First HKUST Undergraduate Math Competition – Junior Level

April 27, 2013

Directions: This is a three hour test. No calculators are allowed. **For every problem, provide complete details of your solution**.

Problem 1. Write down the Taylor series of e^x about 0 and use it to compute

$$\lim_{n\to\infty} n\sin(2\pi e n!).$$

Problem 2. Let D_n denote the determinant of the $(n-1)\times(n-1)$ matrix

$$\begin{pmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 5 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 6 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & n+1 \end{pmatrix}.$$

Determine whether the set $\left\{\dfrac{D_n}{n!} : n = 2, 3, 4, \ldots\right\}$ is bounded or not.

Problem 3. Let $f, g : [0,1] \to \mathbb{R}$ be functions such that

$$f(0) < g(0) < g(1) < f(1).$$

If f is continuous and g is increasing, then prove that there exists $w \in [0,1]$ such that $f(w) = g(w)$.

Problem 4. Let

$$A = \int_0^1 x^x\, dx \quad \text{and} \quad B = \int_0^1 \int_0^1 (xy)^{xy}\, dy\, dx.$$

Determine which of the following is true: $A < B$, $A = B$ or $A > B$.

Problem 5. Determine whether or not there exists an infinitely differentiable function $f(x)$ such that for every real x and for every positive integer n,

$$1 \le f^{(n)}(x) + f^{(n+1)}(x) + \cdots + f^{(3n)}(x) \le 3,$$

where $f^{(n)}$ denotes the n-th derivative of f.

Problem 6. Let H be an inner product space. Let n be a positive integer less than the dimension of H. If V and E are dimension n and $n-1$ subspaces of H respectively, prove that there exists a nonzero $v \in V$ orthogonal to every $x \in E$.

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