First HKUST Undergraduate Math Competition – Senior Level

April 27, 2013

Directions: This is a three hour test. No calculators are allowed. **For every problem, provide complete details of your solution**.

Problem 1. For $x \geq 1$, let $f(x)$ be the unique value $c \geq 1$ for which $c^c = x$. Determine the value of $\int_0^e f(e^t)\,dt$ with proof.

Problem 2. Let A and B be 3×2 and 2×3 matrices respectively. If

$$AB = \begin{pmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{pmatrix},$$

then determine the rank of AB and determine all possible answers for BA.

Problem 3. Prove that there exists a continuous function f on $[0,1]$ satisfying the integral equation

$$f(x) + \int_0^1 \frac{f(y)}{2 + (xy)^\pi}\,dy = \int_0^x \int_0^y \frac{1}{2 + t^{2\pi}}\,dt\,dy.$$

Problem 4. Suppose p is an odd prime number. Prove that

$$\sum_{j=0}^{p} \binom{p}{j}\binom{p+j}{j} \equiv 2^p + 1 (\text{mod } p^2),$$

where the binomial coefficient $\binom{m}{n} = \frac{m!}{n!(m-n)!}$.

Problem 5. Let $h : \mathbb{C} \to \mathbb{C}$ be defined by $h(z) = \sum_{n=1}^{5} \frac{1}{n^z}$. Prove that for all $t \in \mathbb{R}$, $h(1 + it) \neq 0$.

Problem 6. Let F and K be subfields of the complex number $\mathbb{C}$ such that K is a finite algebraic extension of F. Let $\zeta \in \mathbb{C}$. If ζ is not a root of any nonconstant polynomial with coefficients in K, then prove that $[K(\zeta) : F(\zeta)] = [K : F]$.

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