Third HKUST Undergraduate Math Competition – Junior Level

April 25, 2015

Directions: This is a three hour test. No calculators are allowed. **For every problem, provide complete details of your solution**.

Problem 1. Given the value of $I = \displaystyle\int_0^{+\infty} \frac{\ln x}{1+x^2}\,dx$ is a number. Find that number.

Problem 2. Let P_n be the number of permutations $x_1, x_2, x_3, \ldots, x_n$ of the integers $1, 2, 3, \ldots, n$ such that

$$1x_1 \le 2x_2 \le 3x_3 \le \cdots \le nx_n.$$

(a) Find the values of P_1, P_2, P_3 and P_4.

(b) Find the value of P_{20}.

Problem 3. Let n be a positive integer and A be an $n \times n$ matrix over complex numbers. If $A^j = 0$ for some positive integer j, then prove that $A^n = 0$.

Problem 4. For every positive integer n, let $\langle n \rangle$ be the closest integer to $\sqrt{n}$. Determine the value of

$$\lim_{n\to\infty} \sum_{j=1}^{n} \frac{2^{\langle j \rangle} + 2^{-\langle j \rangle}}{2^j}.$$

Problem 5. For all real numbers x and y satisfying $y > x > 0$, prove that $y^{x^y} > x^{y^x}$.

Problem 6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $f, \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ and $\dfrac{\partial^2 f}{\partial x\,\partial y}$ are continuous and each of them at every point of $\mathbb{R}^2$ is not zero. If $f\dfrac{\partial^2 f}{\partial x\,\partial y} = \dfrac{\partial f}{\partial x}\dfrac{\partial f}{\partial y}$ at all points in $\mathbb{R}^2$, then prove that there exist functions $g, h : \mathbb{R} \to \mathbb{R}$ such that $f(x,y) = g(x)h(y)$ at every $(x,y) \in \mathbb{R}^2$.

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