Solution of the Second HKUST Undergraduate Math Competition – Senior Level

1. Below we will write each term $a_k x^k$ as a difference of two increasing polynomials.

   For a_0, let $g_0(x) = x + a_0$ and $h_0(x) = x$, then $a_0 = g_0(x) - h_0(x)$.

   Let $k \in \{1, 2, \dots, n\}$. For $a_k x^k$ with k odd, if $a_k \geq 0$, then let $g_k(x) = a_k x^k$ and $h_k(x) = 0$ Otherwise $a_k < 0$, then let $g_k(x) = 0$ and $h_k(x) = -a_k x^k$. In both cases, we have $a_k x^k = g_k(x) - h_k(x)$.

   For $a_k x^k$ with k even, observe that the polynomial $p_k(x) = x^{k+1} + x^k + x^{k-1}$ is increasing since $p_k{}'(x) = (k+1)x^k + kx^{k-1} + (k-1)x^{k-2} = x^{k-2}\big((k+1)x^2 + kx + (k-1)\big) \geq 0$ due to the discriminant of the quadratic factor is $k^2 - 4(k^2-1) = -3k^2 + 4 < 0$.

   If $a_k \geq 0$, then let $g_k(x) = a_k(x^{k+1} + x^k + x^{k-1})$ and $h_k(x) = a_k(x^{k+1} + x^{k-1})$. Otherwise $a_k < 0$, then let $g_k(x) = -a_k(x^{k+1} + x^{k-1})$ and $h_k(x) = -a_k(x^{k+1} + x^k + x^{k-1})$. In both cases, we have $a_k x^k = g_k(x) - h_k(x)$.

   Finally $f = g - h$, where $g(x) = g_0(x) + g_1(x) + g_2(x) + \cdots + g_n(x)$ and $h(x) = h_0(x) + h_1(x) + h_2(x) + \cdots + h_n(x)$ are (strictly) increasing.

2. Let $\mathbb{Q}$ be the set of all rational numbers. For every real number r, define $Q_r = \mathbb{Q} \cap (-\infty, r)$. Since $\mathbb{Q}$ is countable, there exists a bijection $f : \mathbb{Q} \to \mathbb{N}$. Let $S_r = \{f(x) : x \in Q_r\}$. Then $S_r \subseteq \mathbb{N}$. If $a < b$, then $Q_a \subseteq Q_b$, which implies $S_a \subseteq S_b$. So $\mathcal{S} = \{S_r : r \in \mathbb{R}\}$ is such an uncountable set.

3. Let $u = (1+xy)^{-1}$. Formally, we guess the inverse of $1 + yx$ by a geometric series argument as follows:

   $$\frac{1}{1+yx} = 1 - yx + yxyx - yxyxyx + yxyxyxyx - \cdots = 1 - y(1 - xy + xyxy - xyxyxy + \cdots)x = 1 - y\Big(\frac{1}{1+xy}\Big)x.$$

   This suggests $1 - yux$ should be the inverse of $1 + yx$. For this, we just need to check

   $$(1+yx)(1-yux) = 1 + yx - yux - yxyux = 1 + yx - y(1+xy)ux = 1 + yx - yx = 1,$$

   $$(1-yux)(1+yx) = 1 + yx - yux - yuxyx = 1 + yx - yu(1+xy)x = 1 + yx - yx = 1.$$

4. The probability of getting a first tail on the k-th toss is $1/2^k$. There are 6^k possible sequences that can result from rolling a fair die k times. The number of non-decreasing sequences of length k using the numbers 1 through 6 is $\binom{k+5}{5}$, as this equals the number of ways to place k identical balls in 6 boxes. Therefore, the overall odds of getting a non-decreasing sequence of any length are

   $$\sum_{k\geq 1} \frac{1}{2^k} \cdot \frac{\binom{k+5}{5}}{6^k} = \sum_{k\geq 1} \frac{\binom{k+5}{5}}{12^k} = \frac{1}{5!} \sum_{k\geq 1} \frac{(k+5)(k+4)(k+3)(k+2)(k+1)}{12^k}.$$

   If we replace $\frac{1}{12^k}$ by x^k, then we get for $|x| < 1$,

   $$\begin{aligned}
   &\frac{1}{5!} \sum_{k\geq 1} (k+5)(k+4)(k+3)(k+2)(k+1)x^k \\
   =&\frac{1}{5!} \sum_{k\geq 1} \frac{d^5}{dx^5} x^{k+5} = \frac{1}{5!} \frac{d^5}{dx^5} \sum_{k\geq 1} x^{k+5} \\
   =&\frac{1}{5!} \frac{d^5}{dx^5} \Big(\frac{1}{1-x} - 1 - x - x^2 - x^3 - x^4 - x^5\Big) \\
   =&\frac{1}{5!} \Big(\frac{5!}{(1-x)^6} - 5!\Big) = \frac{1}{(1-x)^6} - 1.
   \end{aligned}$$

   Therefore, the answer is $\big(\frac{12}{11}\big)^6 - 1$.

5. Assume the conclusion is false. Then there exists $\varepsilon > 0$ such that there exists $x_j \to +\infty$ satisfying $f(x_j)/x_j \geq \varepsilon$. If $1 \leq n \leq f(x_j)$, then since b_n is strictly decreasing, $b_n \geq b_{f(x_j)} \geq 1/x_j$. Now for every positive integer N,

$$\begin{aligned}\sum_{n=N}^{\infty} b_n &\geq \sup\Big\{\sum_{n=N}^{f(x_j)} b_n : f(x_j) \geq N\Big\} \geq \sup\Big\{\frac{f(x_j)-N}{x_j} : f(x_j) \geq N\Big\} \\ &\geq \sup\Big\{\varepsilon - \frac{N}{x_j} : f(x_j) \geq N\Big\} = \varepsilon,\end{aligned}$$

which contradicts $\sum_{n=1}^{\infty} b_n$ converges.

6. (*Solution based on the work of Pang Lok Wing*) Let a_{ij} be the (i,j)-entry of A. Let E_{kl} denote the $n \times n$ matrix with a 1 in the (k,l)-entry and 0 in the other entries. For $M_n(\mathbb{R})$, we will use the ordered basis

$$B = \{E_{11}, \ldots, E_{1n}, E_{21}, \ldots, E_{2n}, \ldots, E_{n1}, \ldots, E_{nn}\}.$$

Now $T_A(E_{kl}) = AE_{kl}A^t = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ik}a_{jl}E_{ij}$. So, with respect to B, the entry in the (i,j) row, (k,l) column of the matrix of T_A is $a_{ik}a_{jl}$. If we divide the $n^2 \times n^2$ matrix of T_A into $n \times n$ blocks, then the matrix is

$$\begin{pmatrix} a_{11}A & \cdots & a_{1n}A \\ \vdots & \ddots & \vdots \\ a_{n1}A & \cdots & a_{nn}A \end{pmatrix}.$$

In particular, the entry in the (i,j) row, (i,j) column of the matrix of T_A is $a_{ii}a_{jj}$. So

$$\operatorname{Tr}(T_A) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ii}a_{jj} = \sum_{i=1}^{n} a_{ii} \sum_{j=1}^{n} a_{jj} = (\operatorname{Tr} A)^2.$$

For determinant, we factor the matrix of T_A as

$$\begin{pmatrix} a_{11}A & \cdots & a_{1n}A \\ \vdots & \ddots & \vdots \\ a_{n1}A & \cdots & a_{nn}A \end{pmatrix} = \begin{pmatrix} A & & 0 \\ & \ddots & \\ 0 & & A \end{pmatrix} \begin{pmatrix} a_{11}I & \cdots & a_{1n}I \\ \vdots & \ddots & \vdots \\ a_{n1}I & \cdots & a_{nn}I \end{pmatrix}.$$

The first matrix on the right side has n blocks of A on the diagonal and 0 matrix on the other blocks. Hence, the determinant of the first matrix on the right side is $\operatorname{Det}(A)^n$. For the second matrix on the right side, for all $1 \leq i < j \leq n$, interchange the (i,j) row with the (j,i) row, then interchange the (i,j) column with the (j,i) column. This will change the second matrix to the first matrix. As this is an even number of interchanges, we get the determinant of the second matrix is also $\operatorname{Det}(A)^n$. So $\operatorname{Det}(T_A) = \operatorname{Det}(A)^{2n}$.

More succintly, in terms of tensor product, the above solution can be simply stated as follows. Let V be the vector space of column vectors with n entries, then we can identify $M_n(\mathbb{R})$ with $V \otimes V$. The operator T_A is the operator $A \otimes A$ on $V \otimes V$. Therefore,

$$\operatorname{Tr}(T_A) = \operatorname{Tr}(A \otimes A) = \operatorname{Tr}(A)\operatorname{Tr}(A) = \operatorname{Tr}(A)^2.$$

$$\operatorname{Det}(T_A) = \operatorname{Det}(A \otimes A) = \operatorname{Det}(A)^n \operatorname{Det}(A)^n = \operatorname{Det}(A)^{2n}.$$