## Second HKUST Undergraduate Math Competition – Junior Level

April 26, 2014

**Directions**: This is a three hour test. No calculators are allowed. **For every problem, provide complete details of your solution**.

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**Problem 1.** Prove that every real-coefficient polynomial $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ on the interval $(-\infty, +\infty)$ is the difference of two polynomials $g$ and $h$, each of which is increasing, where a function $p$ is increasing means for all real numbers $a$ and $b$, $a < b$ implies $p(a) \le p(b)$.

**Problem 2.** Let $A, B$ be invertible $n \times n$ complex matrices. Suppose that $AB = \lambda BA$ for some complex number $\lambda$.

(1) Prove that if $v$ is an eigenvector of $A$, then so is $Bv$.

(2) Prove that $\lambda$ is a root of unity.

**Problem 3.** Prove that if $y = f(x)$ satisfies the differential equation

$$\frac{dy}{dx} = \frac{1}{3 + 2\sin y + x^2}$$

on the interval $(-\infty, +\infty)$, then $f(x)$ must be bounded on the interval $(-\infty, +\infty)$.

**Problem 4.** Evaluate the following sums showing all details:

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{i+j}(i-j)}{(i-j)^2 - \frac{1}{4}} \qquad \text{and} \qquad \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{(-1)^{i+j}(i-j)}{(i-j)^2 - \frac{1}{4}}.$$

**Problem 5.** For what values of $a > 1$ is

$$\int_a^{a^2} \frac{1}{x} \ln \frac{x-1}{32}\, dx$$

minimum? Here $\ln x$ is the natural logarithmic function.

**Problem 6.** A $13 \times 13$ grid of lights can be controlled according to a series of switches. For any $9 \times 9$ or $2 \times 2$ square of lights there is a switch that reverses each of those $9^2$ or $2^2$ lights. Initially, all $13^2$ lights are off. Determine whether or not it is possible to achieve every lighting configuration using some combination of switches.

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