**Third HKUST Undergraduate Math Competition – Senior Level**

April 25, 2015

**Directions**: This is a three hour test. No calculators are allowed. **<u>For every problem, provide complete details of your solution</u>**.

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**Problem 1.** A point $(x, y)$ in the coordinate plane is called a *<u>lattice</u>* point if both coordinates $x$ and $y$ are integers. Prove that no three lattice points in the plane are the vertices of an equilateral triangle.

**Problem 2.** For every real number $a$, define $x_1 = a$ and $x_{n+1} = f(x_n)$ for $n = 1, 2, 3, \ldots$, where $f(x) = 1 + \dfrac{x^2}{1+x^2}$. Determine the set of all real numbers $a$ such that the sequence $x_1, x_2, x_3, \ldots$ defined above is convergent.

**Problem 3.** Let $A$ be a $n \times n$ positive definite matrix over $\mathbb{R}$. Prove that there exists a $n \times n$ symmetric matrix $B$ such that $A = B^2$. Let $C$ and $D$ be $n \times n$ positive definite matrices. Prove that all the eigenvalues of $CD$ are real and positive.

**Problem 4.** Let $a$ and $b$ be positive integers such that for every positive integer $n$, $b^n + n$ is divisible by $a^n + n$. Prove that $a = b$.

**Problem 5.** Let $f : \mathbb{R} \to (0, +\infty)$ be continuous and $\displaystyle\int_{-\infty}^{+\infty} f(x)\, dx = 1$. Let $0 < \alpha < 1$. If $[a, b]$ is an interval of *minimal length* such that $\displaystyle\int_a^b f(x)\, dx = \alpha$, then prove that $f(a) = f(b)$.

**Problem 6.** Let $D = \{z \in \mathbb{C} : |z| < 1\}$. Let $f : D \to D$ be analytic (or holomorphic) and satisfy $f(0) = 0$. If there exists a real number $r \in (0, 1)$ such that $f(r) = f(-r) = 0$, then prove that for all $z \in D$,

$$|f(z)| \leq |z| \left| \frac{z^2 - r^2}{1 - r^2 z^2} \right|.$$

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