero R&D effort). This agrees with the financial intuition that as the calendar time is approaching the expiration date of the technology, the potential profit becomes negligibly small while there is a fixed cost to keep the project if not abandoned. As a result, it is optimal to abandon the project. When $\tau > \tau^*$, the firm sets a higher optimal R&D effort when the calendar time is further from the expiration date. This is because higher expected profit $W(x, \tau)$ is generated from the project as τ increases. On the other hand, since $W(x, \tau)$ is bounded above by $\frac{x}{r-\mu}$ for a fixed value of x, so u^* tends to some asymptotic level as τ increases to some sufficiently high value ($u^* \to 1.96$ as shown in

Figure 1).

Next, we explore the dependence of u^* on the current value of the state variable x with varying values of knowledge stock z. It is shown in Figure 2 that u^* increases almost linearly with respect to x until up to the level that is capped by the maximum value of allowable R&D expenditure (in our calculations, we set $\hat{u}^* = \sup Q = 10$). We also show various curves of u^* against x at varying values of the knowledge stock z. For a given fixed value of x, when z assumes a lower value, the firm increases the control u^* to speed up R&D in order to increase the expected value of profit from the R&D project.

Alternatively, we show the dependence of u^* on z with two different choices of cost functions and varying values of x. In Figure 3a, we plot u^* against z with the cost function $c(u) = 10 + \dfrac{u^2}{2}$. It is observed that that u^* decreases almost linearly as the knowledge stock z increases. When z stays above some threshold value, u^* becomes identically zero indicating that the firm chooses optimally to stop putting any extra R&D effort (though the R&D project is not abandoned). Our calculations shows that (i) at $x = 18.75$, $u^* = 0$ when $z \geq 20.31$, (ii) at $x = 31.25$, $u^* = 0$ when $z \geq 34.38$. This is coined the "pure knowledge effect" phenomenon by Doraszelski (2003), where the firm chooses optimally to reduce R&D effort as the knowledge stock increases. Similarly, we plot u^* against z with the cost function $c(u) = 10 + \dfrac{u^{1.5}}{2}$ in Figure 3b. Here, the elasticity of the cost function is 1.5 (exponent of the power function in the cost function). We observe that the dependence of u^* on z no longer exhibits linear decline, as what has been observed under the quadratic cost function. Instead, the optimal control is seen to decrease and tend to some asymptotic level with increasing knowledge stock.

We also investigate how u^* is affected by market uncertainty as proxied by the volatility parameter σ of the stochastic state variable x_t. We modify the cost function to be $c(u) = 10 + \dfrac{u^2}{2}$ and the set of admissible controls to be $Q = [0, 10]$ in the numerical calculations that generate Figure 4. The optimal control is seen to decrease with increasing volatility, indicating that the firm chooses to expend less in R&D effort under a higher level of market uncertainty. The impact of σ on u^* to be almost insignificant as revealed by the curves in Figure 4. This result is expected since saving of the maintenance cost of R&D from an early abandonment is not sensitive to volatility of the profit flow. In other word, the optimal abandonment policy is almost unrelated to market uncertainty as characterized by volatility.

Lastly, we investigate the optimal stopping rule of abandonment by plotting the optimal abandonment threshold x^* against time to expiry τ with varying values of fixed cost c_0 in the cost function $c(u) = c_0 + \frac{u^2}{2}$. In the numerical calculations that generate the separating boundary curve shown in Figure 5, we take the set of admissible control to be $Q = [0, 10]$ and $z = 50$. The optimal abandonment boundary separates the continuation region in the above and the abandonment region below. When τ becomes smaller than some threshold value, it becomes optimal for the firm to abandon the R&D project at any level of x. As a result, x^* becomes infinitely large in value. When the calendar time is sufficiently far from the expiration of the technology, x^* does not depend sensibly on τ. When the fixed cost

c_0 is higher, it is more likely for the firm to exercise the abandonment right at a higher abandonment threshold x^*, so x^* is expected to be an increasing function of c_0. The plots of x^* against τ with $c_0 = 5, 10$ and 20 in Figure 5 are seen to agree well with this economic intuition.

6 Conclusion

We have developed a finite-time real option model with stochastic control that explores the optimal strategy of R&D effort adopted by a firm engaging in R&D in the development of an innovative product. Our R&D stochastic control model includes market uncertainty and technological uncertainty, and the firm is allowed to adopt its optimal strategy of R&D effort as control together with the right to abandon the R&D project. While most real option models on R&D assume perpetuity for simplicity of analysis, we consider the more realistic modeling where the new product is based on technologies that have a finite life span. Also, we model the hazard rate of arrival to depend on the current R&D effort and knowledge accumulation in the R&D process, so the hazard rate is non-memoryless. We manage to present the HJB formulation of the stochastic control model in combination with the linear complementarity formulation of the optimal stopping rule of abandonment. An efficient finite difference algorithm coupled with policy iteration and penalty approximation has been developed to solve for the optimal control strategy of R&D effort. Special attention has been taken in the choice of discretization of the HJB equation so that convergence of the numerical solution to the viscosity solution of the HJB equation is guaranteed.

We performed extensive numerical studies on the optimal control of R&D effort with respect to market conditions and knowledge stock. We found that when the calendar time is approaching the expiration date of the technologies, it is optimal for the firm to abandon the R&D project at any level of the profit flow rate. Hence, for finite time horizon R&D models, the right of abandonment is worthy. Taking the stochastic profit flow rate as a proxy of the market conditions, the firm increases its R&D effort with an increasing level of profit flow rate and decreasing volatility of the stochastic profit flow rate. Also, our model exhibits a phenomenon similar to the "pure knowledge effect", where the firm may choose optimally to put off R&D effort when the knowledge stock reaches certain threshold value (though the R&D project is kept in progress). This threshold value depends on the current level of the stochastic profit flow rate.

As potential future works, one may consider extending the existing framework of stochastic control R&D model with knowledge stock to R&D race between two competing firms. Also, we may include the spillover effects of knowledge accumulation from one firm to its rival. Under a two-firm R&D race model, one has to analyze various types of strategic equilibriums to be adopted by the two firms and the associated first-mover and second-mover advantages.

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APPENDICES

A. Derivation of the linear asymptotic boundary condition

When $x \to \infty$, we have $u = \hat{u}^*$ and the abandonment right becomes worthless. The governing equation (3.2) reduces to

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}x^2\frac{\partial^2 V}{\partial x^2} + \mu x\frac{\partial V}{\partial x} + u\frac{\partial V}{\partial z} - [h(t) + r]V = c(u) - h(t)W(x,t).$$

Substituting $V(x,z,t) = C_1(z,t)x + C_2(z,t)$ into the above differential equation and using $\tau = T - t$ as the temporal variable, we obtain the following decoupled pair of differential equations for $C_1(z,\tau)$ and $C_2(z,\tau)$

$$\begin{aligned}
\frac{\partial C_1}{\partial \tau} - \hat{u}^*\frac{\partial C_1}{\partial z} &= -(a\hat{u}^* + bz + r - \mu)C_1 + (a\hat{u}^* + bz)\frac{1 - e^{-(r-\mu)\tau}}{r - \mu}, \\
\frac{\partial C_2}{\partial \tau} - \hat{u}^*\frac{\partial C_2}{\partial z} &= -(a\hat{u}^* + bz + r)C_2 - c(\hat{u}^*),
\end{aligned}$$

with terminal conditions: $C_1(z,0) = C_2(z,0) = 0$. The general solution of C_1 and C_2 is found to be

$$\begin{aligned}
C_1(z,\tau) &= F_1(z,\tau)\int_0^\tau \frac{1 - e^{-(r-\mu)s}}{r - \mu}\,\frac{a\hat{u}^* + bv + b\hat{u}^* s}{F_1(z,s)}\ ds, \\
C_2(z,\tau) &= -F_2(z,\tau)\int_0^\tau \frac{c(\hat{u}^*)}{F_2(z,s)}\ ds,
\end{aligned}$$

where

$$\begin{aligned}
F_1(z,s) &= \exp\left(\int [a\hat{u}^* + b(v - \hat{u}^* s) + r - \mu]\ ds\right), \\
F_2(z,s) &= \exp\left(\int [a\hat{u}^* + b(v - \hat{u}^* s) + r]\ ds\right), \\
v &= z + \hat{u}^*\tau.
\end{aligned}$$

One can obtain the analytic expressions shown in eqs. (3.4a,b) by performing the integration procedures at relative ease, thanks to the exponential functional forms of $F_1(z,s)$ and $F_2(z,s)$ and the linear functions in the respective numerator of the integrand functions.

B. Proof of Proposition 1

To ensure the convergence of the numerical scheme (4.2) to the viscosity solution of the HJB formulation (4.1), it suffices to show that the numerical scheme satisfies the following three properties: consistency (pointwise), monotonicity and l_∞-stability.

1. *Consistency*
   The numerical scheme (4.2) is said to be consistent if for any smooth test function $V(x,z,\tau)$ having bounded partial derivatives of all orders, we have

$$\lim_{\delta \to 0} |F(V) - G(\bullet)| = 0.$$

Consider

$$
\begin{aligned}
& |F(V(x_j, z_k, \tau_{n+1})) - G_{j,k}^{n+1}| \\
\leq \ & |\frac{\partial V}{\partial \tau}(x_j, z_k, \tau_{n+1}) - \frac{V_{j,k}^{n+1} - V_{j,k}^{n}}{\Delta \tau}| \\
& + \sup_{u \in Q, A \in \{0,1\}} |\frac{\sigma^2}{2} x_j^2 \frac{\partial^2 V}{\partial x^2}(x_j, z_k, \tau_{n+1}) + \mu x_j \frac{\partial V}{\partial x}(x_j, z_k, \tau_{n+1}) \\
& \qquad + u \frac{\partial V}{\partial z}(x_j, z_k, \tau_{n+1}) - [h(\tau_{n+1}) + r] V(x_j, z_k, \tau_{n+1}) \\
& \qquad - \frac{A}{\varepsilon} V(x_j, z_k, \tau_{n+1}) + h(\tau_{n+1}) W(x_j, \tau_{n+1}) - c(u) \\
& \qquad - [a_j V_{j+1,k}^{n+1} - (a_j + b_j + c_{j,k}^{n+1}) V_{j,k}^{n+1} + b_j V_{j-1,k}^{n+1} + d_{j,k}^{n+1} V_{j,k+1}^{n+1} + e_{j,k}^{n+1}]| \\
= \ & O(\Delta \tau) + O(\Delta x) + O(\Delta z) = O(\delta).
\end{aligned}
$$

The above order estimates are resulted from the discretization of the differential terms that observes at least first order accuracy and the last equality is due to the choice of the stepwidth parameter and time step as dictated by eq. (4.3). We then have

$$
\lim_{\delta \to 0} |F(V) - G(\bullet)| = 0.
$$

2. *Monotonicity*

The numerical scheme (4.2) is said to be monotone if

$$
G_{j,k}^{n+1}(V_{j,k}^{n+1}, \{Y_{a,b}^{n+1}\}_{(a,b) \neq (j,k)}, Y_{j,k}^{n}) \leq G_{j,k}^{n+1}(V_{j,k}^{n+1}, \{X_{a,b}^{n+1}\}_{(a,b) \neq (j,k)}, X_{j,k}^{n})
$$

where $Y_{j,k}^{n+1} \geq X_{j,k}^{n+1}$, valid for any (j,k). Forsyth and Labahn (2007) provide a nice financial interpretation of this monotonicity condition as a discrete arbitrage inequality. Intuitively, the value function at a node cannot decrease in value if the value function at any neighboring node increases in value. To establish the monotone property, suppose $Y_{j,k}^{n+1} \geq X_{j,k}^{n+1}$ for any j, k and n, and observe that a_j, b_j and $d_{j,k}^{n+1}$ are non-negative coefficients, we have

$$
\begin{aligned}
& G_{j,k}^{n+1}(V_{j,k}^{n+1}, \{Y_{a,b}^{n+1}\}_{(a,b) \neq (j,k)}, Y_{j,k}^{n}) - G_{j,k}^{n+1}(V_{j,k}^{n+1}, \{X_{a,b}^{n+1}\}_{(a,b) \neq (j,k)}, X_{j,k}^{n}) \\
= & - \{ \sup_{u_{j,k}^{n+1}|_Y \in Q, A_{j,k}^{n+1}|_Y \in \{0,1\}} \{a_j Y_{j+1,k}^{n+1} + b_j Y_{j-1,k}^{n+1} - (a_j + b_j + c_{j,k}^{n+1}) V_{j,k}^{n+1} + d_{j,k}^{n+1} Y_{j,k+1}^{n+1} + e_{j,k}^{n+1}\} \\
& - \sup_{u_{j,k}^{n+1}|_X \in Q, A_{j,k}^{n+1}|_X \in \{0,1\}} \{a_j X_{j+1,k}^{n+1} + b_j X_{j-1,k}^{n+1} - (a_j + b_j + c_{j,k}^{n+1}) V_{j,k}^{n+1} + d_{j,k}^{n+1} X_{j,k+1}^{n+1} + e_{j,k}^{n+1}\}\} \\
\leq & - \sup_{u_{j,k}^{n+1} \in Q, A_{j,k}^{n+1} \in \{0,1\}} \{a_j \left(Y_{j+1,k}^{n+1} - X_{j+1,k}^{n+1}\right) + b_j \left(Y_{j-1,k}^{n+1} - X_{j-1,k}^{n+1}\right) + d_{j,k}^{n+1} \left(Y_{j,k+1}^{n+1} - X_{j,k+1}^{n+1}\right)\} \\
\leq & \ 0.
\end{aligned}
$$

This gives the desired monotone property of the numerical scheme. As a remark, the first inequality is deduced from an inequality established in Appendix B of Forsyth and Labahn (2007).

3. *l_∞-stability*
The numerical scheme (4.2) is said to be l_∞-stable if

$$||V^{n+1}||_\infty \leq C$$

for $0 \leq n \leq N$, $T = N\triangle\tau$, $\triangle\tau \to 0$, $\triangle x \to 0$ and $\triangle z \to 0$, where $||V^{n+1}||_\infty = \max\limits_{j,k}\{|V_{j,k}^{n+1}|\}$, and C is independent of $\triangle x$, $\triangle z$ and $\triangle\tau$. First, we note that $\max\limits_{j,k}\{|V_{j,k}^{n+1}|\}$ can be attained at a boundary node or an interior node at the time level $\tau = (n+1)\triangle\tau$. Suppose $\max\limits_{j,k}\{|V_{j,k}^{n+1}|\}$ is attained at $j = j_{\max}$, then

$$||V^{n+1}||_\infty = \max_k\{|C_1(z_k, \tau_{n+1})x_{\max} + C_2(z_k, \tau_{n+1})|\} \leq M_x$$

where

$$M_x = \max_{z,\tau}|C_1(z,\tau)x_{\max} + C_2(z,\tau)|,$$

a quantity that is independent of Δx, Δz and $\Delta\tau$. Similarly, suppose $\max\limits_{j,k}\{|V_{j,k}^{n+1}|\}$ is attained at $k = k_{\max}$, then

$$||V^{n+1}||_\infty \leq M_z = \max_{x,\tau}|W(x,\tau)|,$$

which is also independent of Δx, Δz and $\Delta\tau$. Lastly, suppose $\max\limits_{j,k}\{|V_{j,k}^{n+1}|\}$ is attained at an interior node $(j, k, n+1)$, we have

$$\begin{aligned}
||V^{n+1}||_\infty &= V_{j,k}^n + \triangle\tau \sup_{A_{j,k}^{n+1}\in\{0,1\}, u_{j,k}^{n+1}\in Q}\{f_{j,k}^{n+1}(A_{j,k}^{n+1}, u_{j,k}^{n+1})\} \\
&= V_{j,k}^n + \triangle\tau\ f_{j,k}^{n+1}(A_{j,k}^{n+1^*}, u_{j,k}^{n+1^*}),
\end{aligned}$$

where

$$f_{j,k}^{n+1} = a_j V_{j+1,k}^{n+1} - (a_j + b_j + c_{j,k}^{n+1})V_{j,k}^{n+1} + b_j V_{j-1,k}^{n+1} + d_{j,k}^{n+1} V_{j,k+1}^{n+1} + e_{j,k}^{n+1},$$

and $A_{j,k}^{n+1^*}$ and $u_{j,k}^{n+1^*}$ denote the optimal control variables that maximize $f_{j,k}^{n+1}$. Assuming that all quantities in $f_{j,k}^{n+1}$ are now evaluated at $A_{j,k}^{n+1^*}$ and $u_{j,k}^{n+1^*}$, and taking $V_{j,k}^{n+1}$ as $||V^{n+1}||_\infty$, we rearrange the terms in the above equation as follows:

$$\begin{aligned}
&[1 + (a_j + b_j + c_{j,k}^{n+1})\triangle\tau]||V^{n+1}||_\infty \\
= \ &\triangle\tau(a_j V_{j+1,k}^{n+1} + b_j V_{j-1,k}^{n+1} + d_{j,k}^{n+1} V_{j,k+1}^{n+1} + e_{j,k}^{n+1}) + V_{j,k}^n.
\end{aligned}$$

For notational simplicity, we have suppressed the representation of the dependence on $A_{j,k}^{n+1^*}$ and $u_{j,k}^{n+1^*}$ of the various terms in $f_{j,k}^{n+1}$. Let $M_e = \max\limits_{j,k,n}|e_{j,k}^n|$. We then take the absolute value of the quantities on both sides of the above equation and apply the triangle inequality. This gives

$$\begin{aligned} & |1+(a_j+b_j+c_{j,k}^{n+1})\triangle\tau||V^{n+1}||_\infty \\ \le\ & \triangle\tau(a_j|V_{j+1,k}^{n+1}|+b_j|V_{j-1,k}^{n+1}|+d_{j,k}^{n+1}|V_{j,k+1}^{n+1}|)+|V_{j,k}^{n}|+|e_{j,k}^{n+1}|\triangle\tau \\ \le\ & \triangle\tau(a_j||V^{n+1}||_\infty+b_j||V^{n+1}||_\infty+d_{j,k}^{n+1}||V^{n+1}||_\infty)+||V^n||_\infty+M_e\triangle\tau. \end{aligned}$$

Furthermore, by observing the non-negativity of a_j,b_j,$c_{j,k}^{n+1}$,$d_{j,k}^{n+1}$, we have

$$\begin{aligned} & 1+(a_j+b_j+c_{j,k}^{n+1})\triangle\tau||V^{n+1}||_\infty \\ \le\ & \triangle\tau(a_j+b_j+d_{j,k}^{n+1})||V^{n+1}||_\infty+||V^n||_\infty+M_e\triangle\tau. \end{aligned}$$

Rearranging the terms, we obtain

$$[(1+(c_{j,k}^{n+1}-d_{j,k}^{n+1})\triangle\tau]||V^{n+1}||_\infty \le ||V^n||_\infty+M_e\triangle\tau.$$

Thanks to $c_{j,k}^{n+1}\ge d_{j,k}^{n+1}$, we can simplify the inequality as follows:

$$||V^{n+1}||_\infty \le [1+(c_{j,k}^{n+1}-d_{j,k}^{n+1})\triangle\tau]||V^{n+1}||_\infty \le ||V^n||_\infty+M_e\triangle\tau.$$

Combining the various possibilities that $\max\limits_{j,k}\{V_{j,k}^{n+1}\}$ can be attained at an interior node or a boundary node, we deduce that

$$||V^{n+1}||_\infty \le \max\{||V^n||_\infty+M_e\triangle\tau, M_x, M_z\}.$$

Applying the above inequality inductively until $n=0$ and observing $||V^0||_\infty=0$, we finally obtain

$$||V^{n+1}||_\infty \le \max\{M_z, M_x\}+M_eT.$$

Hence, the l_∞-stability property of the numerical scheme is established.

C. Proof of Proposition 2

First, we would like to show that B_k is a M-matrix for any $u\in Q$ and $A\in\{0,1\}$. Note that a_j, b_j, $c_{j,k}^{n+1}$ are all non-negative for any control u and A, so B_k has positive diagonal entries and non-positive off-diagonal entries. On the other hand, we have

$$|1+(a_j+b_j+c_{j,k}^{n+1})\Delta\tau| > |-a_j\Delta\tau|+|-b_j\Delta\tau|,$$

implying that B_k is diagonally dominant. Hence, B_k is a M-matrix. Besides, the property on boundedness of the matrix norm of B_k^{-1}: $||B_k^{-1}||_\infty\le 1$, is also useful in the proof of the proposition. This matrix norm property can be established via some technical calculations, the details can be found in Leung (2011).

The formal proof starts with the proof of convergence of the iterates $(\mathbf{V}_k^{n+1})^i$. Recall that the i^{th} iterate $(\mathbf{V}_k^{n+1})^i$ is obtained by solving the algebraic system of equations:

$$(B_k)^{i-1}(\mathbf{V}_k^{n+1})^i = (\mathbf{h}_k^{n+1})^{i-1}.$$

The boundedness of $(\mathbf{V}_k^{n+1})^i$ is easily seen since

$$||(\mathbf{V}_k^{n+1})^i||_\infty = ||[(B_k)^{i-1}]^{-1}(\mathbf{h}_k^{n+1})^{i-1}||_\infty \leq ||(\mathbf{h}_k^{n+1})^{i-1}||_\infty \leq \sup_{u,A}\{||\mathbf{h}_k^{n+1}||_\infty\} < \infty,$$

by virtue of the matrix norm property: $||B_k^{-1}||_\infty \leq 1$. We then show the monotonicity property of the iterates, where

$$(\mathbf{V}_k^{n+1})^{i+1} \geq (\mathbf{V}_k^{n+1})^i.$$

We consider

$$\begin{aligned}
&(B_k)^{i-1}\left[(\mathbf{V}_k^{n+1})^{i+1} - (\mathbf{V}_k^{n+1})^i\right] \\
&= (B_k)^{i-1}(\mathbf{V}_k^{n+1})^{i+1} - (\mathbf{h}_k^{n+1})^{i-1} - (B_k)^{i-1}(\mathbf{V}_k^{n+1})^i + (\mathbf{h}_k^{n+1})^{i-1} \\
&= -\left[-(B_k)^{i-1}(\mathbf{V}_k^{n+1})^{i+1} + (\mathbf{h}_k^{n+1})^{i-1}\right] \\
&\geq -\sup_{u,A}\{-B_k(\mathbf{V}_k^{n+1})^{i+1} + \mathbf{h}_k^{n+1}\} \\
&= -\left[-(B_k)^i(\mathbf{V}_k^{n+1})^{i+1} + (\mathbf{h}_k^{n+1})^i\right] = 0.
\end{aligned}$$

Since $(B_k)^{i-1}$ is a M-matrix, so $[(B_k)^{i-1}]^{-1} \geq 0$. We can then conclude that

$$(\mathbf{V}_k^{n+1})^{i+1} \geq (\mathbf{V}_k^{n+1})^i.$$

Since a bounded monotonic sequence converges, so does the iterates $(\mathbf{V}_k^{n+1})^i$. The convergence of the iterates is independent of the value of the initial iterate. We write the limit of the iterates as

$$\mathbf{V}_k^{n+1^*} = \lim_{i\to\infty}(\mathbf{V}_k^{n+1})^i.$$

Finally, we would like to show that $\mathbf{V}_k^{n+1^*}$ is a solution to eq. (4.7). Since B_k is bounded for all possible controls $(u, A) \in Q \times \{0, 1\}$, it is seen that

$$\begin{aligned}
&\lim_{i\to\infty}(B_k)^i[(\mathbf{V}_k^{n+1})^{i+1} - (\mathbf{V}_k^{n+1})^i] \\
&= \lim_{i\to\infty}[-(B_k)^i(\mathbf{V}_k^{n+1})^i + (\mathbf{h}_k^{n+1})^i] \\
&= \lim_{i\to\infty}\sup_{u,A}\{-B_k(\mathbf{V}_k^{n+1})^i + \mathbf{h}_k^{n+1}\} \\
&= \sup_{u,A}\{-B_k\mathbf{V}_k^{n+1^*} + \mathbf{h}_k^{n+1}\} = \mathbf{0}.
\end{aligned}$$

Hence, the limit of the iterates satisfies eq. (4.7).

To complete the proof, it is also necessary to show that convergent sequences of iterates with different initial starting iterates do converge to the same limit. It suffices to show that the solution of eq. (4.7) is unique. Suppose there are two solution tuples $(\hat{\mathbf{V}}_k^{n+1}, \hat{u}, \hat{A})$ and $(\tilde{\mathbf{V}}_k^{n+1}, \tilde{u}, \tilde{A})$ such that

$$-B_k|_{(\hat{u},\hat{A})}\hat{\mathbf{V}}_k^{n+1} + \mathbf{h}_k^{n+1}|_{\hat{u},\hat{A}} = \mathbf{0} \text{ where } (\hat{u}, \hat{A}) = \underset{u\in Q, A\in\{0,1\}}{\operatorname{argmax}}\{-B_k\hat{\mathbf{V}}_k^{n+1} + \mathbf{h}_k^{n+1}\}$$

and

$$-B_k|_{(\tilde{u},\tilde{A})}\tilde{\mathbf{V}}_k^{n+1} + \mathbf{h}_k^{n+1}|_{\tilde{u},\tilde{A}} = \mathbf{0} \text{ where } (\tilde{u}, \tilde{A}) = \underset{u\in Q, A\in\{0,1\}}{\operatorname{argmax}}\{-B_k\tilde{\mathbf{V}}_k^{n+1} + \mathbf{h}_k^{n+1}\}.$$

Observe that

$$B_k|_{(\hat{u},\hat{A})}(\hat{\mathbf{V}}_k^{n+1} - \tilde{\mathbf{V}}_k^{n+1}) = -B_k|_{(\hat{u},\hat{A})}\tilde{\mathbf{V}}_k^{n+1} + \mathbf{h}_k^{n+1}|_{(\hat{u},\hat{A})},$$

so we obtain

$$\hat{\mathbf{V}}_k^{n+1} - \tilde{\mathbf{V}}_k^{n+1} = (B_k|_{(\hat{u},\hat{A})})^{-1}(-B_k|_{(\hat{u},\hat{A})}\tilde{\mathbf{V}}_k^{n+1} + \mathbf{h}_k^{n+1})|_{(\hat{u},\hat{A})}.$$

Since $(\hat{u}, \hat{A})$ may not be the pair of optimal control variables for $\tilde{\mathbf{V}}_k^{n+1}$, we deduce that

$$-B_k|_{(\hat{u},\hat{A})}\tilde{\mathbf{V}}_k^{n+1} + \mathbf{h}_k^{n+1}|_{(\hat{u},\hat{A})} \leq \sup_{u,A}\{-B_k\tilde{\mathbf{V}}_k^{n+1} + \mathbf{h}_k^{n+1}\} = \mathbf{0}.$$

Observing that $B_k^{-1} \geq 0$ (by virtue of the M-matrix property of B_k) and together with the above inequality, we obtain

$$\hat{\mathbf{V}}_k^{n+1} - \tilde{\mathbf{V}}_k^{n+1} \leq \mathbf{0}.$$

The above arguments remain valid when we interchange the role of $\hat{\mathbf{V}}_k^{n+1}$ and $\tilde{\mathbf{V}}_k^{n+1}$; thus, we also obtain

$$\tilde{\mathbf{V}}_k^{n+1} - \hat{\mathbf{V}}_k^{n+1} \geq \mathbf{0}.$$

Putting these results together, we obtain $\hat{\mathbf{V}}_k^{n+1} = \tilde{\mathbf{V}}_k^{n+1}$. Therefore, we conclude that the solution to eq. (4.7) is unique.

number of time steps	numerical value $V(5,5,1)$	difference in numerical solutions	ratio of difference
32	2.5688		
64	2.1951	0.3737	
128	2.0060	0.1891	1.9762
256	1.9107	0.0951	1.9843
512	1.8628	0.0479	1.9895
1024	1.8389	0.0239	2.0042

Table 1: Results of numerical tests that were performed to examine the convergence rate of the numerical algorithm. Linear rate of convergence is confirmed since the ratio of difference in numerical solutions is close to 2 when the number of time steps is doubled.

$x_{\max}$	$z_{\max}$	numerical value $V(50,50,1)$	difference in numerical solutions
100	100	48.3371	
200	200	48.3351	0.0020
400	400	48.3338	0.0013
800	800	48.3329	0.0009

Table 2: Results of numerical tests that were performed to examine the impact of the choices of the upper boundaries, $x_{\max}$ and $z_{\max}$, on accuracy of the numerical solutions. When $x_{\max}$ and $z_{\max}$ are chosen to be about 200, the numerical errors caused by finite truncation of the computational domain is insignificant.

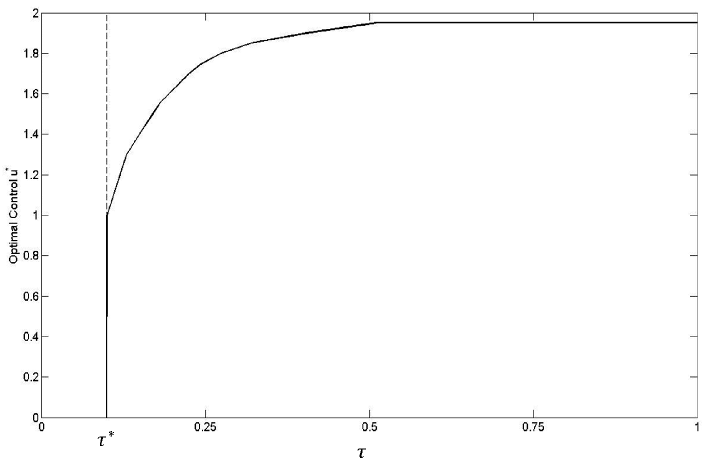

Figure 1: Plot of optimal control u^* against time to expiry τ. When $\tau < \tau^*$, the firm optimally chooses to abandon its R&D at any level of the stochastic state variable. When $\tau > \tau^*$, the optimal control u^* increases with increasing τ and tends to some asymptotic level when τ is sufficiently large.

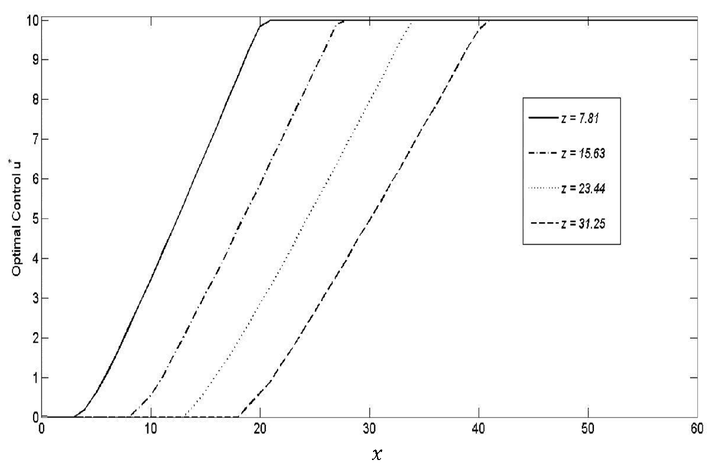

Figure 2: Plot of optimal control u^* against current value of the state variable x with varying values of knowledge stock z. One observes that u^* increases almost linearly with respect to x until up to the level $\hat{u}^* = \sup Q = 10$. On the other hand, the firm optimally increases the R&D effort when the knowledge stock z is lower.

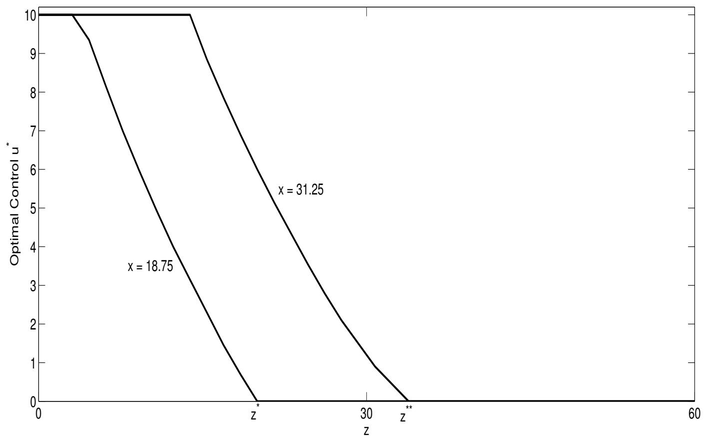

Figure 3a: Plot of the optimal control u^* against knowledge stock z with the cost function $c(u) = 10 + \frac{u^2}{2}$. The optimal control is seen to decrease almost linearly as the knowledge stock increases. When z is greater than some threshold value ($z^* = 20.31$ for $x = 18.75$ and $z^{**} = 34.38$ for $x = 31.25$), the optimal control becomes identically zero indicating that the firm chooses optimally not to put any extra R&D effort (though the R&D project is not abandoned).

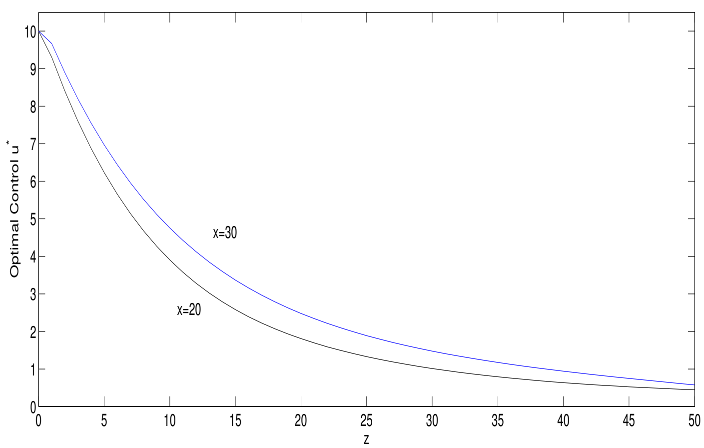

Figure 3b: Plot of the optimal control u^* against knowledge stock z with the cost function $c(u) = 10 + \frac{u^{1.5}}{2}$. One observes that the optimal control decreases and tends to some asymptotic level with increasing level of knowledge stock.

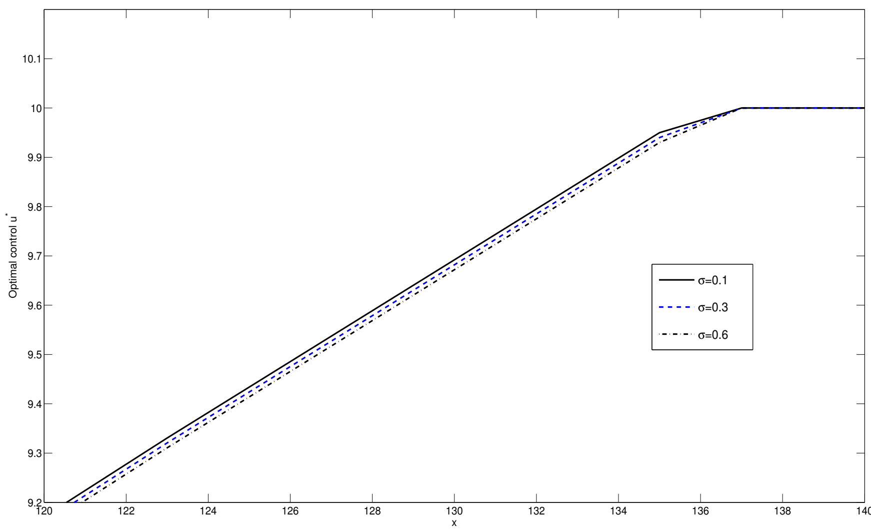

Figure 4: Plot of optimal control u^* against x with varying values of volatility σ. The optimal control is seen to decrease with increasing volatility, indicating that the firm tends to slow down its R&D effort when subject to a higher level of market uncertainty.

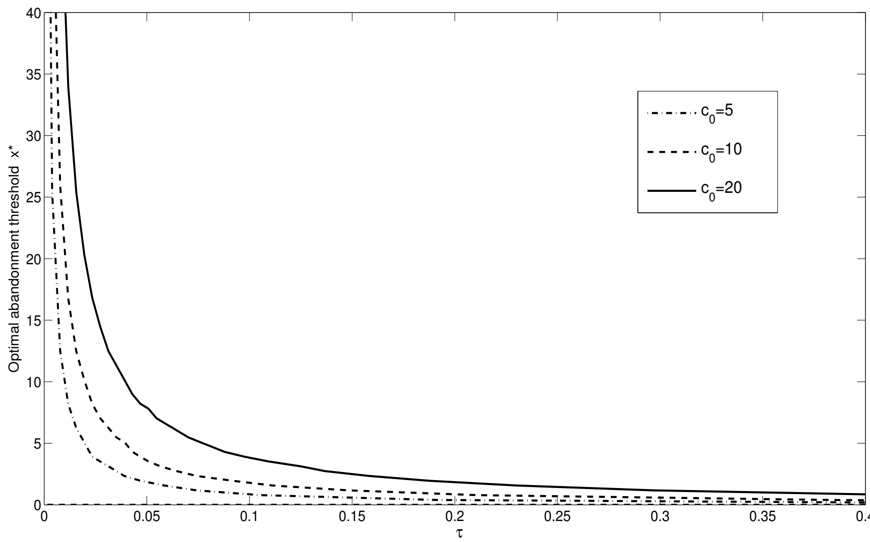

Figure 5: Plot of the optimal abandonment threshold x^* against τ with varying values of the fixed cost c_0 in the cost function $c(u) = c_0 + \dfrac{u^2}{2}$. At a given value of τ, the firm optimally chooses to continue the R&D project when the stochastic state variable x_t assumes a value higher than x^* and abandon the project if otherwise. In the (τ, x)-plane, the region above (below) the optimal abandonment boundary represents the continuation (abandonment) region. The abandonment threshold is seen to be an increasing function of the fixed cost c_0.