Solution of the Third HKUST Undergraduate Math Competition – Junior Level

1. Let $x = \dfrac{1}{y}$, then $I = \displaystyle\int_0^{+\infty} \frac{\ln x}{1+x^2}\,dx = \int_{+\infty}^0 \frac{\ln(1/y)}{1+(1/y)^2}\Big(-\frac{dy}{y^2}\Big) = -\int_0^\infty \frac{\ln y}{1+y^2}\,dy = -I$. So $I = 0$.

   Remark. A number of students used different substitutions, like $x = \tan\theta$ followed by $\phi = \frac{\pi}{2} - \theta$.

2. (a) Let's refer to the condition $1x_1 \le 2x_2 \le 2x_3 \le \cdots \le nx_n$ as (*). For $n = 1$, $(x_1) = (1)$ satisfies (*) and so $P_1 = 1$. For $n = 2$, $(x_1, x_2) = (1,2),(2,1)$ satisfy (*) and so $P_2 = 2$. For $n = 3$, only $(x_1, x_2, x_3) = (1,2,3),(1,3,2),(2,1,3)$ satisfy (*) and so $P_3 = 3$. For $n = 4$, only $(x_1, x_2, x_3, x_4) = (1,2,3,4),(1,3,2,4),(1,3,2,4),(1,2,4,3),(2,1,4,3)$ satisfy (*) and so $P_4 = 5$.

   (b) Observe that in the cases $n = 1$ to 4, n is either x_n or x_{n-1}. We claim this is true for $n > 4$. If $x_n = n$, then we can get the P_{n-1} permutations of $1, 2, \ldots, n-1$ from the case $n-1$ and add $x_n = n$ to the end of these permutations for case n. Also, if $x_{n-1} = n$, then we can get the P_{n-2} permutations of $1, 2, \ldots, n-2$ from the case $n-2$ and add $x_{n-1} = n, x_n = n-1$ to the end of these permutations for case n. So $P_n \ge P_{n-1} + P_{n-2}$.

   Next assume $n = x_{n-k}$ for some $k = 2, \ldots, n-1$. Then at least one of the k numbers $x_{n-k+1}, \ldots, x_n$ is less than or equal to $n-k$. Say for some $x_{n-j} \in \{x_{n-k+1}, \ldots, x_n\}$, we have $x_{n-j} \le n-k$. Then $(n-k)n = (n-k)x_{n-k} \le (n-j)x_{n-j} \le (n-j)(n-k)$. This implies $j = 0$, i.e. $x_n = n-k$. Then $ix_i = (n-k)n$ for $i = n-k, \ldots, n$. In particular, $(n-1)x_{n-1} = (n-k)n$. Since n and $n-1$ are divisors of $(n-k)n$ and $\gcd(n, n-1) = 1$. So $n(n-1)$ is a divisor of $(n-k)n$, which is less than $n(n-1)$ due to $k \ge 2$. This is a contradiction.

   Therefore, $P_n = P_{n-1} + P_{n-2}$. Using $P_3 = 3, P_4 = 5$, we can compute $P_5, P_6, P_7, \ldots$, then finally $P_{20} = 10946$.

3. (*Solution 1*) Let $L : \mathbb{C}^n \to C^n$ be given by $L(x) = Ax$. Let m be the smallest positive integer such that $\ker L^m = \ker L^{m+1}$. From $L^j = 0$, we get $m \le j$. Since

$$0 \subset \ker L \subset \ker L^2 \subset \cdots \subset \ker L^m \subseteq \mathbb{C}^n,$$

   we have $m \le n$. Next for every $k \ge m$, if $x \in \ker L^{k+1}$, then $L^{k+1}(x) = 0$ and $L^{k-m}(x) \in \ker L^{m+1} = \ker L^m$. Hence, $L^k(x) = L^m \circ L^{k-m}(x) = 0$. Thus, $\ker L^{k+1} = \ker L^k$ for every $k \ge m$. Since $n, j \ge m$, $\ker L^n = \ker L^j = \ker L^m$. Since $\ker L^j = \ker 0 = \mathbb{C}^n$, so $\ker L^n = \mathbb{C}^n$ and $A^n = 0$.

   Remark. In place of kernels, ZHU Songhao considered the ranges of A^k and came up with essentially the same proof.

   (*Solution 2*) Let c be an eigenvalue of A with nonzero eigenvector x. Then $Ax = cx$ implies $0 = A^j x = c^j x$. So $c = 0$. Then the characteristic polynomial of A is $p(t) = t^n$. By the Cayley-Hamilton theorem, we have $A^n = p(A) = 0$.

4. Since $(k - \frac{1}{2})^2 = k^2 - k + \frac{1}{4}$ and $(k + \frac{1}{2})^2 = k^2 + k + \frac{1}{4}$, it follows that $\langle n \rangle = k$ if and only if $k^2 - k + 1 \le n \le k^2 + k$. Hence

$$\begin{aligned}
\lim_{n\to\infty} \sum_{j=1}^n \frac{2^{\langle j \rangle} + 2^{-\langle j \rangle}}{2^j} &= \sum_{k=1}^\infty \sum_{\langle j \rangle = k} \frac{2^{\langle j \rangle} + 2^{-\langle j \rangle}}{2^j} = \sum_{k=1}^\infty \sum_{n=k^2-k+1}^{k^2+k} \frac{2^k + 2^{-k}}{2^n} \\
&= \sum_{k=1}^\infty (2^k + 2^{-k})(2^{-k^2+k} - 2^{-k^2-k}) = \sum_{k=1}^\infty (2^{-k(k-2)} - 2^{-k(k+2)}) \\
&= \sum_{k=1}^\infty 2^{-k(k-2)} - \sum_{k=3}^\infty 2^{-k(k-2)} = 3.
\end{aligned}$$

5. Since $\ln x$ is strictly increasing, it is the same as proving $x^y \ln y - y^x \ln x > 0$.

   Case 1 ($x^y \geq y^x$) Then $y > 1$ (otherwise $0 < x < y \leq 1$ implies $x^y < x^x < y^x$, which is a contradiction) and $x^y \ln y > y^x \ln x$.

   Case 2 ($x^y < y^x$) Either $0 < x < 1$ or $x \geq 1$. If $0 < x < 1$, then $x^y \ln y - y^x \ln x > x^y \ln x - y^x \ln x = (x^y - y^x) \ln x > 0$. If $x \geq 1$, then $y > 1$ and $y \ln x < x \ln y$. Hence

$$x^y \ln y - y^x \ln x > x^{y-1} y \ln x - y^x \ln x = \underbrace{\frac{\ln x}{x}}_{\geq 0}(yx^y - xy^x) \geq 0.$$

   Next $yx^y - yxy^x \geq 0$ iff $\ln(yx^y) \geq \ln(xy^x)$ iff $\ln y + y \ln x - \ln x - x \ln y \geq 0$. Now fix x. Let $f(w) = \ln w + w \ln x - \ln x - x \ln w$. Then $f(x) = 0$. If $y > x$, then

$$f'(y) = \frac{1}{y} + \ln x - \frac{x}{y} = \ln x - \frac{x-1}{y} > \ln x - \frac{x-1}{x} = \frac{1}{x}\int_1^x \ln t \, dt \geq 0.$$

   Then $f(y) \geq f(x) = 0$. Therefore, $x^y \ln y - y^x \ln x > 0$.

6. We have $f_{xy}/f_x = f_y/f$. Integrating with respect to y, we get $\ln |f_x| = \ln |f| + \alpha(x)$ for some function $\alpha(x)$. Then $\ln |f_x/f| = \alpha(x)$. Let $\beta(x) = f_x/f$. By continuity, either $\beta(x) = e^{\alpha(x)}$ or $\beta(x) = -e^{\alpha(x)}$. Then

$$\ln |f| = \int \beta(x) dx + \gamma(y) = \delta(x) + \gamma(y).$$

   By continuity, either $f(x, y) = e^{\delta(x)+\gamma(y)}$ or $f(x, y) = -e^{\delta(x)+\gamma(y)}$. The result follows.