Second HKUST Undergraduate Math Competition – Senior Level

April 26, 2014

Directions: This is a three hour test. No calculators are allowed. **For every problem, provide complete details of your solution**.

Problem 1. Prove that every real-coefficient polynomial $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ on the interval $(-\infty, +\infty)$ is the difference of two polynomials g and h, each of which is increasing, where a function p is increasing means for all real numbers a and b, $a < b$ implies $p(a) \leq p(b)$.

Problem 2. Let $\mathbb{N}$ denote the set of all positive integers. Prove that there exists an uncountable set $\mathcal{S}$ of subsets of $\mathbb{N}$ such that if $A, B \in \mathcal{S}$, then either $A \subseteq B$ or $B \subseteq A$.

Problem 3. Suppose R is a ring (possibly commutative or non-commutative). If $x, y \in R$ and the element $1 + xy$ is invertible, then prove that $1 + yx$ is invertible, too.

Problem 4. Alec tosses a fair coin until he gets a tails. If the first tails is on the k-th toss, he then rolls a fair die k times. The probability that his k rolls form a non-decreasing sequence can be written in the form $a^b - c$, where a, b, c are rationals. Compute this probabilty.

Problem 5. Let $b_1, b_2, b_3, \ldots$ be a strictly decreasing sequence and the series $\sum_{n=1}^{\infty} b_n$ converges. For $x > 0$, define $f(x)$ to be the number of n's satisfying $b_n \geq 1/x$. Prove that $\lim_{x \to +\infty} \frac{f(x)}{x} = 0$.

Problem 6. Let $M_n(\mathbb{R})$ be the vector space of all $n \times n$ matrices over the real numbers $\mathbb{R}$. For $A \in M_n(\mathbb{R})$, let A^t denote the transpose of A. Define a linear operators $T_A : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ by

$$T_A(X) = AXA^t.$$

Let $\mathrm{Tr}(A)$ and $\mathrm{Det}(A)$ denote the trace and determinant of A respectively. Prove that the trace of T_A is $\mathrm{Tr}(A)^2$ and the determinant of T_A is $\mathrm{Det}(A)^{2n}$.

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