Solution of the First HKUST Undergraduate Math Competition – Junior Level

1. For all $x \in \mathbb{R}$, $e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}$. So $e = \sum_{j=0}^{\infty} \frac{1}{j!}$. For a positive integer n, $I_n = \sum_{j=0}^{n} \frac{n!}{j!} \in \mathbb{Z}$ and let $a_n = \sum_{j=n+1}^{\infty} \frac{n!}{j!}$. Then $n\sin(2\pi en!) = n\sin(2\pi I_n + 2\pi a_n) = n\sin(2\pi a_n)$. Note

$$\frac{1}{n+1} \le a_n = \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots \le \sum_{k=1}^{\infty} \frac{1}{(n+1)^k} = \frac{1}{n}.$$

By sandwich theorem, $a_n \to 0$ and $na_n \to 1$ as $n \to \infty$. Using $\sin\theta \sim \theta$ as $\theta \to 0$, we get

$$\lim_{n\to\infty} n\sin(2\pi en!) = \lim_{n\to\infty} n\sin(2\pi a_n) = \lim_{n\to\infty} 2\pi n a_n = 2\pi.$$

2. Subtracting the first row from each of the other rows, we get

$$D_n = \det\begin{pmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ -2 & 3 & 0 & 0 & \cdots & 0 \\ -2 & 0 & 4 & 0 & \cdots & 0 \\ -2 & 0 & 0 & 5 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -2 & 0 & 0 & 0 & \cdots & n \end{pmatrix}.$$

For $2 \le i \le n-1$, adding $2/(i+1)$ times the i-th column to the first column, we get

$$D_n = \det\begin{pmatrix} 3+\frac{2}{3}+\frac{2}{4}+\cdots+\frac{2}{n} & 1 & 1 & 1 & \cdots & 1 \\ 0 & 3 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 4 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 5 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n \end{pmatrix} = n!\Big(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\Big).$$

Now $\frac{D_n}{n!} = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$ diverges to $+\infty$ by the p-test, hence it is unbounded.

3. (*Solution 1*) Let $S = \{x \in [0,1] : f(x) \le g(x)\}$. Now $0 \in S$ and S is bounded above by 1. Hence $w = \sup S$ exists. Since $f(0) < g(0) < g(1) < f(1)$ and f is continuous, we get $0 < w < 1$. Since g is monotone, $g(w-) = \lim_{x\to w-} g(x)$ and $g(w+) = \lim_{x\to w+} g(x)$ exist. Being supremum, there exists a sequence $x_n \in S$ converging to w. Since $w > 0$, we have $f(w) = \lim_{n\to\infty} f(x_n) \le \lim_{n\to\infty} g(x_n) = g(w-)$. Next, take a sequence $y_n \in (w,1]$ converging to w. Now $y_n \notin S$ implies $f(w) = \lim_{n\to\infty} f(y_n) \ge \lim_{n\to\infty} g(y_n) = g(w+)$. Finally, $g(w-) \ge f(w) \ge g(w+)$ implies $f(w) = g(w)$.

(*Solution 2 due to Li Zhiming and Tai Ming Fung Philip*) Assume for all $w \in [0,1]$, $f(w) \ne g(w)$. We will construct a sequence of nested intervals $[a_n, b_n]$ such that $f(a_n) < g(a_n) < g(b_n) < f(b_n)$ by math induction.

Let $a_1 = 0$ and $b_1 = 1$. We have $f(a_1) < g(a_1) < g(b_1) < f(b_1)$. Suppose $f(a_k) < g(a_k) < g(b_k) < f(b_k)$. Let $m = (a_k+b_k)/2$. Since $f(m) \ne g(m)$, either $f(m) < g(m)$ or $f(m) > g(m)$. In the former case, let $[a_{k+1}, b_{k+1}] = [m, b_k]$. In the latter case, let $[a_{k+1}, b_{k+1}] = [a_k, m]$. Since $|a_k - b_k| = 1/2^{k-1} \to 0$, by the nested interval theorem, a_k and b_k converge to some $w \in [0,1]$. We are given that $w \ne 0$ or 1. Since f is continuous and g is increasing, taking limit as $k \to \infty$, we get $f(w) \le g(w-) \le g(w+) \le f(w)$. Since $g(w-) \le g(w) \le g(w+)$, we get $f(w) = g(w)$, a contradiction.

4. Fixing x and substituting $u = xy$ in B, we get

$$B = \int_0^1 \int_0^1 (xy)^{xy} \, dy \, dx = \int_0^1 \int_0^x \frac{u^u}{x} \, du \, dx = \int_0^1 \int_u^1 \frac{u^u}{x} \, dx \, du = -\int_0^1 u^u (\ln u) \, du.$$

Then $A - B = \displaystyle\int_0^1 u^u(1 + \ln u) \, du = u^u \Big|_{0+}^{1} = 0$. Therefore, $A = B$.

5. <u>*Lemma*</u> If there exist $M \in \mathbb{R}$ and $\varepsilon > 0$ such that $f^{(n)}(x) > \varepsilon$ for all $x \geq M$, then f is unbounded above.

<u>*Proof.*</u> Let $c_{n-1} = f^{(n-1)}(M)$. Since $f^{(n-1)}(x) > c_{n-1} + \varepsilon x$ for all $x > M$ by the mean value theorem, $f^{(n-1)}$ is unbounded above. Then there exists $M' \in \mathbb{R}$ such that $f^{(n-1)}(x) > \varepsilon$ for all $x \geq M'$. Repeating this $n - 1$ times more, we get f is unbounded above. This proved the lemma.

Now assume such a function $f(x)$ exists. Consider

$$A(x) = f^{(1)}(x) + f^{(2)}(x) + f^{(3)}(x), \quad B(x) = f^{(4)}(x) + f^{(5)}(x) + \cdots + f^{(12)}(x),$$

$$C(x) = f^{(13)}(x) + f^{(14)}(x) + \cdots + f^{(39)}(x), \quad D(x) = f^{(19)}(x) + f^{(20)}(x) + \cdots + f^{(57)}(x).$$

Let $E(x) = A(x) + B(x) + C(x)$. We are given that $1 \leq A(x), B(x), C(x) \leq 3$ (hence $3 \leq E(x) \leq 9$) and $D(x) \geq 1$ for all real x. Now

$$D(x) = A^{(18)}(x) + B^{(18)}(x) + C^{(18)}(x) = E^{(18)}(x).$$

By the lemma, E is unbounded above, a contradiction to $E(x) \leq 9$ for all real x.

6. (*Solution 1 due to Li Siwei and Li Zhiming*) Let $\{v_1, v_2, \ldots, v_n\}$ and $\{e_1, e_2, \ldots, e_{n-1}\}$ be orthonormal bases of V and E respectively. We will show there exists $(c_1, c_2, \ldots, c_n) \in \mathbb{K}^n$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) which is not $(0, 0, \ldots, 0)$ and $v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$ is orthogonal to $e_1, e_2, \cdots, e_{n-1}$. Then v is orthogonal to E.

The conditions $v \neq 0$ and $\langle v, e_i \rangle = c_1 \langle v_1, e_i \rangle + c_2 \langle v_2, e_i \rangle + \cdots + c_n \langle v_n, e_i \rangle = 0$ for $i = 1, 2, \ldots, n-1$ is equivalent to the linear transformation $T : \mathbb{K}^n \to \mathbb{K}^{n-1}$ defined by

$$T \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} \langle v_1, e_1 \rangle & \langle v_2, e_1 \rangle & \cdots & \langle v_n, e_1 \rangle \\ \langle v_1, e_2 \rangle & \langle v_2, e_2 \rangle & \cdots & \langle v_n, e_2 \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle v_1, e_{n-1} \rangle & \langle v_2, e_{n-1} \rangle & \cdots & \langle v_n, e_{n-1} \rangle \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

has a null space not equal to $\{0\}$. This is the case because the range of T cannot be n-dimensional in $\mathbb{K}^{n-1}$. So such a $v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$ exists.

(*Solution 2*) Let $W = V \cap E$. Let V' be the orthogonal complement of W in V. Similarly, let E' be the orthogonal complement of W in E. Since $V' \cap E' \subseteq V \cap E \cap W^{\perp} = \{0\}$, so $V' \cap E' = \{0\}$.

Also, $V' + E' \perp W$ and $\dim V' = \dim E' + 1$. So $\dim(V' + E') = \dim V' + \dim E' = 2(\dim V') - 1$, which implies the orthogonal complement M of E' in $V' + E'$ has dimension equal $\dim V'$. Since $\dim V' + \dim M > \dim(V' + E')$, there exists a nonzero $v \in V' \cap M$. Then $v \in V' \subseteq V$ and $v \in M \subseteq V' + E'$ implies $v \perp \operatorname{span}(E' \cup W) = E$.