Solution of the First HKUST Undergraduate Math Competition – Senior Level

1. Note $y = f(e^x) \Leftrightarrow y^y = e^x \Leftrightarrow y\ln y = x$. Then $dx = (\ln y + 1)dy$. So

$$\int_{x=0}^{x=e} f(e^x)\,dx = \int_{y=1}^{y=e} y(\ln y + 1)\,dy = \int_{y=1}^{y=e} (y\ln y + y)\,dy = \Big(\frac{y^2}{2}\ln y - \frac{y^2}{4}\Big)\Big|_{y=1}^{y=e} + \frac{y^2}{2}\Big|_{y=1}^{y=e} = \frac{3e^2-1}{4}.$$

2. (From linear algebra, the inequalities $\mathrm{rank}(XY) \le \mathrm{rank}(X)$ and $\mathrm{rank}(XYZ) \le \mathrm{rank}(Y)$ are useful.)

(*Solution 1*) Since the first two rows of AB are linearly independent, so $2 \le \mathrm{rank}(AB) \le \mathrm{rank}(A) \le 2$. Hence $\mathrm{rank}(AB) = 2$.

Next to get BA, we note $\mathrm{rank}(BA) \ge \mathrm{rank}(A(BA)B) = \mathrm{rank}((AB)^2)$. Now

$$(AB)^2 = \begin{pmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{pmatrix}^2 = \begin{pmatrix} 72 & 18 & -18 \\ 18 & 45 & 36 \\ -18 & 36 & 45 \end{pmatrix} = 9AB.$$

Since BA is a 2×2 matrix and $\mathrm{rank}(9AB) = 2$, so $\mathrm{rank}(BA) = 2$. Hence BA is invertible. Finally $(BA)^3 = B(ABAB)A = B(AB)^2A = 9BABA = 9(BA)^2$. Cancelling $(BA)^2$, we get $BA = 9I$.

(*Solution 2 due to Lau Lap Ming*) Since the first two rows of AB are linearly independent, so $2 \le \mathrm{rank}(AB) \le \mathrm{rank}(A) \le 2$. Then $\mathrm{rank}(AB) = \mathrm{rank}(A) = 2$.

Next $\det(AB - tI) = -t(t-9)^2$, so the eigenvalues of AB are 0 and 9. If λ is an eigenvalue of BA with eigenvector $v \ne 0$, then $AB(Av) = A(BAv) = A(\lambda v) = \lambda Av$. Since A is 3×2 and of rank 2, A is injective. Hence, $Av \ne 0$ and λ is an eigenvalue of AB. This implies the only possible eigenvalues of BA are 0 or 9. From row operations on the matrix of AB, we see the eigenspace of AB for the eigenvalue 0 is spanned by $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ and the eigenspace V of AB for the eigenvalue 9 is spanned by $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$.

Restricting to V, $AB : V \to \mathbb{R}^2 \to V$ is bijective since $AB = 9I$ on V. So the linear maps $B : V \to \mathbb{R}^2$ and $A : \mathbb{R}^2 \to V$ must be bijective. In particular, $B(V) = \mathbb{R}^2$. Then for every $x \in \mathbb{R}^2$, there exists $v \in V$ such that $Bv = x$. So we have $BAx = B(ABv) = B(9v) = 9Bv = 9x$. Therefore, $BA = 9I$.

3. (This is an existence problem with solution to be found among continuous functions on $[0,1]$. In a course on metric spaces, a key theorem on existence problem is the contractive mapping theorem.) Define $T : C[0,1] \to C[0,1]$ by $(Tf)(x) = \int_0^x \int_0^y \frac{1}{2+t^{2\pi}}\,dt\,dy - \int_0^1 \frac{f(y)}{2+(xy)^\pi}\,dy$. Since $C[0,1]$ is a complete metric space with $d(f,g) = \|f-g\|_\infty$ and

$$|Tf(x) - Tg(x)| = \Big|\int_0^1 \frac{f(y)}{2+(xy)^\pi}\,dy - \int_0^1 \frac{g(y)}{2+(xy)^\pi}\,dy\Big| \le \frac{1}{2}\|f-g\|_\infty.$$

By the contractive mapping theorem, there exists $f \in C[0,1]$ such that $Tf = f$ and we are done.

4. (For a binomial coefficient problem, we should think about the binomial expansion of $(1+x)^n$.) Observe the sum $I = \sum_{j=0}^{p} \binom{p}{j}\binom{p+j}{j} = \sum_{j=0}^{p} \binom{p}{j}\binom{p+j}{p}$ is the coefficient of x^p in the expansion of

$$\sum_{j=0}^{p} \binom{p}{j}(1+x)^{p+j} = \Big(\sum_{j=0}^{p}\binom{p}{j}(1+x)^j\Big)(1+x)^p = (2+x)^p(1+x)^p.$$

Expanding $(2+x)^p(1+x)^p$, we get $I=\sum_{k=0}^{p}\binom{p}{k}\binom{p}{p-k}2^k$. Since p divides $\binom{p}{k}$ for $0<k<p$, we have

$$I\equiv\binom{p}{0}\binom{p}{p}2^0+\binom{p}{p}\binom{p}{0}2^p=2^p+1(\mathrm{mod}\ p^2).$$

5. Note $\mathrm{Re}\,\frac{1}{n^{1+it}}=\mathrm{Re}\,e^{-\ln n-it\ln n}=\frac{\cos(t\ln n)}{n}$ and $x+x^2=\Big(x+\frac{1}{2}\Big)^2-\frac{1}{4}\geq-\frac{1}{4}$. Let $w=t\ln 2$. Then

$$\begin{aligned}\mathrm{Re}\,h(1+it)&=1+\frac{\cos w}{2}+\frac{\cos(t\ln 3)}{3}+\frac{\cos 2w}{4}+\frac{\cos(t\ln 5)}{5}\\&\geq 1+\frac{\cos w}{2}-\frac{1}{3}+\frac{2\cos^2 w-1}{4}-\frac{1}{5}\\&=1-\frac{1}{3}-\frac{1}{4}-\frac{1}{5}+\frac{\cos w+\cos^2 w}{2}\geq\frac{13}{60}-\frac{1}{8}>0.\end{aligned}$$

6. There are two solutions to this algebra problems. The first one is by linear algebra techniques. The second one is by field theory techniques.

(*Solution 1 due to Lau Lap Ming*) Recall $n=[K:F]$ is the dimension of K as a vector space over F. Let $v_1,v_2,\cdots,v_n$ be a basis of K over F. We claim it is also a basis of $K(\zeta)$ over $F(\zeta)$. Then $[K(\zeta):F(\zeta)]=n=[K:F]$.

Suppose there are $c_1,c_2,\ldots,c_n\in F(\zeta)$ such that $c_1v_1+c_2v_2+\cdots+c_nv_n=0$. Now each c_i is of the form $p_i(\zeta)/q_i(\zeta)$, where p_i and q_i are polynomials with coefficients in F and $q_i(\zeta)\neq 0$. By taking common denominators, we may assume all $q_i(\zeta)$ are equal, say to $q(\zeta)$. Let $p_i(\zeta)=a_{i0}+a_{i1}\zeta+\cdots+a_{im}\zeta^m$, where $a_{ij}\in F$ and m is the maximum degree of $p_1,p_2,\ldots,p_n$. Then

$$0=q(\zeta)\sum_{i=1}^{n}c_iv_i=\sum_{i=1}^{n}\Big(\sum_{j=0}^{m}a_{ij}\zeta^j\Big)v_i=\sum_{j=0}^{m}\Big(\sum_{i=1}^{n}a_{ij}v_i\Big)\zeta^j.$$

Since ζ is not a root of nonconstant polynomials over K, we have all $\sum_{i=1}^{n}a_{ij}v_i=0$ for $j=0,1,\ldots,m$. By the linear independence of v_i in K, we get all $a_{ij}=0$, which imply all $c_i=0$. So $v_1,v_2,\cdots,v_n$ are linearly independent in $K(\zeta)$.

Next, if $b_0+b_1\zeta+\cdots+b_k\zeta^k\in K(\zeta)$ with $b_i\in K=\mathrm{span}\{v_1,v_2,\ldots,v_n\}$, then write each $b_i=\beta_{i1}v_1+\beta_{i2}v_2+\cdots+\beta_{in}v_n$, where $\beta_{ij}\in F$. Then $b_0+b_1\zeta+\cdots+b_k\zeta^k=\sum_{i=0}^{k}\sum_{j=1}^{n}\beta_{ij}v_j\zeta^i=\sum_{j=1}^{n}\Big(\sum_{i=0}^{k}\beta_{ij}\zeta^i\Big)v_j$ is in the span of $v_1,v_2,\ldots,v_n$ over $F(\zeta)$. The claim is proved.

(*Solution 2*) Since K is obtained by adjoining finitely many algebraic elements to F, inductively, we may suppose $K=F(\alpha)$ for some algebraic $\alpha\in\mathbb{C}$ over F. Let

$$f(x)=x^n+a_{n-1}x^{n-1}+\cdots+a_0\in F[x]$$

be the minimal polynomial of α over F. Clearly, $f(x)\in F(\zeta)[x]$ annihilates α. It is enough to show $f(x)\in F(\zeta)[x]$ is also the minimal polynomial of α over $F(\zeta)$ because then

$$[K(\zeta):F(\zeta)]=[F(\zeta)(\alpha):F(\zeta)]=n=[K:F].$$

Suppose $g(x)\overset{(*)}{=}x^m+g_{m-1}(\zeta)x^{m-1}+\cdots+g_0(\zeta)\in F(\zeta)[x]$ is another polynomial such that $g(\alpha)=0$, where $g_i(\zeta)\in F(\zeta)$. Since $F\subseteq\mathbb{C}$, F is an infinite field, one can find $u\in F$ such that the product $p(\zeta)$ of the denominators of $g_i(\zeta)\in F(\zeta)$ do not annihilate u when ζ is replaced by u. Since $p(\zeta)(\alpha^m+g_{m-1}(\zeta)\alpha^{m-1}+\cdots+g_0(\zeta))=0$, the polynomial $p(x)(\alpha^m+g_{m-1}(x)\alpha^{m-1}+\cdots+g_0(x))$ is the zero polynomial in $F(\alpha)[x]=K[x]$. Since $p(u)\neq 0$, we get

$$\alpha^m+g_{m-1}(u)\alpha^{m-1}+\cdots+g_0(u)=0,$$

where $g_i(u)\in F$. Since f is the minimal polynomial of α over F, this implies $m\geq n$. Therefore, $f(x)\in F(\zeta)[x]$ is the minimal polynomial of α over $F(\zeta)$.