Solution of the Second HKUST Undergraduate Math Competition – Junior Level

1. Below we will write each term $a_k x^k$ as a difference of two increasing polynomials.

   For a_0, let $g_0(x) = x + a_0$ and $h_0(x) = x$, then $a_0 = g_0(x) - h_0(x)$.

   Let $k \in \{1, 2, \dots, n\}$. For $a_k x^k$ with k odd, if $a_k \geq 0$, then let $g_k(x) = a_k x^k$ and $h_k(x) = 0$ Otherwise $a_k < 0$, then let $g_k(x) = 0$ and $h_k(x) = -a_k x^k$. In both cases, we have $a_k x^k = g_k(x) - h_k(x)$.

   For $a_k x^k$ with k even, observe that the polynomial $p_k(x) = x^{k+1} + x^k + x^{k-1}$ is increasing since $p_k{}'(x) = (k+1)x^k + kx^{k-1} + (k-1)x^{k-2} = x^{k-2}\big((k+1)x^2 + kx + (k-1)\big) \geq 0$ due to the discriminant of the quadratic factor is $k^2 - 4(k^2 - 1) = -3k^2 + 4 < 0$.

   If $a_k \geq 0$, then let $g_k(x) = a_k(x^{k+1} + x^k + x^{k-1})$ and $h_k(x) = a_k(x^{k+1} + x^{k-1})$. Otherwise $a_k < 0$, then let $g_k(x) = -a_k(x^{k+1} + x^{k-1})$ and $h_k(x) = -a_k(x^{k+1} + x^k + x^{k-1})$. In both cases, we have $a_k x^k = g_k(x) - h_k(x)$.

   Finally $f = g - h$, where $g(x) = g_0(x) + g_1(x) + g_2(x) + \cdots + g_n(x)$ and $h(x) = h_0(x) + h_1(x) + h_2(x) + \cdots + h_n(x)$ are (strictly) increasing.

2. (1) By assumption, $v \neq 0$, $Av = cv$ for some nonzero scalar c by invertibility of A. Now $ABv = \lambda BAv = \lambda cBv$. So Bv is an eigenvector of A with eigenvalue λc.

   (2) *Solution 1* By induction $B^m v$ is an eigenvector of A with eigenvalue $\lambda^m c$ for any positive integer m. Since A has at most n eigenvalues, so $\lambda^m c = \lambda^{m'} c$ for some $m' > m$. Then λ is an $(m' - m)$-th root of unity.

   Solution 2 We have $\det(\lambda BA) = \det(AB)$. So $\lambda^n \det(B)\det(A) = \det(A)\det(B)$, which is nonzero due to invertibility of A and B. So $\lambda^n = 1$.

3. (*Solution due to Kwok Ling Hon*) For every $x \in (-\infty, +\infty)$, $|f'(x)| = \dfrac{1}{|3 + 2\sin f(x) + x^2|} \leq \dfrac{1}{1 + x^2}$. Let $c = f(0)$. Now

$$|f(x) - f(0)| = \Big|\int_0^x f'(t)\,dt\Big| \leq \max\Big\{\int_{-\infty}^0 |f'(t)|\,dt, \int_0^\infty |f'(t)|\,dt\Big\} \leq \int_0^\infty \frac{dt}{1 + t^2} = \frac{\pi}{2}.$$

   So, $f(x)$ is always in $\Big[c - \dfrac{\pi}{2}, c + \dfrac{\pi}{2}\Big]$.

4. By interchanging i and j in one of the sums, it is clear that the second sum is the negative of the first sum. So it suffices to compute the first sum.

$$\sum_{j=0}^{\infty} \frac{(-1)^{i+j}(i-j)}{(i-j)^2 - \frac{1}{4}} = \frac{1}{2}\sum_{j=0}^{\infty}(-1)^{i+j}\Big(\frac{1}{i - j + \frac{1}{2}} + \frac{1}{i - j - \frac{1}{2}}\Big) =$$

$$\frac{(-1)^i}{2}\Big[\Big(\frac{1}{i + \frac{1}{2}} + \frac{1}{i - \frac{1}{2}}\Big) - \Big(\frac{1}{i - \frac{1}{2}} + \frac{1}{i - \frac{3}{2}}\Big) + \Big(\frac{1}{i - \frac{3}{2}} + \frac{1}{i - \frac{5}{2}}\Big) - \cdots\Big].$$

   This is clearly a telescoping series whose sum is $(-1)^i/(2i+1)$. Therefore, the first sum is

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{i+j}(i-j)}{(i-j)^2 - \frac{1}{4}} = \sum_{i=0}^{\infty} \frac{(-1)^i}{2i+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

$$= \int_0^1 (1 - x^2 + x^4 - x^6 + \cdots)\,dx = \int_0^1 \frac{1}{1 + x^2}\,dx = \text{Arctan}\,1 = \frac{\pi}{4}.$$

5. Define $f(a) = \int_a^{a^2} \frac{1}{x} \ln \frac{x-1}{32}\, dx$ and let $F(x)$ be an antiderivative of $g(x) = \frac{1}{x} \ln \frac{x-1}{32}$ for $x \in (0, +\infty)$. By the fundamental theorem of calculus, $f(a) = F(a^2) - F(a)$. Hence,

$$\begin{aligned} f'(a) &= 2aF'(a^2) - F'(a) = \frac{2}{a} \ln \frac{a^2-1}{32} - \frac{1}{a} \ln \frac{a-1}{32} \\ &= \frac{\ln\{(a-1)(a+1)^2\} - \ln 32}{a}. \end{aligned}$$

It follows that $f'(a) < 0$ if $a \in (1, 3)$ and $f'(a) > 0$ if $a \in (3, +\infty)$. Then $f(a)$ achieves its minimum for $a = 3$.

6. It is not possible. The total number of lighting configurations in the grid is 2^{169}. We will show that the number of configurations achievable using the switches is less than 2^{169}.

   We first observe that we never need to use a given switch more than once. If we flip it an even number of times, it has no net effect; if we flip it an odd number of times, it has the same effect as flipping it once. Secondly, we observe that the order in which we flip switches has no effect, since all the matters for any given light is the number of times it is reversed. Together, these observations imply that all achievable configurations can be obtained by flipping some subset of the switches one time each, in any order.

   The total number of switches is $(13-9+1)^2 + (13-2+1)^2 = 5^2 + 12^2 = 169$, which means there are 2^{169} subsets of switches we may choose from. In order to show that the number of available lightings is strictly less than 2^{169}, we need to find two unequal sets of switches that result in the same lighting. One example is as follows: we can get all lights except the center 5×5 square to be on in 2 ways. One is to light the lower-left and upper-right 9×9 squares, then light the upper-left and lower-right 4×4 squares (using the 2×2 squares). This same lighting can be achieved by rotating the above example by $90°$.

   Thus, there are at most $2^{169} - 1$ available patterns.